6-1-2015

Portfolio Optimization: A Modeling Perspective

Camarie Campfield
Western Oregon University, Ccampfield11@wou.edu

6-1-2015

Portfolio Optimization: A Modeling Perspective -

Camarie Campfield
Western Oregon University, Ccampfield11@wou.edu

Portfolio Optimization: A Modeling Perspective

By

Camarie Campfield

An Honors Thesis Submitted in Partial Fulfillment
of the Requirements for Graduation from the
Western Oregon University Honors Program

Dr. Matthew Nabity,
Thesis Advisor

Dr. Gavin Keulks,
Honors Program Director

Western Oregon University

June 2015

Acknowledgments

To my thesis advisor, Dr. Matthew Nabity, thank you for the motivation, thoughtful insight, and time that you have dedicated to the completion of this project.

To my parents, thank you for your encouragement and instilling in me the confidence that I can do whatever I set my mind to.

To Dr. Gavin Keulks, thank you for your guidance and support you have provided not only for this thesis, but throughout my collegiate career.

Contents

Abstract

Investing is critical in the business world and is an avenue to make profit for many. Making the decisions of what to invest in involves intricate mathematics in order to reduce risk. We investigate portfolio optimization, which is a branch of economic and financial modeling that typically has the goal of maximizing an investment's expected return. We explore a linear programming approach to a decision model for a first time investor. Our results are compared to our expectation and different outcomes are computed based on adjusting our models used for calculating rates of return and failure rates in order to best capture reality. We then explore how changing our constraint of confidence in our investment affects the distribution of the model.

1. Introduction

Investment is defined as putting money to use, by purchase or expenditure, in something offering potential profitable returns, as interest, income, or appreciation in value. Why do people invest? The answer seems obvious and is built into the definition– to make money. In the world we live in today, everything has a price. Money is how we can live; it provides the means to food, shelter, and all the luxuries we seek. Sure money doesn't make you happy, but it does make certain things in life easier.

In America, the inflation rate per year is approximately two percent while savings accounts typically have an annual percentage yield around six-hundredths of a percent. Although there once was a time when savings accounts could be used as a method of generating profit, with current economic conditions money sitting in a savings account is actually depreciating in value. Many people turn to investing their money in hopes to at least compete with the inflation rate.

The number of options for investments is overwhelming. "Thousands of stocks, thousands of bonds, and many other alternatives are worthy of consideration" [8]. That list of alternatives includes tax-deferred retirement and education accounts such as Roth IRAs and 403b plans respectively, target-date funds, and utilizing a financial manager. Francis goes on to say that "Portfolios are the objects of choice. The individual assets that go into a portfolio are inputs, but they are not the objects of choice on which an investor should focus. The investor should focus on the best possible portfolio that can be created... A portfolio is simply a list of assets. But managing a portfolio requires skills" [8]. It is important to consider that the individual investments alone are not going to make or break you, what is important to consider is the mix of investments as a whole and how the combination will perform to create your portfolio. So how does someone figure out what investments they want in their portfolio when there is such an abundance of options? One option is to just go with your instincts and hope everything turns out for the best, but that seems rather risky; another option is to follow the method dominating trading on Wall Street and use mathematical models as a base for making decisions. Investing is risky, but that risk can be reduced with the accurate use of mathematical models that are current and built upon good assumptions.

2. Sufficient Background Information

Portfolio Optimization is a mathematical approach to aid in making the decision of what mix of assets to invest in, according to certain criteria. These criteria are left up to the investor, but typically one considers the desired rate of return for the portfolio, the time desired for dispersion, and the level of risk the investor is willing to accept. These methods allow for you to have meaning and purpose behind your investments because of research done to make the investment with as little risk as possible.

It is a common belief of economists that diversification plays a key role in reducing risk in investing. Investments fail to produce profits frequently enough, but if you spread out your money over a multitude of investments, how likely is it that they will all fail? However, you must know how to properly diversify a portfolio. Markowitz sums this idea up nicely by saying, "A portfolio with sixty different railway securities, for example, would not be as well diversified as the same size portfolio with some railroad, some public utility, mining, various sort of manufacturing, etc. The reason is that it is generally more likely for firms within the same industry to do poorly at the same time than for firms in dissimilar industries" [11]. Markowitz is advocating for not only investing in many different opportunities, but investing across industries. This is because different industries have different economic characteristics and lower covariances than businesses within the same industry. Covariance provides a measure of the strength of correlation between things, or in other words, how much a change in one thing directly affects the performance of another thing. So in the case of investments, lower covariance is good because it means lower dependency. That way if one investment fails, it does not necessarily affect the others.

Mathematical modeling is "based on the desire to understand some behavior or phenomenon in the real world" [10]. Models are meant to approximate a real world system and are based on assumptions which aim to reduce the number of factors under consideration to make finding a solution feasible. Mathematical modeling is used in various disciplines, such as engineering, physics, ecology, and economics; it is a very useful tool for making predictions and providing insight about the real world. However, it is important to remember that mathematical modeling is an experiment on mathematical representations of the real world. There is no best model, only better models. Howard Emmons once said that the challenge in mathematical modeling is "not to produce the most comprehensive descriptive model, but to produce the simplest possible model that incorporates the major features of the phenomenon of interest" [10].

Many approaches of mathematical modeling draw on mathematics involving linear algebra and probability. Many people think that probability is the chance of something happening. However, a better definition would be that probability is the numerical likelihood that something will occur. The difference is that the latter definition iterates that probability is just a representation of reality; it provides no guarantees. If you flip a fair quarter, it is common knowledge the probability that you get heads is 50%, or 1 out of 2. However, time and time again someone flips a quarter twice and gets two tails. That result does not change the probability, but it does show that the probability figure is not always a guarantee. At the same time, probability still is highly valuable and a valid tool. If someone were to flip that same quarter a thousand times, their percentage of getting heads would near the 50% ratio.

A common technique in modeling is optimization, specifically linear programming. Linear programming is a mathematical method for maximizing or minimizing a linear function subject to linear constraints. Dantzig describes it as “Part of a great revolutionary development which has given mankind the ability to state general goals and to lay out a path of detailed decisions to take in order to ‘best’ achieve its goals when faced with practical situations of great complexity” [6]. Linear programming was developed in 1947 because of the need to solve complex planning problems in wartime operations. Many industries began using this method to allocate their resources in an optimal way. The industries included airline crew scheduling, shipping or telecommunication networks, oil refining and blending, and stock and bond portfolio selection. Linear programming was so useful because it could be applied diversely. George B. Dantzig and John von Neumann are often credited as the founders of linear programming. Dantzig is given credit for the Simplex method, which is a method still utilized today that makes use of a step-by-step tableau system, while von Neumann was responsible for the theory of duality [12].

Linear programming is a highly useful tool for finding optimum solutions, but the process of actually finding those solutions can become rather complex extremely quickly. Computers can be used to find the solutions, but there are other methods that simplify the process so that it can still be done by hand. The Simplex method is one algorithm used for linear programming. A linear program has the important property that the points satisfying the constraints form a convex set, which means that any two points within the set can be joined by a straight-line segment in which all of its points lie within the set. The Simplex method is able to work because minimum and maximum values always occur on one of the extreme points due to the set being convex. This means that there is a finite number (except in special cases where an extreme point occurs along an edge of the feasible region resulting in an infinite number) of feasible solutions to the problem; however finite is still quite large for the typical linear program. The Simplex method works by utilizing extreme points. If the initial extreme point chosen is not a minimum or maximum, then the edge containing the point decreases or increases respectively, and the simplex algorithm applies this insight by continuing along the polytope to the next extreme point. This process is an effective way of finding optimum solutions [7].

While math is a helpful tool for these decisions, by no means can it predict the future. Ballentine points out that there is a "very substantial gap between asset allocation theory and the real world" [2]. He considers portfolio construction to be an art rather than a science and believes that to be successful in investment you need to have experience, keen judgment and skill, but mostly flat-out luck. While Ballentine seems to be at the extreme point of view placing very little worth in mathematics, he does have a point that great care must be taken while constructing models. This is why models are constantly changing and evolving; mathematicians continue to strive towards creating a model that can portray the best grasp on reality to aid in investment-making decisions. Each model takes into account a different set of assumptions and a different way to account for them in the model; the models will continue to evolve and become better portrayals of reality.

3. Leonard's Portfolio

Gallin and Shapiro address one approach to portfolio optimization in "Optimal Investment under Risk" [9]. Their approach combines linear programming and probability theory to figure out an optimal strategy. They illustrate their method through a typical investment problem; an investor named Leonard has \$12,000 he wishes to invest so that his money does not waste away in a savings account. Leonard has found three potential investment opportunities each with its own risk and expected return associated. His first option is a Broadway musical production with a 25% failure rate, but expected double return when successful; we'll refer to this opportunity as investment option A. Another option is an opportunity with molybdenum futures with failure rate of 20% and average return of 75% (option B). His last option is an oil development scheme with failure rate of 10% and a potential return of 50% (option C). Leonard wants to make his choice for investment in the best way possible, so he does some research.

Note how in this article the probabilities are just given to us with minimal explanation of how they were found. The article says that the probabilities were based on an investigation of past performances of similar ventures. One of the things I will explore is where they find the numbers they use to calculate the probability. Models are only as good as the numbers they are based on, so having accurate figures is important. In order to make the probabilities and models as accurate as possible, they need to be based on accurate assumptions that are effective in narrowing down the problem, but not overgeneralizing.

Leonard looks into the expected return for each of these possible investments in order to better compare and decide upon the investment that will most likely return revenue with little risk associated. The expected return for each investment is calculated by subtracting the probability of a loss (which is the failure rate) from the probability of a gain (which is the success rate multiplied by the average return). For example, the expected return for option C is $0.9 \cdot 0.5 - 0.1 = 0.35$. Calculated similarly, the expected return for option A is 0.5 and for option B is 0.4. Expected return can be a useful tool when investments are made day after day because then the actual return will begin to be comparable to the expected return. Just like the probability of flipping a head on a quarter, reality and theory come closer together upon repetition. However, if the investment is a one time deal then the return is much more variable. Investing in the opportunity with the highest expected return often means involving high risk that you will lose money. Based on comparing expected returns, Leonard should invest all of his money in the Broadway option since it has the highest return, but that would mean running a risk of 25% of losing all of his money. Since Leonard is only making this investment once, he isn't sure that expected return alone is the best basis for his decision.

Investment	Failure Rate	Rate of Return	Expected Return
A	25%	100%	50%
B	20%	75%	40%
C	10%	50%	35%

Table 1: Leonard's Expected Return

Gallin and Shapiro then look at another method of figuring out what Leonard should invest in. Leonard tells his investment advisor that he wants no more than 10% risk of losing money on his investment. An objective function is what is trying to be maximized or minimized in the situation, so in this case it is the gain on investment that is trying to be maximized. Constraints are the conditions set in place that the solution must follow. An objective function is then formulated for the expected gain with the given constraints of positive investments (since you cannot invest negative dollars into any of the options), his $\$12,000$, and the probability of gain being greater than or equal to 90%. The objective function is

$$\text{Max } E(G) = 0.50x + 0.40y + 0.35z \tag{3.1}$$

subject to

$$x, y, z \geq 0$$

$$x + y + z \leq 12000$$

$$P(G \geq 0) \geq 0.90$$

This results in eight possible outcomes for the three investments, the different combinations of success and failures per event. For example, there is a 0.54 probability that all three investments will be successful. Since we have made the assumption that each event is independent from one another, we calculate this probability by multiplying each individual event's probability of success, so $0.75 \cdot 0.8 \cdot 0.9 = 0.54$. Then the net gain is $1.00x + 0.75y + 0.50z$, which is the combination of each investment's percentage return based on the amount of dollars invested in it. Similarly if we want to look at the net gain when A fails but B and C are successful, we have a probability of $0.25 \cdot 0.8 \cdot 0.9 = 0.18$ of this occurring and a net gain of $-x + 0.75y + 0.50z$. In words, Leonard would have an 18% chance of losing x dollars, but gaining $0.75y + 0.50z$ dollars. We can do this for each of the eight scenarios.

Case	A	B	C	Probability	Net Gain
1	S	S	S	54.0%	x+0.75y+0.50z
2	S	S	F	6.0%	x+0.75y-z
3	S	F	S	13.5%	x-y+0.50z
4	S	F	F	1.5%	x-y-z
5	F	S	S	18.0%	-x+0.75y+0.50z
6	F	S	F	2.0%	-x+0.75y-z
7	F	F	S	4.5%	-x-y+0.50z
8	F	F	F	0.5%	-x-y-z

Table 2: Leonard's Net Gain

Since the events are disjoint and form a partition of all the possibilities, we can employ the Total Probability Theorem from the probability world to get the equation below [3].

$$\begin{aligned} P(G \geq 0) = P(SSS) \cdot P(G \geq 0|SSS) + P(SSF) \cdot P(G \geq 0|SSF) + ... \\ + P(G \geq 0) \cdot P(G \geq 0|FFF) \end{aligned}$$

This is the total probability of having a positive gain. It is calculating the probabilities of having a positive gain for each of the eight situations individually and summing them together since they are mutually exclusive. The conditional probabilities, for example $P(G \geq 0|SSS)$, are the probabilities that the gain will be positive given that that is the event to occur, in this case that all three investments are successful. The conditional probabilities can therefore only take on the values of 0 and

1; either the event will realize a positive net gain or a negative one. The conditional probability is dependent on the actual net gain, which is dependent on the values invested in each of the choices. So we have

$$P(G \geq 0) = p_1\varepsilon_1 + p_2\varepsilon_2 + ... + p_8\varepsilon_8$$

where, for example, $p_3 = P(SFS) = 0.135$ and ε_i is either 0 or 1 depending on whether $x - y + 0.50z \leq 0$ or $x - y + 0.50z \geq 0$, respectively.

Net Gain	$P(G \geq 0)\|Case)$	Probability
9750	1	54.0%
6750	1	6.0%
1000	1	13.5%
-2000	0	1.5%
-250	0	18.0%
-3250	0	2.0%
-9000	0	4.5%
-12000	0	0.5%

Table 3: Leonard's Likelihood

So if we assume that Leonard invests $\$5,000$ in x and y, and $\$2,000$ in z (for a total equaling his allotted $\$12,000$), the net gain for each instance occurring can be calculated. If A was to fail and B and C were to be successful, we would have a net gain of $-5000 + .75 \cdot 5000 + .50 \cdot 2000 = -250$. In other words, a net loss of \$250. Therefore, the conditional probability of having a positive net gain given that A fails, and B and C are successful, would be 0. If we work out the actual net gain for each combination of events being successful/failure, we would find that our net gain is positive for SSS, SSF, and SFS (1,2, and 3). However, the probabilities of any of those three events occurring only account for $0.54 + 0.06 + 0.135 = 0.735$ of the possibilities, and our risk constraint is 90% certainty. Therefore, allocating the money as we originally planned won't work.

From where we are, the problem Gallin and Shapiro present can be viewed geometrically. Rather than guessing different values to invest in each opportunity and work through to see if any of them fit our constraints, we can solve the problem with our new set of constraints. The problem according to those guidelines is to maximize a linear objective function over a constraint set which is the union between two convex polyhedral sets. These types of problems are easy to solve. First, the individual sets must be maximized. Then the solution is given by finding the greater value of the objective function for each of the optimal solutions. The simplex algorithm can be used to find the maximum values.

So going back to Table 2, since we know $P(G \geq 0)$. In fact, we know $P(G \geq 0) \geq 0.90$, we will need either p_1, p_2, p_3, p_5 or p_1, p_3, p_5, p_7. So our risk constraint of $P(G \geq 0.90)$ is equal to the disjunction of two different sets of joint linear inequalities. Thus we rewrite our problem with that risk constraint.

From here, to solve this problem we can view it as maximizing a linear objective function over a constraint set S which is a union of $S_1 \cup S_2$ of convex polyhedral sets. Thus we can maximize the function separately over S_1 and S_2 and choose whichever point gives the greater value at points $P_1 \in S_1$ or $P_2 \in S_2$. There are multiple ways to solve this kind of problem, one of the more popular being the Simplex Method. The Simplex Method is so popular to use on smaller dimension problems because of its efficiency. It generally takes no more than 2 or 3 times the number of equality constraints of iterations to find a maximal point [4].

Our new problem is to solve

$$\text{Max } E(G) = 0.50x + 0.40y + 0.35z \tag{3.2}$$

$$\text{subject to}$$

$$x, y, z \geq 0$$

$$x + y + z \leq 12000$$

$$x + 0.75y + 0.50z \geq 0$$

$$x + 0.75y - z \geq 0$$

$$x - y + 0.50z \geq 0$$

$$-x + 0.75y + 0.50z \geq 0$$

as well as solving this:

$$\text{Max } E(G) = 0.50x + 0.40y + 0.35z \tag{3.3}$$

$$\text{subject to}$$

$$x, y, z \geq 0$$

$$x + y + z \leq 12000$$

$$x + 0.75y + 0.50z \geq 0$$

$$x - y + 0.50z \geq 0$$

$$-x + 0.75y + 0.50z \geq 0$$

$$-x - y + 0.50z \geq 0$$

Then we choose from the two whichever gives the maximum value for expected gain. One of the niceties of the Simplex Method is that it can be done by hand with a series of tableaus. However, the other convenience is that it can also be handed off to a software program like Matlab and we can compute a solution in a fraction of time compared to using tableaus. Doing this, we find the solution to 3.2 to be $E(G) = 5,224$ at the point $P_1 = (4941, 5647, 1412)$ and to 3.3 to be $E(G) = 4,800$ at the point $P_2 = (4000, 0, 8000)$. Thus we would choose P_1 since the expected gain is greater at that point. Remember, P_1 is the number of dollars we should invest in each investment opportunity to get our maximum return, so this tells us to invest \$4,941 in the Broadway musical, \$5,647 in the molybdenum futures, and \$1,412 in the oil development scheme.

Gallin and Shapiro also discuss how the model they have illustrated can be generalized to all optimization problems. If instead of only having three investment options, Leonard had ten, we could account for this by extending our objective function to include ten variables. Any finite number of investment opportunities with a given return can be expressed as a fraction of the amount invested. In general, the return is a random variable that takes on known values with known probabilities. If the investments are not independent of each other, then the joint distribution function for the returns is also needed. The risk constraint that Leonard mandated can be replaced by another condition depending on the situation. The expected net gain can then be calculated from the rate of return, amount invested in each option, and the total amount of capital. The problem then becomes to maximize the expected gain subject to the constraints. Then one of the techniques of solving stochastic problems can be used to find the solution [9].

4. Analyzing the Model

A model is only as good as the assumptions it is built upon, so let's look at those present in Leonard's situation. In Leonard's problem, we are given failure rates and expected return rates that were assumed to be true; these came to us with very little explanation of how they were calculated. This assumption directly affects the accuracy of our output, but it does not change the construction of our model if they were to change. It would just mean substituting in different values to our model. Often, failure rates and expected return rates are calculated by analyzing historical performance. There is no guarantee that something will continue to perform in the same way, but such is the gamble of the stock market. When it comes to investments, everyone is comfortable taking on different risks. A big factor that relates to comfort levels is one's age. When people are young and just beginning their portfolios they generally have a large assortment between high and low risk investments. This is because they can afford more risk at that point in their lifetime. If they were to take a large hit on their portfolio, they would have plenty of time over their life to recover and make up for their loss. As people get older and closer to retirement, they move many of their investments into much lower risk genres because they no longer have time to recover from a major loss in their portfolio, and ideally they will already have enough money saved up that they do not need to take on the higher risk in hopes of higher returns.

There is also the assumption that there are only two possible outcomes for each investment, it either fails and you lose everything or it succeeds and you realize 100% of the expected return. This isn't very realistic of the world, since it is very possible that you could take a hit on an investment and lose some capital without losing it all, you may break even, or you could turn a profit without getting as much as you expected to. However, this perspective of the investments either succeeding or failing is how we are able to come up with the eight different cases that we had and calculate the probability of each occurring. Another assumption is that the success rates are independent from one another. This is likely to be true in our real world investments as well, unless we are choosing stocks in a related field. Arguments can be made that the entire economy is related and therefore any stock's performance would have an affect on another stock. However, arguments such as these can be dismissed because even if there is a relationship there, its effects are minimal and therefore negligible.

The beauty of modeling is the intricate balance in making sure our model is accurate and a good prediction of what is likely to occur, while keeping a computation that is achievable. We could continue to add in different assumptions and constraints to try and make the model even more closely tied to reality, but those come at a cost of making our model more complex. The ideal model will portray everything substantial to the outcome, while still being simple enough in computation.

5. Our Portfolio

Investing is extremely relevant in our era, and crucial to young people. The earlier you begin to invest, the exponential effects it has later on in your life. A famous example of this is the story where you double your money every day. You start with a penny, and by day ten you still only have $5.12, but do not be fooled. By the end of the month you will have over $5 million. Just missing one day, brings you down to $2.6 million. The effects of one day in time are dramatic, but so are the effects of waiting to invest until you are older. Many young people, including myself, are getting ready to graduate from college and enter the real world, and to plan for their future financially. We have so many decisions to make, and what we decide now has lasting consequences on our life. Time and time again we hear how much greater our investments will become the earlier we make them.

So, let us consider a model that will be more realistic for our generation than Leonard's. We will keep our portfolio small at three investments. If we were to add more options, as is typical for portfolios, it would increase the computational complexity, but not change the method of solving the problem. We will choose investments that are appealing and relevant to our generation: stock in Apple, Inc. and Whitestone Real Estate Investment Trust (REIT), and a Roth IRA.

IRA stands for Individual Retirement Account. Essentially, it is a savings account with benefits of tax breaks. Another major benefit is that typically employers will offer a match on investments in your IRA up to a certain cap. This is likely one of the greatest investments you will ever see based on that fact, once you allocate a certain amount of your paycheck and your employer matches your contribution, you just invested with 0% risk and an immediate 100% return. You will not be able to beat those figures. Having an IRA is like having an entire separate portfolio in itself, since it includes investments in stocks, bonds, mutual funds, and other assets. However, this portfolio someone else manages for you and there is little to no risk associated with it. Since risk and return go hand in hand, the minimal risk results in lower return levels (without factoring in employer's matches). However, this adds to the diversification of our portfolio by giving us an investment on the safer side. As a first time investor with not a lot of money, a safer option holds an appeal. There are several types of IRAs- traditional IRAs, Roth IRAs, SEP IRAs, and SIMPLE IRAs. Each IRA comes with limits of what can be invested in it per year and penalties if you withdraw money before a designated retirement age. A Roth IRA is a retirement savings account that allows your money to grow tax-free, which is why we have selected it over the other types of IRAs. You fund a Roth with after-tax dollars, meaning you have already paid taxes on the money you put into it. In return for no up-front tax break, your money grows tax free, and when you begin to make withdrawals at retirement, you do not have to pay taxes on the distributions.

Stock is a share of a company held by an individual or group that entitles that individual or group to partial ownership of the company. Selling stock is one way corporations are able to raise capital. Stock prices fluctuate on a daily basis due to market and company performance. A real estate investment trust, REIT, is a company that owns, and typically operates, income-producing real estate or real estate-related assets. REITs provide a way for individual investors to earn a share of the income produced through commercial real estate ownership, without requiring that they have enough funds to purchase real estate on their own. We have selected stock in Whitestone REIT to add to our portfolio.

Now we have our investments, but we still need our figures for our model so that we can calculate what mix of these investments will maximize our investment without maximizing our risk. IRAs are considered extremely safe with no risk, so we will use zero as our rate of risk, and an average rate of return of 7%. When it comes to our stock in Apple and Whitestone, the figures are not as easy to come by. We will base our rate of return off of historical data, as is typical for calculating these rates. We will also use a figure called beta to measure risk. Volatility is the amount by which investment returns vary over a certain time period. A larger value for volatility implies greater variability which means more risk which means higher chance of selling a low point. Volatility can be determined by analyzing the historical information. In order to find our measures of risk, we will use the beta values which have already been calculated for us on any of the financial sites. Beta is a measure of volatility that is calculated using regression analysis. Beta is a representation of a security's response to swings in the market. A beta of value 1 implies that the securities price will move with the market. Less than 1 implies that the security is less volatile than market, or more stable and therefore safer. Greater than 1 means security price is more volatile than the market [5].

The beta values for Apple and Whitestone are 1.06 and 0.62, respectively [1]. In their current state, these measures are unhelpful. We need to fit the beta values to some percentage corresponding to the risk, but how do we translate these values into percentages? This is not done for us as the risk percentages were given to us in Leonard's situation, so we must figure out a way of our own. We will start by figuring out our range for beta. According to Crowell, values for beta typically range from 0.5-2.5 [5]. We also make the assumption that the risk associated with stocks typically range from 10% to 50%. There is always some risk associated with stock investment no matter how good of a stock it is. The market is always subject to crashing such as in 2007. Additionally, we figure very few stocks exceed 50% risk. To get our risk from beta values to percentages, first we will try to fit a linear equation to model the relationship of the points $(0.5, 0.10)$ and $(2.5, 0.50)$. This can be seen in Figure 1.

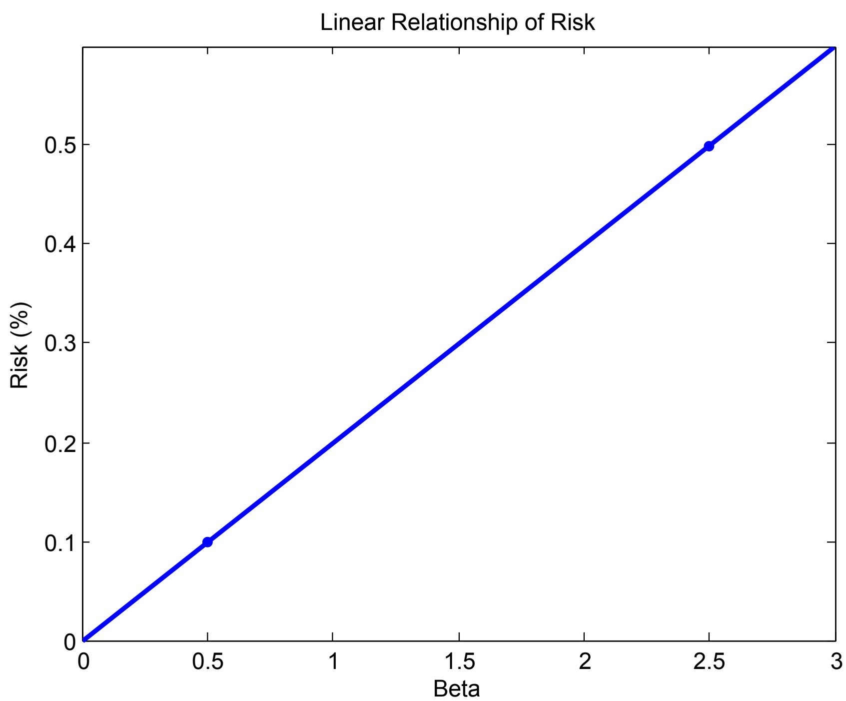

Figure 1: Linear Regression for β values

So, for our equation where β is the beta value and R is the percentage of risk, we have $R = 0.2\beta$. This gives us risk values of 21.2% and 12.4% for Apple and Whitestone, respectively. Comparing this to our initial risks in Leonard's dilemma, this seems reasonable since the risks ranged from 10% to 25%. It also reflects what we would expect from our beta values, the stock in Apple is moderately risky, since beta was greater than one, but the stock in Whitestone is much safer since beta was less than one.

Now we need to figure out what rate of return we can expect for these stocks. To get an idea of this, we will examine historical data for past selling prices. We will use data from the last five years to get an idea of what to expect. See Table 4 for the adjusted closing prices [1]. We are using the adjusted prices because they take into account dividend shares and stock splits to make historical prices comparable to current prices. Whitestone was not a stock until August 26, 2010, so we will use its first opening price as our beginning point.

Investment	1/4/10	8/26/10	1/3/11	1/3/12	1/2/13	1/2/14	12/31/14
Whitestone		7.93	10.51	9.63	12.19	12.29	14.93
Apple	28.84		44.41	55.41	74.64	77.09	109.95

Table 4: Historical Prices

We will use Equation 5.1 to calculate our annual rates of return for each of the five years.

$$\text{Annual Rate of Return} = \frac{\text{Closing Price} - \text{Opening Price}}{\text{Opening Price}} \tag{5.1}$$

Table 5 summarizes Apple and Whitestone's annual rates of return. We will then average the five annual rates of return and use the averages in our model. This gives us rate of returns of 14.6% and 31.9% for Whitestone and Apple respectively.

Investment	2010	2011	2012	2013	2014
Whitestone	0.325	-0.084	0.266	0.008	0.215
Apple	0.534	0.245	0.347	0.033	0.427

Table 5: Yearly Average Returns

Since Whitestone's risk is greater than our IRA's and less than Apple's, we would expect its rate of return to follow the same relationship, as we can see that it does. We take that as a good sign. Now we have calculated what appear to be reasonable values for risk and return, or in the case of our IRA found reasonable values. This is much different from Leonard's problem where we were just given the values for this. These values are critical to the solution because they will determine how you are to distribute your money. It is important that they are as accurate a prediction of the future as they can be.

Now that we have the values, we will solve this problem in the same way as Leonard to figure out how we should invest. First, we will calculate the expected return for each of these investments, see Table 6 for the values.

	Investment	Beta	Failure Rate	Rate of Return	Expected Return
A	Roth IRA	N/A	0.0%	7.0%	7.0%
B	Whitestone	0.62	12.4%	14.6%	0.4%
C	Apple	1.06	21.2%	31.9%	3.9%

Table 6: Expected Return

Thus we get Table 7 that shows the eight different scenarios that can play out. Since we have assumed a 0% failure rate for our IRA, this gives the four events including the IRA investment failing a 0% probability of occurring.

Case	A	B	C	Probability	Net Gain
1	S	S	S	69.0%	0.070x+0.146y+0.319z
2	S	S	F	18.6%	0.070x+0.146y-z
3	S	F	S	9.8%	0.070x-y+0.319z
4	S	F	F	2.6%	0.070x-y-z
5	F	S	S	0.0%	-x+0.146y+0.319z
6	F	S	F	0.0%	-x+0.146y-z
7	F	F	S	0.0%	-x-y+0.319z
8	F	F	F	0.0%	-x-y-z

Table 7: Net Gain

As before with Leonard's problem, since the model is designed for a young investor, we would like 90% certainty that we will have a positive gain. As we can see, this is achieved in combinations of Cases 1, 2, and 3 or Cases 1, 2, and 4. Thus our problem becomes to solve for each of the following problems below (with the different constraints ensuring that 90% confidence is achieved) and then choosing whichever of the two outputs that gives a higher expected return.

$$\text{Max } E(G) = 0.070x + 0.004y + 0.039z \tag{5.2}$$

$$\text{subject to}$$

$$x, y, z \geq 0$$

$$x + y + z \leq 12000$$

$$\text{and either}$$

$$0.070x + 0.146y + 0.319z \geq 0$$

$$0.070x + 0.146y - z \geq 0$$

$$0.070x - y + 0.319z \geq 0$$

$$\text{or}$$

$$0.070x + 0.146y + 0.319z \geq 0$$

$$0.070x + 0.146y - z \geq 0$$

$$0.070x - y - z \geq 0$$

Solving this with the Simplex method in Matlab, our solution for either set of constraints is to put $12,000 into the IRA with an expected return of $840. So we have our answer, we will put all $12,000 into the IRA and expect to get a return of $840. Is that really the best distribution of our money though? A common belief of financial advisors is that diversification in a portfolio is critical to our success. So what if the assumptions that we made and based our figures off of that led to our model telling us to invest all $12,000 into the IRA did not account for the need for diversification? We also assumed 0% risk for the IRA which resulted in the expected return being much higher than either Whitestone or Apple. Is that actually reasonable considering the rates of returns and how much more Whitestone and Apple are expected to yield? Also, looking at Table 5, we can see that the five year average rates of returns for Whitestone and Apple are much lower than their average rate of return for just 2014, which is the most recent representation of their performance. Should we be considering the average rate of return over the last five years or is the current year's rate a better capture of reality? These questions are all things to consider. Our model has given us the distribution it has because of the assumptions and computations we made, but we want to be sure that each of those is really the best picture of reality. So now, we will explore what our model tells us to do when we experiment with some of our previous assumptions. Our goal is to get the most realistic model that we can.

5.1. Annual Rates of Return

Comparing our rates of return, we notice that they are much lower than rates from Leonard's portfolio. Looking at Table 5, we can see that each of our stocks had higher rates of returns in 2014 than they did over the average of the five years. In particular, years 2011 and 2013 were not great for either of them. This could be for many reasons, but those two years greatly bring down our average. So instead of using the five year average return, we will use just the rate of return achieved in 2014. Thus our new figures include a 21.5% return for Whitestone and a 42.6% return for Apple. These figures produce Table 8.

	Investment	Beta	Failure Rate	Rate of Return	Expected Return
A	Roth IRA	N/A	0.0%	7.0%	7.0%
B	Whitestone	0.62	12.4%	21.5%	6.4%.
C	Apple	1.06	21.2%	42.6%	12.4%.

Table 8: Expected Return for the 2014 Rate of Return Model

In the same manner as before, we will look at the eight different scenarios that may occur with our investments. Table 9 gives the expected net gain and likelihood for each situation to occur.

Case	A	B	C	Probability	Net Gain
1	S	S	S	69.0%	0.070x+0.215y+0.426z
2	S	S	F	18.6%	0.070x+0.215y-z
3	S	F	S	9.8%	0.070x-y+0.426z
4	S	F	F	2.6%	0.070x-y-z
5	F	S	S	0.0%	-x+0.215y+0.426z
6	F	S	F	0.0%	-x+0.215y-z
7	F	F	S	0.0%	-x-y+0.426z
8	F	F	F	0.0%	-x-y-z

Table 9: Net Gain for the 2014 Rate of Return Model

Based off of these figures, our new problem becomes

$$\text{Max } E(G) = 0.070x + 0.064y + 0.124z \tag{5.3}$$

$$\text{subject to}$$

$$x, y, z \geq 0$$

$$x + y + z \leq 12000$$

$$\text{and either}$$

$$0.070x + 0.215y + 0.426z \geq 0$$

$$0.070x + 0.215y - z \geq 0$$

$$0.070x - y + 0.426z \geq 0$$

$$\text{or}$$

$$0.070x + 0.215y + 0.426z \geq 0$$

$$0.070x + 0.215y - z \geq 0$$

$$0.070x - y - z \geq 0$$

Again using the Simplex method to solve, our first constraints give us to put \$9,971 into our IRA, \$1,096 into Whitestone, and \$934 into Apple with a return of \$884. The second set of constraints gives us \$9,326 into the IRA, \$1,663 into

Whitestone, and $\$1,010$ into Apple with an overall expected return of $\$885$. Therefore we would choose to follow the distribution given by the second set of constraints. This still gives us to put the majority of our funds into the IRA, the allocation continues to favor the IRA because of its failure rate of 0%. So this begs the question, is the 0% really an accurate capture of the risk associated with an IRA?

5.2. Quadratic Relationship between Beta and Risk

Up to this point, we have used a linear relationship to connect our beta values to risk in the form of percentages. The linear model was useful because it offered a vast array of options with modest effort. Now we will explore a different relationship between beta and risk, and fit a quadratic to our model. We will build our quadratic model off of assumptions of points connecting beta values to risk percentages. We will assume $(0, 0.02)$ is a point due to the minimal, yet apparent, risk of the IRA. We want no beta value to correlate to a small amount of risk, since risk is apparent in all investments and we do not want our model to be biased towards the IRA. The next point we will use is the same as in Case 1, $(0.5, 0.10)$. Our final point needed to formulate a quadratic will be $(2, 0.75)$. This point represents much higher risk corresponding a value of beta than we have considered before. Solving this quadratic gives us the equation $y = 0.137x^2 + 0.092x + 0.02$ where y is the risk in percentage form and x is the beta value.

Using this formula to find our failure rates and again using 2014's rate of return, we are able to fill out Table 10 with the expected return for each investment.

	Investment	Beta	Failure Rate	Rate of Return	Expected Return
A	Roth IRA	N/A	2.0%	7.0%	4.9%
B	Whitestone	0.62	12.9%	21.5%	5.8%.
C	Apple	1.06	27.1%	42.6%	4.0%.

Table 10: Expected Return for the Quadratic Model

Once again, we then construct Table 11 which gives the probability and net gain

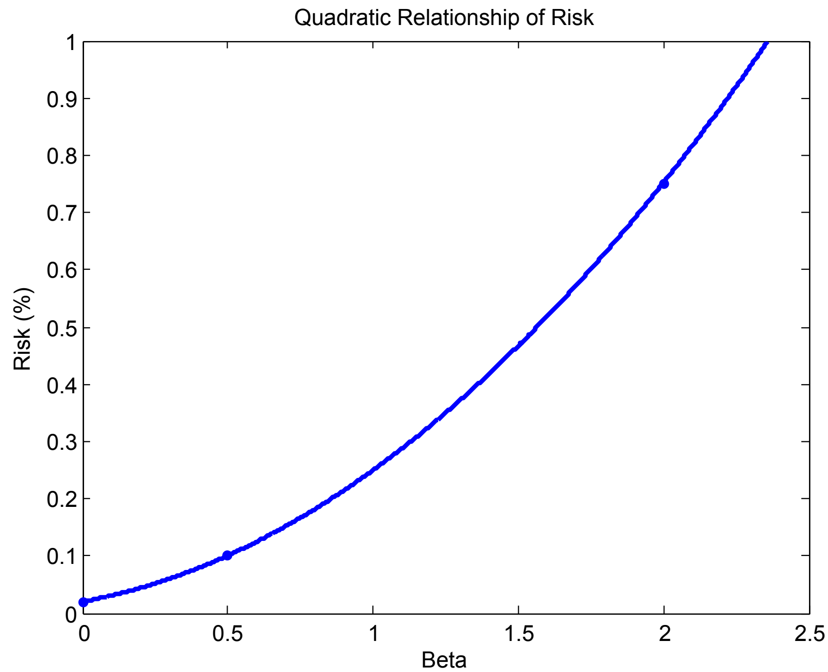

Figure 2: Quadratic Regression for β values

for each possible outcome for our investments.

Case	A	B	C	Probability	Net Gain
1	S	S	S	62.2%	0.070x+0.215y+0.426z
2	S	S	F	23.1%	0.070x+0.215y-z
3	S	F	S	9.2%	0.070x-y+0.426z
4	S	F	F	3.4%	0.070x-y-z
5	F	S	S	1.3%	-x+0.215y+0.426z
6	F	S	F	0.5%	-x+0.215y-z
7	F	F	S	0.2%	-x-y+0.426z
8	F	F	F	0.1%	-x-y-z

Table 11: Net Gain for the Quadratic Model

From Table 11, we will now formulate our objective function and constraints.

$$\text{Max } E(G) = 0.049x + 0.058y + 0.040z \tag{5.4}$$

subject to

$$x, y, z \geq 0$$

$$x + y + z \leq 12000$$

and either

$$0.070x + 0.215y + 0.426z \geq 0$$

$$0.070x + 0.215y - z \geq 0$$

$$0.070x - y + 0.426z \geq 0$$

or

$$0.070x + 0.215y + 0.426z \geq 0$$

$$0.070x + 0.215y - z \geq 0$$

$$0.070x - y - z \geq 0$$

$$-x + 0.215y + 0.426z \geq 0$$

Solving this objective function with the Simplex method, each of our sets of constraints result in the same distribution, put $\$11,215$ into the IRA and \$785 into stock in Whitestone with an expected return of \$595.

Table 12 below summarizes the distributions of our funds given by each of the models.

Model	Roth IRA	Whitestone	Apple	Expected Return
Linear Model	\$12,000	\$0	\$0	\$840
Annual Rate of Return	\$9,326	\$1,663	\$1,010	\$885
Quadratic Relationship	\$11,215	\$785	\$0	\$595

Table 12: Results

So overall, the result with the highest expected return came from our second model. The real question is which model is the best model, and the answer of that is it is hard to know. Based on our knowledge of the importance of diversification, it would seem logical to follow the distribution of our funds given by our second model. As an early investor with not a lot of money to spare, a large portion of our money is kept in the safest option, while we still are putting money into higher risk funds to hopefully achieve higher returns. Portfolios are always disposing of and adding new assets, and if we so chose we could adapt this model and update it with new figures to continue to help us make the decisions of how to invest our funds.

5.3. Changing the Constraints

Up to this point, we have imposed the constraint of 90% certainty that we will have a positive net gain in each of the different models we looked at. Now we will explore what happens when we vary that constraint. We will use our second model that was based off of the linear relationship of beta and risk and the rates of return given by 2014. Rather than having 90% certainty, first we will look at how lowering that to 70% affects our distribution that our model tells us. This is significantly reducing our confidence in the model and allowing much more risk. With our new constraint, we restate our optimization problem with the same objective function. We change the constraints to reflect the lower level of certainty that can be achieved by adding up the individual probabilities of an event occurring, see Table 9 for a reminder of these values. Since our risk constraint is reduced, we now have three different cases that achieve this. We will solve the problem for each as before, and choose the distribution with the highest expected gain. Our problem is to

$$\text{Max } E(G) = 0.070x + 0.064y + 0.124z \tag{5.5}$$

subject to

$$x, y, z \geq 0$$

$$x + y + z \leq 12000$$

and either

$$0.070x + 0.215y + 0.426z \geq 0$$

$$0.070x + 0.215y - z \geq 0$$

or

$$0.070x + 0.215y + 0.426z \geq 0$$

$$0.070x - y + 0.426z \geq 0$$

or

$$0.070x + 0.215y + 0.426z \geq 0$$

$$0.070x - y - z \geq 0$$

The results of the three different optimization problems are listed below in Table 13. As we can see, the distribution given by Constraint 2 yields the highest expected return, which tells us to invest all $\$12,000$ into stock in Apple. Since we lowered our required percent certainty of a positive gain in our model, it is now permitting us to take on much more risk and invest all of our money in the highest risk category, since it in turn offers the highest rate of return.

Constraint	Roth IRA	Whitestone	Apple	E(G)
1	$0	$9,877	$2,123	$895
2	$0	$0	$12,000	$1,488
3	$11,215	$0	$785	$882

Table 13: Results

Again changing our risk constraint, the optimization problem below reflects an 80% certainty of a positive net gain.

$$\text{Max } E(G) = 0.070x + 0.064y + 0.124z \tag{5.6}$$

$$\text{subject to}$$

$$x, y, z \geq 0$$

$$x + y + z \leq 12000$$

$$\text{and either}$$

$$0.070x + 0.215y + 0.426z \geq 0$$

$$0.070x + 0.215y - z \geq 0$$

$$\text{or}$$

$$0.070x + 0.215y + 0.426z \geq 0$$

$$0.070x + 0.215y - z \geq 0$$

$$0.070x - y - z \geq 0$$

Solving this, we have the results listed below in Table 14. The two constraints result in close expected gains, but we will follow the distribution given by the first. Note that both of these expected returns are significantly less than what we had in our previous model that told us to invest all $\$12,000$ in Apple. We can also see that the distribution given by the first set of constraints moved towards Whitestone, which is considered a safer investment.

Constraint	Roth IRA	Whitestone	Apple	E(G)
1	$0	$9,877	$2,123	$895
2	$11,215	$0	$785	$882

Table 14: Results

Finally, we will explore what happens to our model when we impose a 95% risk constraint. The problem is to

$$
\begin{aligned}
\text{Max } E(G) = 0.070x + 0.064y + 0.124z \qquad (5.7) \\
\text{subject to} \\
x, y, z \geq 0 \\
x + y + z \leq 12000 \\
0.070x + 0.215y + 0.426z \geq 0 \\
0.070x + 0.215y - z \geq 0 \\
0.070x - y + 0.426z \geq 0
\end{aligned}
$$

This gives the result in Table 15.

Constraint	Roth IRA	Whitestone	Apple	E(G)
1	\$9,970	\$1,096	\$934	\$884

Table 15: Results

Again, we can see that our model has shifted to favor the safer investment, the Roth IRA.

Risk Constraint	Roth IRA	Whitestone	Apple	Expected Return
70%	\$0	\$0	\$12,000	\$1,488
80%	\$0	\$9,877	\$2,123	\$895
90%	\$9,326	\$1,663	\$1,010	\$885
95%	\$9,971	\$1,096	\$ 934	\$884

Table 16: Results

Table 16 summarizes the distributions given by all four risk constraints (the 90% results coming from our original model). Notice as we increased the level of certainty we wanted in our model, the distributions shifted from Apple towards the IRA, or from our riskiest investment towards our safest investment. This makes sense if you think about it, Apple offers the highest potential rate of return so our model will favor it until it is deemed too risky. So as we impose a constraint of a "safer" investment, we also are reducing the expected gain. In investments it will always be important to weigh the level of risk you are willing to take against the rate of return levels you are willing to give up.

6. Future Work

There are many ways to adapt the model Gallin and Shapiro have constructed to make it a "better" picture of reality. Models are built upon assumptions, so to better a model is to make the assumptions a more accurate representation of reality. Currently, one perspective the model has is that an investment can either fail and all money is lost, or succeed and gain the entire expected realization. In the real world, losing everything in an investment is on the far end of the spectrum and actually gaining the expected return is not always likely. In all reality, an investment may break-even, only lose a portion of the investment, turn a small profit in an investment, or even realize more than what was expected. Adapting the model to capture all the possible ways an investment could perform would be complex because that would include an infinite number of possibilities from losing everything to gaining an unlimited return. An alternative to this that would still bring our model closer to reality would be to add in the possibility of break-even points. This would change our model from the success-fail outlook it has, to each investment having three possibilities. We would then need a measure for not only the likelihood of an investment succeeding or failing, but also for the chances of breaking-even. This new structure of three performance possibilities would change the eight possible scenarios that we had that were the different combinations of success of failures. Therefore, we would need to be able to change our calculation of probabilities our how we insure the risk constraint that we impose. Once we figured out how to incorporate one more possibility of an investment performing, then how to incorporate even more performance possibilities could be explored.

Our model also makes the assumption that each of our investments are independent from one another. If we wanted to choose investments that were related, or found that the investments we have already chosen have an effect on one another, we could incorporate this into our model by taking into consideration their covariances, measures of how closely related they are. We could then use their respective covariances to account for the effects one investment's performance has on another.

A main factor when looking at portfolio optimization is always risk. This is because risk and return have a strong relationship, high risk is associated with higher returns and low risk is associated with lower returns. Our model currently has the objective function with the goal of maximizing our portfolio's return. In order to incorporate risk in our model, it is added in as a constraint. A different take on portfolio optimization could set up our model to have an objective function to minimize risk. This would completely change the model and give a fresh perspective of the problem with the same overall goal of finding the best balance in a portfolio between rate of return and risk.

Rather than changing the construction of the model, we could also continue to explore how experimenting with the constraints affects our results. By changing the different levels of certainty we wanted of having a positive net gain, we were changing the constraints of our model. This in turn, changed the feasible region and we therefore were given different results. We were able to see how changing the desired level of certainty drastically changed our distributions. We could also explore changing our other constraints, such as the amount of money we have available to invest and if that were to change the overall ratios of our investment. We could also change the kind of constraints we impose, like limiting the amount of dollars an investor is willing to lose rather than a percentage certainty of a positive net gain that an investor is comfortable with. Changing the different constraints we impose gives us insight on how our model works in different scenarios.

In our portfolio, we made two major calculations to get figures for rate of return and risk to use in our model. We calculated our rates of return using the method outlined in Equation 5.1, often referred to as the arithmetic average. This is not the only way to calculate rate of return. Other methods include the geometric average and the Sharpe ratio. The geometric average rate of return is calculated by first adding one to each of the annual rates of return, multiplying all of those numbers together, and then raising the product to the power of one divided by the number of annual rates of return considered. Finally, subtract one from that result and you have your geometric average. This way of calculating the rate of return offers the benefit of taking into consideration the effect of previous year's performance. For example, if you had \$100 but suffered a -50% rate of return, you would be down to only \$50. In the next year if you realize a 50% rate of return, then you will have \$75. If you calculated the average rate of return using the arithmetic method, it would give a 0% rate of return, which would lead one to believe you had the same amount of money that you started with. The geometric average reflects that even though your two rates of return are 50% and -50%, they do not cancel each other out. The Sharpe ratio is a way to calculate rates of return that takes into consideration risk in the calculation. Although the calculation is a little more difficult to explain, the ratio is the average return earned in excess per unit of deviation in an investment asset. While our method does not offer the benefits that the geometric average or Sharpe ratio do, it does still fulfill the purpose of trying to predict a future annual rate of return based on historical performances. Determining which calculation to use is dependent on what the investor thinks is most likely to occur given the circumstances.

We calculated risk by using beta values and making assumptions to model them into risk percentages. We touched on where beta comes from earlier, but now we will analyze it further. There are two types of risk, systematic and unsystematic.

Systematic risk is the inherent risk involved in the market, in other words it cannot be avoided no matter what you do. Unsystematic risk is the risk associated with securities that can be minimized with proper diversification. It is widely accepted in the economics field that having approximately 60 securities in your portfolio minimizes the unsystematic risk. This is a much larger amount than the three we chose for our portfolio, but our model could be broadened to include as many securities. Beta is a measure of the systematic risk, or the risk associated with the market. So since we used that as our basis for risk calculation, we assumed unsystematic risk would not have an effect.

Beta is calculated using regression analysis, specifically by taking the covariance of the security's return with the market's return and dividing by the market's return. The covariance and market return values in the calculation also have some variation in them. They are both built off of assumptions that economists have differing views of. The different assumptions result in different values and therefore different valuations for beta. One criticism of beta's calculation is that there is just not enough access to good data to calculate it upon, even though the theory behind the calculation is generally accepted as sound.

Another major criticism of beta is that there is no world's market to base the calculation off of. Beta is constructed to provide a measure of risk by comparing a security's return to that of its market's return. Without having a world's market, there is no single market return that can be compared to every investment on a standardized basis. Critics say that unless the security in question correlates with the market it is being compared to, beta is a useless number. The market used in the calculation of beta is the S&P 500, or Standard and Poors 500. The S&P 500 is an index made up of 500 different stocks that are chosen based on liquidity, size, and industry. So in the case of Apple, which is in the S&P 500, the beta value is reasonably accepted as an accurate measure of risk. However in the case of Whitestone, it could be argued that beta loses much of its meaning since Whitestone is not a stock in the S&P 500.

This does not mean that our model is useless, rather we are just highlighting the areas that it could be improved. There is no standardized way of calculating risk when it comes to securities because even though everyone has access to the same historical data, the way it is interpreted to predict the future has an element of subjectivity. It is a matter of trying to best guess the future based on the mathematical tools you have available along with some insight on where you believe the market is going. We began our representation of risk with beta values that have some controversy surrounding their value, and we attempted to interpret them in a meaningful way to represent risk. Our method of using two different approaches, linear and quadratic, to model the relationship between beta and risk highlighted the subjectivity surrounding beta's value. With these two different perspectives, we were able to gain insight on how the changing representations of risk affect the diversification that our model gives us.

7. Conclusion

Mathematical models are not decision-making models in themselves, but rather information models which provide financial information about the suitability or not of undertaking an investment. They provide the means to take values for risk and rates of return and add meaning to them. Modeling is an art. Much of it is based on trying to make the best possible guesses about the future based on what you know of the past. That is why mathematical models have no guarantees and are ever-changing; they are just the predictions we come up with based on what we determine to be the best assumptions at a given time.

These models are tools that give us insight on how to make smart choices. Since so much of the model is dependent on assumptions that are made, it is highly important to make good assumptions that portray a good representation of the real world. We have explored Gallin and Shapiro's article that details Leonard's investment dilemma, and we have explored a model of our own.

We began by choosing what investments to include in our portfolio, an IRA and stock in Whitestone and Apple. We then computed rates of return and risk for each investment to use in our model. We explored three different models by varying how the figures that we used were computed. The first model we looked at modeled rate of return by averaging annual rates of return for the last five years and utilized linear regression to model risk. For the second model, we used the same figure for risk as before, but only the most recent annual rate of return. In the final model, we used the same annual rate of return as in the second model, but measured risk through quadratic regression rather than linear.

The complexity of modeling figures for risk and rates of return was apparent in our different methods. While those values do not change the way that the model operates, they do change the distribution that the model gives. Therefore, it was of the utmost importance that those values were the best predictions of their future value that they could be. This is why we explored using different values for rate of return and the associated risk, changing them allowed us to see the effect they have on the diversification of the portfolio. Ultimately, it is important to not choose the model that gives the best expected return, but the model that you think truly best represents the investments' performances.

Investing will continue to be relevant and an avenue for generating profits as time goes on, even if the investments take on different forms. Models are constantly adapting and will continue to do so to keep up with our financial market and to provide us with even better insight. Having knowledge on how mathematical models work, even if it is a small sample of all the financial models that exist, will help you understand the careful decisions that must be made when it comes to finances. Portfolio optimization is a way to solve how to best diversify your portfolio, but that diversification is completely dependent on the figures you use in your model, so they need to be as accurate as possible. To be a successful investor, you need to be able to make good predictions of the future based on careful analysis of historical performance and have a little luck.

References

[1] Yahoo finance.

[2] Roy Ballentine. Portfolio optimization theory versus practice. *Journal of Financial Planning*, 2013.

[3] Dimitri P. Bertsekas and John N. Tsitsiklis. *Introduction to Probability, Second Edition*. Athena Scientific, New Hampshire, 2002.

[4] Miguel A. Carreira-Perpinan. Simplex method. *MathWorld– A Wolfram Web Resource*.

[5] Richard A. Crowell. Risk measurement: Five applications. *Financial Analysts Journal*, 29(4):81–87, 1973.

[6] George B. Dantzig. Linear programming. *Operations Research*, 50(1):42–47, 2002.

[7] George B. Dantzig, Alex Orden, and Philip Wolfe. The generalized simplex method for minimizing a linear form under linear inequality restraints. 1955.

[8] Jack Clark Francis and Dogcheol Kim. *Modern Portfolio Theory: Foundation, Analysis, and New Developments*. Wiley, New Jersey, 2013.

[9] Daniel Gallin and Edwin Shapiro. Optimal investment under risk. *Mathemactics Magazine*, 49(5):235–238, 1976.

[10] Frank R. Giordano, Maurice D. Weir, and William P. Fox. *A First Course in Mathematical Modeling, Second Edition*. Brooks/Cole Publishing Company, California, 1997.

[11] Harry Markowitz. Portfolio selection. *The Journal of Finance*, 7(1):77–91, 1952.

[12] Michael L. Overton. Linear programming. 1997.