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By

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1. INTRODUCTION

Linear independence is one of the main defining characteristics in vector space theory as it guarantees a unique representation in terms of basis vectors. Further, we can expand the concept of linear independence with the study of frames, which generalize the idea of a basis while allowing for more desirable traits. In this talk we examine certain collections of functions, both finite-dimensional and infinite-dimensional, and the necessary conditions for linear independence within. In closing, we take a look at linear independence as applied to wavelet theory.

2. Preliminary Remarks

Definition 2.1 *A vector space is a set V over a field $\mathbb{F}$ along with an addition on V and a scalar multiplication on V such that commutativity, associativity and distributive properties hold along with the existence of an additive and multiplicative identity and additive inverse.*

Definition 2.2 *A sequence of vectors $v_1, \ldots, v_m \in V$ is linearly independent if the only solution to $a_1 v_1 + \cdots + a_m v_m = 0$ is $a_1 = \cdots = a_m = 0$.*

Definition 2.3 *A basis $\{v_k\}$ is a set of vectors in V that is linearly independent and spans V.*

Linear independence guarantees that the representation of v in terms of the basis vectors is unique.

Proposition 2.4 *A set $v_1, \ldots, v_n$ in V is a basis of V if and only if every $v \in V$ can be written uniquely in the form $v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$, where $a_1, \ldots, a_n \in \mathbb{F}$.*

Proof: In the forward direction, suppose $\{v_1, v_2, \ldots, v_n\}$ is a basis for V and let v be an element of V. Since $\{v_i\}_{i=1}^n$ is a basis, it spans V, and hence there exists $\{a_i\}_{i=1}^n$ such that $v = a_1 v_1 + \cdots + a_n v_n$. Now to show uniqueness, suppose by way of contradiction there exists $\{b_i\}_{i=1}^n$ such that $a_i \neq b_i$ for some $j \in \{1, 2, \ldots, n\}$ and $v = b_1 v_1 + \cdots + b_n v_n$. So then

$$v - v = (a_1 - b_1)v_1 + (a_2 - b_2)v_2 + \cdots + (a_j - b_j)v_j + \cdots + (a_n - b_n)v_n$$

If $a_j = b_j$ and $a_i = b_i$ for all $i = j$, then $(a_j - b_j)v_j = \odot$, which contradicts the assumption that $a_j = b_j$ and $v_j = \odot$ (since a basis is linearly independent).
If other $a_i = b_i$, we would have $\odot = (a_j - b_j)v_j + (a_i - b_i)v_i$, which gives us $v_j = \frac{(b_i - a_i)v_i}{a_j - b_j}$, which contradicts the hypothesis that $\{v_i\}_{i=1}^n$ is a basis, hence linearly independent.
In the backward direction, suppose that for each v in V, there exists a unique set $\{a_i\}_{i=1}^n$ such that $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$ (i.e., $\{v_i\}_{i=1}^n$ is a spanning set). It remains to show that $\{v_i\}_{i=1}^n$ is linearly independent.
Let $\{a_i\}_{i=1}^n$ be such that $\odot = a_1v_1 + a_2v_2 + \cdots + a_nv_n$ and $\odot = 0v_1 + 0v_2 + \cdots + 0v_n$. Since by assumption the set $\{a_i\}_{i=1}^n$ must be unique, we have $a_i = 0$ for all $i \in \{1, 2, \ldots, n\}$. Thus $\{v_i\}_{i=1}^n$ is linearly independent, and so it follows that $\{v_i\}_{i=1}^n$ is a basis. ❑

Definition 2.5 *For v in V, the norm of v is defined by $\|v\| = \sqrt{\langle v, v\rangle}$, which satisfies the following properties:*

1. $\|x\| \geq 0$
2. $\|x\| = 0 \Leftrightarrow x = \odot$
3. $\|x + y\| \leq \|x\| + \|y\|$
4. $\|cx\| = |c| \cdot \|x\|$

for each x, y in V, c in $\mathbb{F}$.

Definition 2.6 *A sequence x in a normed vector space V is defined to be a Cauchy sequence if for all $\varepsilon > 0$ there exists j such that for all $m, n > j$, $\|x_n - x_m\| < \varepsilon$.*

Definition 2.7 *A Banach space B is a vector space on which every Cauchy sequence (x_n) converges to an element x in B.*

Definition 2.8 *In general, for an interval $[a, b]$ and for two arbitrary functions $f, g \in R[a, b]$, the inner product of f and g is*

$$\langle f, g\rangle = \int_a^b f(x)\, g(x)\, dx.$$

Definition 2.9 *We say a set S is countable if either S is finite or if it is in one-to-one correspondence with the set of natural numbers (also called countably infinite).*

It immediately follows from this definition that one such example of a countable set is the set of natural numbers.

Definition 2.10 *A subset S of T is dense in T if every neighborhood of any point x of T contains points of S.*

If we consider the set of rationals $\mathbb{Q}$, we see that $\mathbb{Q}$ is a countable subset of $\mathbb{R}$. Further, given any neighborhood of any point $x \in \mathbb{R}$, there exists a rational number in that neighborhood.

Lemma 2.11 *If $x, y \in \mathbb{R}$ such that $x < y$, then there exists $r \in \mathbb{Q}$ such that $x < r < y$.*

Proof: In the first case, assume $x \geq 0$. Then by the Archimedean Principle, $\exists n \in \mathbb{Z} \ni n > \frac{1}{y-x}$.
Then $\frac{1}{q} < y - x$. Now if we consider the set of integers m where $y \leq \frac{m}{n}$, we know again by the Arcimedean Principle that this is a nonempty set of positive integers. Therefore we know there exists some element p in our set.
So then $\frac{p-1}{n} < y \leq \frac{p}{n}$. It follows that

$$\begin{aligned} x &= y - (y - x) \\ &< \frac{p}{n} - \frac{1}{n} \\ &= \frac{p-1}{n} \end{aligned}$$

Therefore $r = \frac{p-1}{n}$ is between x and y.
In the second case, let us assume $x < 0$. Then we can find an integer n such that $n > -x$. From this we have $n + x > 0$, and so there is a rational r such that $n + x < r < n + y$. Therefore $r - n$ is a rational lying between x and y.

Thus we have the existence of a rational between any two real numbers. ❑

3. Sine and Cosine Functions

We will now consider linear independence for a set of trigonometric functions: consider

$$\{\cos(\lambda_k x)\}_{k=1}^{n} \cup \{\sin(\mu_\ell x)\}_{\ell=1}^{m} \tag{3.1}$$

where λ_k, μ are parameters such that $k = 1, 2, \ldots, n$ and $\ell = 1, 2, \ldots, m$ for $n, m \in \mathbb{Z}^+$.

We must impose conditions on our parameters λ_k and μ to guarantee the functions in this set are linearly independent.

Example 3.1 *Consider numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ for which $\lambda_1 = -\lambda_2$. Since $\cos(x) = \cos(-x)\, \forall x \in \mathbb{R}$, we can choose coefficients a_1, a_2 such that*

$$1 \cdot \cos(\lambda_1 x) - 1 \cdot \cos(\lambda_2 x) + 0 \cdot \cos(\lambda_3 x) + \cdots + 0 \cdot \cos(\lambda_n x) = 0$$

for all $x \in \mathbb{R}$. Since not all coefficients are zero, the functions $\cos(\lambda_k x)$ are linearly dependent on any interval.

This shows us that we must impose the condition $|\lambda_k| = |\lambda_j|$ when $k = j$ for the functions $\cos(\lambda_k x)$ to possibly be linearly independent.

When we consider $\sin(\mu\ x)$, we must note that $\sin(0) = 0$. Thus we must impose another condition: $\mu\ = 0$ for all ℓ.

These two conditions alone, in fact, are sufficient for the set in (**??**) to be linearly independent on an arbitrary interval.

Theorem 3.2 *If $\{\lambda_k\}_{k=1}^n$ and $\{\mu\ \}_{=1}^m$ are sets of real numbers such that $\mu\ = 0$ for each ℓ and such that $|\lambda\ | = |\lambda_j|$ and $|\mu\ | = |\mu_j|$ when $\ell = j$, then the set of functions*

$$\{\cos(\lambda_k x)\}_{k=1}^n \cup \{\sin(\mu\ x)\}_{=1}^m$$

is linearly independent on any interval I.

Proof: We may assume that the λ_k and μ are nonnegative and satisfy $0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n$ and $0 < \mu_1 < \mu_2 < \ldots \mu_m$. Now suppose that for coefficients c_k and d_l,

$$\sum_{k=1}^{n} c_k \cos(\lambda_k x) + \sum_{=1}^{m} d\ \sin(\mu\ x) = 0 \tag{3.2}$$

for all x in I. We want to show that $c_k = d \ = 0$ for all k and ℓ. Let $\lambda_n > \mu_m$, where λ_n is the largest λ-value and μ_m is the largest μ-value. When we differentiate both sides of equation (**??**) $4N$ times for arbitrary $N \in \mathbb{N}$ (since cosine and sine are functions of period 4, we have

$$\sum_{k=1}^{n} c_k \lambda_k^{4N} \cos(\lambda_k x) + \sum_{=1}^{m} d \ \mu^{4N} \sin(\mu \ x) = 0$$

on I. When we multiply by $\left(\frac{1}{\lambda_n^{4N}}\right)$, we get

$$\sum_{k=1}^{n} c_k \left(\frac{\lambda_k}{\lambda_n}\right)^{4N} \cos(\lambda_k x) + \sum_{=1}^{m} d \ \left(\frac{\mu}{\lambda_n}\right)^{4N} \sin(\mu \ x) = 0.$$

We may now rewrite this as

$$c_n \cos(\lambda_n x) = -\sum_{k=1}^{n-1} c_k \left(\frac{\lambda_k}{\lambda_n}\right)^{4N} \cos(\lambda_k x) - \sum_{=1}^{m} d \ \left(\frac{\mu}{\lambda_n}\right)^{4N} \sin(\mu \ x). \tag{3.3}$$

Fix $x \in I \ni \cos(\lambda_n x) = 0$.

We now see that (**??**) reduces to one fewer terms. We want to show that this result is sufficiently small. Since c_k, d are fixed coefficients, and $\left(\frac{\lambda_k}{\lambda_n}\right), \left(\frac{\mu_\ell}{\lambda_n}\right) < 1$,

$$\lim_{N\to\infty} -\sum_{k=1}^{n-1} c_k \left(\frac{\lambda_k}{\lambda_n}\right)^{4N} \cos(\lambda_k x) - \sum_{=1}^{m} d \ \left(\frac{\mu}{\lambda_n}\right)^{4N} \sin(\mu \ x) = 0.$$

Thus the right side of (**??**) converges to 0 as $N \to \infty$, and so then $c_n = 0$.

Now let $\mu_m > \lambda_n$, where μ_m is the largest μ-value and λ_n is the largest λ-value. When we differentiate both sides of Equation (**??**) $4N$ times for arbitrary $N \in \mathbb{N}$, we have

$$\sum_{k=1}^{n} c_k \lambda_k^{4N} \cos(\lambda_k x) + \sum_{=1}^{m} d \ \mu^{4N} \sin(\mu \ x) = 0$$

on I. When we multiply by $\left(\frac{1}{\mu_m^{4N}}\right)$, we get

$$\sum_{k=1}^{n} c_k \left(\frac{\lambda_k}{\mu_m}\right)^{4N} \cos(\lambda_k x) + \sum_{=1}^{m} d \ \left(\frac{\mu}{\mu_m}\right)^{4N} \sin(\mu \ x) = 0.$$

We may now rewrite this as

$$d_m \sin(\mu \ x) = -\sum_{k=1}^{n} c_k \left(\frac{\lambda_k}{\mu_m}\right)^{4N} \cos(\lambda_k x) - \sum_{=1}^{m-1} d \ \left(\frac{\mu}{\mu_m}\right)^{4N} \sin(\mu \ x). \tag{3.4}$$

Fix $x \in I \ni \sin(\mu\ x) = 0$. Again, since $c_k, d\ $ are fixed and $\left(\frac{\lambda_k}{\mu_m}\right), \left(\frac{\mu_\ell}{\mu_m}\right) < 1$, we show that the result is sufficiently small:

$$\lim_{N\to\infty} -\sum_{k=1}^{n} c_k \quad \frac{\lambda_k}{\mu_m} \quad {}^{4N} \cos(\lambda_k x) - \sum_{=1}^{m-1} d \quad \frac{\mu}{\mu_m} \quad {}^{4N} \sin(\mu\ x) = 0.$$

Therefore the right side of (**??**) converges to 0 as $N \to \infty$, and it follows that $d_m = 0$. Assuming all λ-values are unique as compared to all μ-values, repeating the above arguments shows that $c_k = d\ \ = 0\ \forall k, \ell$.

The case where particular λ-values coincide with certain μ-values is trivial. Suppose without loss of generality that $\lambda_n = \mu_m$. Then by similar argument,

$$c_n \cos(\lambda_n x) + d_m \sin(\mu_m x) = -\sum_{k=1}^{n-1} c_k \quad \frac{\lambda_k}{\lambda_n} \quad {}^{4N} \cos(\lambda_k x) - \sum_{=1}^{m-1} d \quad \frac{\mu}{\mu_m} \quad {}^{4N} \sin(\mu\ x)$$

for all $x \in I$. Again, as $N \to \infty$,

$$c_n \cos(\lambda_n x) + d_m \sin(\lambda_n x) = 0 \tag{3.5}$$

on I. Differentiating (**??**), we get

$$\lambda_n (d_m \cos(\lambda_n x) - c_n \sin(\lambda_n x)) = 0. \tag{3.6}$$

Since $\lambda_n = \mu_m = 0$, Equations (**??**) and (**??**) tell us that $c_n = d_m = 0$. ❏

Definition 3.3 *A Fourier series is an expansion of a* 2π*-periodic function* $f : \mathbb{R} \to \mathbb{C}$ *in terms of the functions*

$$1, \cos x, \cos 2x, \ldots, \cos nx, \ldots, \sin x, \sin 2x, \ldots, \sin nx, \ldots \tag{3.7}$$

We can see now that any finite subfamily of the expansion in (**??**) is linearly independent by Theorem **??**.

Since Fourier series are often used in complex form, let us recall the complex exponential function $e^{i\lambda x}, \lambda \in \mathbb{R}$, which is defined by

$$e^{i\lambda x} = \cos(\lambda x) + i \sin(\lambda x).$$

From here, we can prove that a family of complex exponentials is linearly independent if no λ-value is repeated:

Corollary 3.4 *If* $\{\lambda_k\}_{k=1}^{n}$ *is a set of real numbers for which* $\lambda \;\, = \lambda_j$ *when* $\ell = j$, *then the set of complex exponentials* $\{e^{i\lambda_k x}\}_{k=1}^{n}$ *is linearly independent on an arbitrary interval.*

Proof: From the set of complex exponentials along with $a_1, a_2, \ldots, a_n$, consider $a_1 e^{i\lambda_1 x} + a_2 e^{i\lambda_2 x} + \cdots + a_n e^{i\lambda_n x} = 0$. For any single element of the form $e^{i\lambda_k x} = \cos(\lambda_k x) + i\sin(\lambda_k x)$, both the cosine and sine elements would need to equal 0 for the sum to equal 0. However, this is impossible, since no combination $\lambda_k x$ will produce 0 for both cosine and sine at the same time. Therefore, the only solution to $a_1 e^{i\lambda_1 x} + a_2 e^{i\lambda_2 x} + \cdots + a_n e^{i\lambda_n x} = 0$ is $a_1 = a_2 = \ldots a_n = 0$. So by definition, $\{e^{i\lambda_k x}\}_{k=1}^{n}$ is linearly independent. ❑

Following this notation, the Fourier series of a periodic function $f : \mathbb{R} \to \mathbb{C}$ of period 2π that is integrable on $[-\pi, \pi]$ can be written as

$$f(x) \sim \sum_{k\in\mathbb{Z}} c_k e^{ikx}, c_k = \frac{1}{2\pi} \;{}_{-\pi}^{\;\pi} f(x)\, e^{-ikx} dx. \tag{3.8}$$

It is important to note here that it is sufficient in our studies to consider Lebesgue integrability as a generalization of Riemann integrability. In short, Lebesgue integrability allows us to integrate many functions that are not Riemann integrable.

4. Bases in Hilbert Spaces

We will now give a precise interpretation of the Fourier series (**??**) in terms of orthonormal bases in Hilbert spaces.
From basic linear algebra, we know that a finite-dimensional vector space V with an inner product $\langle\cdot,\cdot\rangle$ has many useful properties. One such property is that a vector space V as mentioned above has orthonormal bases. When we denote an orthonormal basis by $\{e_k\}_{k=1}^{n}$, we can easily define each f in V in terms of those vectors e_k:

$$f = \sum_{k=1}^{n} \langle f, e_k\rangle e_k.$$

We may further choose to work with a certain type of Hilbert spaces called Banach spaces, which offer the useful property that each Cauchy sequence is convergent.

The convenient properties of finite-dimensional vector spaces can cause us to inquire as to whether similar properties hold for infinite-dimensional vector spaces equipped with inner products. Indeed, we are able to extend most of the results from finite-dimensional linear algebra to the infinite-dimensional if we focus on inner product spaces with the property that each Cauchy sequence (relative to the norm arising from the inner product) is convergent.

Definition 4.1 *A Hilbert space $\mathcal{H}$ is a Banach space equipped with an inner product from which the norm $\|\cdot\|$ on $\mathcal{H}$ is derived: $\|x\| = \sqrt{|\langle x, x\rangle|}$.*

We will now be considering the vector space

$$L^2[-\pi,\pi] = \left\{ f : [-\pi,\pi] \to \mathbb{C} \text{ with } \int_{-\pi}^{\pi} |f(x)|^2 dx < \infty \right\}.$$

Definition 4.2 *The inner product on the Hilbert space $L^2[-\pi,\pi]$ of Lebesgue integration, where f and g are defined on $[-\pi,\pi]$, is given by*

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\,\overline{g(x)}dx.$$

We will now assume that our Hilbert spaces $\mathcal{H}$ are *separable* in the metric space topology associated with $||\cdot||$.

Definition 4.3 *A metric space X is said to be separable if it has a subset S with a countable number of points and which is dense in X.*

Since the set $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$, we know that $\mathbb{R}$ is separable.[1] Assuming that our Hilbert spaces are separable provides us with a finite or countably infinite orthonormal basis.

From here on, we will restrict our attention to complex separable Hilbert spaces $\mathcal{H}$, with the inner product $\langle\cdot,\cdot\rangle$ chosen to be linear in the first entry.

Definition 4.4 *A family $\{f_k\}_{k\in\mathbb{Z}}$ of elements in $\mathcal{H}$ is called a basis for $\mathcal{H}$ if for each $f \in \mathcal{H}$ there exist unique scalar coefficients $c_k(f)$ such that*

$$f = \sum_{k\in\mathbb{Z}} c_k(f)\, f_k. \tag{4.1}$$

Definition 4.5 *A sequence of functions $\{f_k\}$ is said to converge in the norm to an integrable function f_0 if $\lim_{k\to\infty} \|f_k - f_0\|_2 = 0$.*

[1] $L^\infty(\mathbb{R})$, the space of all functions bounded "almost everywhere" on $\mathbb{R}$ is a non-separable metric space.

We must keep in mind that pointwise convergence does not imply convergence in the norm, and the converse is found to be false as well. The following examples illustrate this fact.

Example 4.6 *Consider the function f_n defined on the interval $[0,1]$ by*

$$f_n : f_n(x) = \begin{array}{ll} 0 & x = 0, x > \frac{1}{n} \\ n & 0 < x \leq \frac{1}{n} \end{array}$$

Then we set up our norm

$$\|f_n - f_0\|_2^2 = \int_{[0,1]} f_n - f_0 = \int_{[0,1]} f_n.$$

Now by the construction of our function our integral is

$$\int_0^{\frac{1}{n}} f_n = \int_0^{\frac{1}{n}} n = 1$$

for all $n \in \mathbb{N}$. Therefore $\|f_n - f_0\|_2^2 \to 0$ and thus f_n converges in the norm.

Example 4.7 *Consider*

$$f_n = \sqrt{nxe^{-nx^2}}$$

on $[0,1]$. We know f_n is convergent pointwise to zero.

$$\|f_n - f_0\|^2 = \int_0^1 f_n^2(x)\,dx = -\frac{1}{2}e^{-nx^2}\Big|_0^1 = \frac{1}{2}\left(1 - e^{-n}\right) \to \frac{1}{2}$$

Therefore $\|f_n - f_0\| \to \frac{1}{2} = 0 \Rightarrow f_n$ does not converge in the norm at all.

Another interesting example of this arises from the construction of Cantor's set C, which is constructed by methodically removing the inner thirds of intervals. We start with the interval $[0,1]$ and remove the middle third, $(1/3, 2/3)$. This will give us two intervals for the next step, from which we again remove the middle thirds of each. The process continues, and Cantor's set will consist of the points remaining. For our purposes, we will examine its complement C^C:

Example 4.8 *Let $U_k = \cup_{p=1}^{k} \cup_{j=1}^{2p-1} I(p,j)$ denote the union of the intervals removed from $[0,1]$ by the kth stage of the construction of the Cantor set C. The total length of the intervals comprising U_k is $d_k = 1 - (2/3)^k$. For each $k \in \mathbb{N}$, define*

$$f_k(x) = \begin{array}{ll} 1 & \textit{if } x \in U_k \\ 0 & \textit{if } x \in U_k^c \cap [0,1]. \end{array}$$

We note that since f_k can only be 0 or 1, $f_k^2(x) = f_k(x)$ for all $x \in [0,1]$. We now want to show that $\{f_k\}$ converges in the norm to the function f_0 which has value one on the entire interval $[0,1]$.

$$\begin{aligned}
\|1 - f_k\|_2^2 &= \int_0^1 [1 - f_k(x)]^2 dx \\
&= \int_0^1 [1 - 2f_k(x) + f_k^2(x)] dx \\
&= \int_0^1 [1 - f_k(x)] dx \\
&= 1 - d_k \\
&= \frac{2}{3}^{k}.
\end{aligned}$$

Thus $\lim_{k\to\infty} \|1 - f_k\|_2^2 = \lim_{x\to\infty} (2/3)^k = 0$. *So by Definition* **??**, *$\{f_k\}$ converges in the norm. We may now notice that although $\{f_k\}$ converges in the norm to 1 on* $[0,1]$ *and converges pointwise to 1 on the complement of C in* $[0,1]$, *the sequence $\{f_k\}$ fails to converge pointwise to 1 at the infinitely many points in C.*

Equation (**??**) actually implies that the convergence is in the norm of $\mathcal{H}$, since

$$\left\| f - \sum_{|k|\leq N} c_k(f) f_k \right\| \to 0$$

as $N \to \infty$.

The coefficients $c_k(f)$ depend on the set $\{f_k\}$, and it is often useful to consider an orthonormal basis - a basis $\{f_k\}_{k\in\mathbb{Z}}$ for which $\langle f_j, f_k\rangle$ equals one if $k = j$ and zero if $k = j$. Indeed, if $\{f_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis,

$$\begin{aligned}
\langle f, f_{k_0}\rangle &= \left\langle \sum_{k\in\mathbb{Z}} c_k f_k, f_{k_0} \right\rangle \\
&= \int_{-\pi}^{\pi} \sum_{k\in\mathbb{Z}} c_k f_k \overline{f_{k_0}} \\
&= \sum_{k\in\mathbb{Z}} c_k \int_{-\pi}^{\pi} f_k \overline{f_{k_0}} \\
&= \sum_{k\in\mathbb{Z}} c_k \langle f_k, f_{k_0}\rangle \\
&= \sum_{k\in\mathbb{Z}} c_k \begin{matrix} 1 & k = k_0 \\ 0 & k = k_0 \end{matrix} \\
&= c_{k_0}.
\end{aligned}$$

So then

$$c_k\left(f\right) = \langle f, f_k \rangle.$$

Following this, the expansion (**??**) takes the convenient form

$$f = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k. \tag{4.2}$$

In the context of Fourier series, the relevant vector space is

$$L^2[-\pi, \pi] = \left\{ f : [-\pi, \pi] \to \mathbb{C} \text{ with } \int_{-\pi}^{\pi} |f\left(x\right)|^2 dx < \infty \right\}.$$

We now want to define what it means for functions to be equal almost everywhere.

Definition 4.9 *We say functions on $[-\pi, \pi]$ are equal almost everywhere when the following holds:*

$$\int_{\mathbb{R}} |f - g|^2 = 0.$$

If so, we say that f is equivalent to g and write $f \sim g$. This yields an equivalence relation.

By considering that $\langle f, g \rangle = \int f\overline{g}$ is an inner product on $L^2(X)$ and the norm is defined by $|f| = \sqrt{\langle f, f \rangle}$, we now know $L^2(X)$ is a Banach space. Note that elements of $L^2(X)$ are not functions, but equivalence classes. Now from choosing functions that are equal almost everywhere and outfitting the resulting space with the inner product, $L^2[-\pi, \pi]$ becomes a Hilbert space.

When studying Fourier series, one of the main results we have is that the set of functions $\{(1/\sqrt{2\pi})e^{ikx}\}_{k\in\mathbb{Z}}$ forms an orthonormal basis for $L^2[-\pi, \pi]$. Thus the expansion (**??**) corresponds to the Fourier series in (**??**). So it follows that the exact meaning of (**??**) is that for $f \in L^2[-\pi, \pi]$,

$$\left\| f - \sum_{|k|\leq N} c_k e^{ik\cdot} \right\|_{L^2[-\pi,\pi]} \to 0$$

as $N \to \infty$, which translates to

$$\int_{-\pi}^{\pi} \left| f\left(x\right) - \sum_{|k|\leq N} c_k e^{ikx} \right|^2 dx \to 0$$

as $N \to \infty$. All the complex exponentials e^{ikx} $(k \in \mathbb{Z})$ have period 2π. However, for arbitrary real λ_k the system of complex exponentials $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ may not have a common period. When we study such complex exponentials, we are looking at a branch of mathematics known as nonharmonic Fourier analysis. Since this usually requires a strong knowledge of complex analysis, we will only pull out a few interesting results that are fairly simple to obtain. These results are most often acquired when we consider the numbers λ_k as small perturbations of k (if $|k - \lambda_k|$ is uniformly small for $k \in \mathbb{Z}$). This is exemplified in the "Kadec 1/4-Theorem."

Theorem 4.10 *If* $\sup_{k\in\mathbb{Z}} |k - \lambda_k| < 1/4$, *then* $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ *is a basis for* $L^2(-\pi, \pi)$.

The hypothesis in Kadec's 1/4-theorem is just a sufficient condition, not a necessary one. To see this, consider when the supremum over $\mathbb{Z}$ is replaced with the supremum over $\mathbb{Z}\backslash\{k_0\}$ for some $k_0 \in \mathbb{Z}$ and if $\lambda_{k_0} = \lambda_k$ for all $k \in \mathbb{Z}\backslash\{k_0\}$. Then $\{e^{i\lambda_k x}\}_{k\in\mathbb{Z}}$ remains a basis for $L^2[-\pi, \pi]$. This shows that the parameter in *one* of the exponential functions can be perturbed by an arbitrary amount.[**?**]

5. Gabor Systems and Wavelets

We will now consider some systems of functions that are not within classical analysis but have grown in popularity. The functions in these systems belong to the function space

$$L^2(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \text{ with } \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty\}.$$

Mimicking the construction of $L^2[-\pi, \pi]$, the vector space $L^2(\mathbb{R})$ becomes a Hilbert space if we identify functions that are equal almost everywhere and equip the space with the inner product defined by

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}dx.$$

However, in this case, the complex exponential functions are not in $L^2(\mathbb{R})$:

$$\int_{-\infty}^{\infty} |e^{i\lambda x}|^2 dx = \int_{-\infty}^{\infty} dx = \infty$$

for any $\lambda \in \mathbb{R}$.

Alternatively, if $g \in L^2(\mathbb{R})$, then the function $x \mapsto e^{i\lambda x} g(x-\mu)$ is also in $L^2(\mathbb{R})$ for all $\lambda, \mu \in \mathbb{R}$, since

$$\int_{-\infty}^{\infty} |e^{i\lambda x} g(x-\mu)|^2 dx = \int_{-\infty}^{\infty} |g(x-\mu)|^2 dx = \int_{-\infty}^{\infty} |g(x)|^2 dx < \infty.$$

We notice that the graph of the function $x \mapsto e^{i\lambda x} g(x-\mu)$ is simply a translation and modulation of the graph of g μ units to the right. We can check and see that multiplying g by $e^{i\lambda x}$ corresponds to a translation of the Fourier transform of g, which is defined below.

Definition 5.1 *If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then its Fourier transform $\hat{f} : \mathbb{R} \to \mathbb{C}$ is defined by*

$$\hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \gamma x} dx.$$

Then the function $h(x) = e^{i\lambda x} f(x)$ has

$$\hat{h}(\gamma) = \hat{f}\left(\gamma - \frac{\lambda}{2\pi}\right).$$

Proof:

$$\begin{aligned} \hat{h}(\gamma) &= \int_{-\infty}^{\infty} h(x) e^{-2\pi i \gamma x} dx \\ &= \int_{-\infty}^{\infty} e^{i\lambda x} f(x) e^{-2\pi i \gamma x} dx \\ &= \int_{-\infty}^{\infty} f(x) e^{-2\pi i \left(\gamma - \frac{\lambda}{2\pi}\right) x} dx \\ &= \hat{f}\left(\gamma - \frac{\lambda}{2\pi}\right). \end{aligned}$$

❏

Now if we let a, b be positive numbers and let $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we may consider collections of functions of the form

$$\{e^{2\pi i m b x} g(x - na)\}_{m,n \in \mathbb{Z}}.$$

This type of a system is called a *regular* Gabor system. If I is a finite index set and $\{(\lambda_n, \mu_n)\}_{n \in I}$ is an arbitrary family of points in $\mathbb{R}^2$, then an irregular Gabor system is the collection of functions given by

$$\{e^{2\pi i \lambda_n x} g(x - \mu_n)\}_{n \in I}.$$

Regular Gabor systems (and certain irregular Gabor systems) consist of infinitely many functions. Such a system is linearly independent if each finite subfamily is linearly independent.

While results involving linearly independent Gabor systems are complicated, Heil, Ramanathan, and Topiwala proved that an irregular Gabor system is linearly independent under particular conditions on g and the points $\{(\lambda_n, \mu_n)\}_{n\in I}$. The following conjecture is based upon the results:

Conjecture 5.2 *A Gabor system* $\{e^{2\pi i\lambda_n x} g(x - \mu_n)\}_{n\in I}$ *with* $g \in L^2(\mathbb{R}) \setminus \{0\}$ *is linearly independent provided that the points* (λ_n, μ_n) *for* $n \in I$ *are distinct.*

Later, Linnell proved the conjecture for regular Gabor systems, but the methods used do not apply to irregular Gabor systems. In short, this conjecture remains to be proven or disproved.

Gabor systems are widely used in signal analysis, although the more popular system of functions in this area of study is the wavelet system.

Definition 5.3 *For a given function* $\psi \in L^2(\mathbb{R})$*, the associated wavelet system consists of the functions*

$$\psi_{j,k}(x) := 2^{j/2}\psi\ \ 2^j x - k$$

where $j, k \in \mathbb{Z}$*. In words, we say that the wavelet system above is generated by the function* ψ*.*

Linearly dependent wavelet systems exist.

Example 5.4 *Let* $\phi = \chi[0, 1)$*, where* ϕ *is the characteristic function of the interval [0,1). Then by Definition* **??***,*

$$\phi_{0,0} = 2^0\phi(x) = 2\chi_{[0,1)}(x).$$

Now

$$\phi_{1,0} = \sqrt{2}\phi(2x) = \sqrt{2}\chi_{[0,1/2)}(x)$$

and

$$\phi_{1,1} = \sqrt{2}\phi(2x - 1) = \sqrt{2}\chi_{[1/2,1)}(x).$$

Therefore,

$$\phi_{0,0} = \frac{1}{\sqrt{2}}\left(\phi_{1,0} + \phi_{1,1}\right).$$

Particular linearly independent wavelet systems are key in wavelet theory. Indeed, most useful wavelet systems are based on multiresolution analysis, which consists of a set of conditions implying that a certain function ϕ, the scaling function, satisfies the equation

$$\phi(x) = \sum_{k\in\mathbb{Z}} c_k \phi(2x-k), \tag{5.1}$$

where for technical reasons we want the sequence $\{c_k\}_{k\in\mathbb{Z}}$ to belong to the vector space

$$\ell^2 := \quad \{a_k\}_{k\in\mathbb{Z}} : \sum_{k\in\mathbb{Z}} |a_k|^2 < \infty \quad .$$

When only finitely many coefficients c_k in (**??**) are nonzero, we notice that this equation implies that the wavelet system $\{\phi_{j,k}\}_{j,k\in\mathbb{Z}}$ is linearly dependent. Actually the subsystem $\{\phi_{j,k}\}_{j\in\{0,1\},k\in\mathbb{Z}}$ is already linearly dependent.

In multiresolution analysis, we want to construct a function ψ that generates an orthonormal basis $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$. Typically, ψ is defined in terms of the scaling function ϕ as in (**??**):

$$\psi(x) = \sum_{k\in\mathbb{Z}} d_k \phi(2x-k)$$

for particular coefficients d_k. Now we can see that two wavelet systems are necessary in constructing our function.
It is common practice to verify (**??**) in the Fourier domain. It is known that (**??**) holds for some coefficient sequence $\{c_k\}_{k\in\mathbb{Z}} \in \ell^2$ if and only if

$$\hat{\phi}(2\gamma) = \sum_{k\in\mathbb{Z}} \tilde{c}_k e^{2\pi i k\gamma} \hat{\phi}(\gamma) \tag{5.2}$$

holds on $\mathbb{R}$ for some sequence $\{\tilde{c}_k\}_{k\in\mathbb{Z}} \in \ell^2$. This type of condition holds, for example, for even-order B-splines.

Example 5.5 *The first-order B-spline is defined as $B_1 = \chi_{[-1/2,1/2]}$. Otherwise B-splines are defined inductively by convolution:*

$$B_{n+1}(x) = B_n * B_1(x) = \int_{-\infty}^{\infty} B_n(x-t) B_1(t)\, dt.$$

Now we have

$$\hat{B}_1(\gamma) = \int_{-\infty}^{\infty} B_1(x) e^{-2\pi i\gamma x} dx$$

$$
\begin{aligned}
&= \int_{-1/2}^{1/2} e^{-2\pi i\gamma x}dx \\
&= \left.\frac{ie^{-2\pi i\gamma x}}{2\pi\gamma}\right|_{-1/2}^{1/2} \\
&= \left.\frac{\sin(2\pi\gamma x)}{2\pi\gamma} + \frac{i\cos(2\pi\gamma x)}{2\pi\gamma}\right|_{-1/2}^{1/2} \\
&= \left(\frac{\sin(\pi\gamma)}{2\pi\gamma} + \frac{i\cos(\pi\gamma)}{2\pi\gamma}\right) - \left(\frac{\sin(-\pi\gamma)}{2\pi\gamma} + \frac{i\cos(-\pi\gamma)}{2\pi\gamma}\right) \\
&= \left(\frac{\sin(\pi\gamma)}{2\pi\gamma} - \frac{\sin(-\pi\gamma)}{2\pi\gamma}\right) + \left(\frac{i\cos(\pi\gamma)}{2\pi\gamma} - \frac{i\cos(-\pi\gamma)}{2\pi\gamma}\right) \\
&= \frac{\sin(\pi\gamma)}{\pi\gamma}.
\end{aligned}
$$

Now we know that convolution is associative by a simple change of variables $t \to x - t$ *resulting in*

$$B_n * B_1(x) = \int_{-\infty}^{\infty} B_n(t)\, B_1(x-t)\, dt = B_1 * B_n(x).$$

So then by associativity of convolution, we find that

$$\hat{B}_n(\gamma) = \left(\hat{B}_1(\gamma)\right)^n = \left(\frac{\sin(\pi\gamma)}{\pi\gamma}\right)^n,$$

since

$$
\begin{aligned}
\hat{B}_n(\gamma) &= \int_{-\infty}^{\infty} B_n(x)e^{-2\pi i\gamma x}dx \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} B_{n-1}(x-t)B_1(t)dt\right)e^{-2\pi i\gamma x}dx \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} B_{n-2}(x-t)B_1(t)dt\right)B_1(t)dt\right]e^{-2\pi i\gamma x}dx \\
&\vdots \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} B_1(x-t)(B_1(t))^{n-1}e^{-2\pi i\gamma x}dtdx
\end{aligned}
$$

Here we may apply Fubini's theorem since $B_1(t) \in L^2(\mathbb{R})$ *and* $B_1(x-t)e^{-2xi\gamma x} \in L^2(\mathbb{R})$. *Thus*

$$\hat{B}_n(\gamma) = \int_{-\infty}^{\infty} B_1(t)dt \int_{-\infty}^{\infty} B_1(t)dt \dots \int_{-\infty}^{\infty} B_1(t)dt \int_{-\infty}^{\infty} B_1(x-t)e^{-2\pi i\gamma x}dx$$

$$
\begin{aligned}
&= \quad {}_{-1/2}^{1/2} e^{-2\pi i\gamma x}dx \quad {}_{-1/2}^{1/2} e^{-2\pi i\gamma x}dx \ldots \quad {}_{-1/2}^{1/2} e^{-2\pi i\gamma x}dx \\
&= \quad (\hat{B}_1(\gamma))^n.
\end{aligned}
$$

Now

$$\hat{B}_n\left(2\gamma\right) = \left(\hat{B}_1\left(2\gamma\right)\right)^n = \quad \frac{\sin\left(2\pi\gamma\right)}{2\pi\gamma} \quad ^n = \quad \frac{2\cos\left(\pi\gamma\right)\sin\left(\pi\gamma\right)}{2\pi\gamma} \quad ^n = \left(\cos\left(\pi\gamma\right)\right)^n \hat{B}_n\left(\gamma\right).$$

Going back to Equation (**??**)*, we meet the condition by considering when the order is even:*

$$\left(\cos\left(\pi\gamma\right)\right)^n = \quad \frac{e^{i\pi\gamma}+e^{-i\pi\gamma}}{2} \quad ^n = \sum_{k=-n/2}^{n/2} c_k e^{2\pi i k\gamma}$$

for a certain set of coefficients $\{c_k\}_{k=-n/2}^{n/2}$*. Thus by satisfying* (**??**)*, we see that the wavelet system generated by an even order B-spline is linearly dependent. All B-splines whether even or odd can be written explicitly in terms of piecewise polynomial functions. When* $n > 1$*, the B-spline* B_n *is a continuous piecewise polynomial, where the highest order of the polynomials involved is* $n-1$*. For example,*

$$B_2\left(x\right) = \begin{cases} 1+x & \textit{if } x \in [-1,0] \\ 1-x & \textit{if } x \in [0,1] \\ 0 & \textit{otherwise;} \end{cases}$$

$$B_3\left(x\right) = \begin{cases} \frac{1}{2}x^2 + \frac{3}{2}x + \frac{9}{8} & \textit{if } x \in [-\frac{3}{2}, -\frac{1}{2}] \\ -x^2 + \frac{3}{4} & \textit{if } x \in [-\frac{1}{2}, \frac{1}{2}] \\ \frac{1}{2}x^2 - \frac{3}{2}x + \frac{9}{8} & \textit{if } x \in [\frac{1}{2}, \frac{3}{2}] \\ 0 & \textit{otherwise} \end{cases}$$

For B_2*, our scaling function* (**??**) *is*

$$B_2\left(x\right) = \frac{1}{2}B_2\left(2x-1\right) + B_2\left(2x\right) + \frac{1}{2}B_2\left(2x+1\right).$$

Although B_3 *does not satisfy* (**??**) *since it is an odd-order B-spline, it does satisfy a similar equation in which the translation step of the wavelet system is* $1/2$ *instead of* 1*:*

$$B_3\left(x\right) = \frac{1}{4}B_3 \quad 2x - \frac{3}{2} \quad + \frac{3}{4}B_3 \quad 2x - \frac{1}{2} \quad + \frac{3}{4}B_3 \quad 2x + \frac{1}{2} \quad + \frac{3}{4}B_3 \quad 2x + \frac{3}{2} \quad .$$

6. Frames in $L^2(\mathbb{R})$

In order to reach our goal of discussing Gabor systems and wavelet systems in terms of series expansions of functions in $L^2(\mathbb{R})$, we must first cover the topic of frames.

If a function f is in $L^2(\mathbb{R})$ and if $\{f_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$, then the expansion (**??**) holds in $L^2(\mathbb{R})$ with

$$c_k(f) = \int_{-\infty}^{\infty} f(x)\,\overline{f_k(x)}dx.$$

While an orthonormal basis is automatically linearly independent, (**??**) may still hold for families of functions that are not linearly independent. We may find representations of type (**??**) by considering frames.

Definition 6.1 *A family of functions $\{f_k\}_{k\in\mathbb{Z}}$ in $L^2(\mathbb{R})$ is a frame for $L^2(\mathbb{R})$ if there exist positive constants A and B such that*

$$A\int_{-\infty}^{\infty}|f(x)|^2dx \le \left|\int_{-\infty}^{\infty} f(x)\,\overline{f_k(x)}dx\right|^2 \le B\int_{-\infty}^{\infty}|f(x)|^2dx \tag{6.1}$$

holds for all $f \in L^2(\mathbb{R})$.

An orthonormal basis is automatically a frame. Conversely, a frame $\{f_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis when two conditions are met: Equation (**??**) must hold with $A = B = 1$ and

$$\int_{-\infty}^{\infty}|f_k(x)|^2dx = 1$$

for each k. In simpler terms, a frame $\{f_k\}_{k\in\mathbb{Z}}$ is a basis if, for $k \in \mathbb{N}$,

$$\sum_{k=1}^{\infty} c_k f_k = 0 \Rightarrow c_k = 0. \tag{6.2}$$

The sum in (**??**) can actually be seen as an infinite-dimensional version of linear independence.

We can prove that a frame $\{f_k\}_{k\in\mathbb{Z}}$ leads to a basis of type (**??**), where the coefficients $c_k(f)$ in the expansions of f have the form

$$c_k(f) = \int_{-\infty}^{\infty} f(x)\overline{h_k(x)}dx \tag{6.3}$$

for a certain family of functions $\{h_k\}_{k\in\mathbb{Z}}$ in $L^2(\mathbb{R})$. However, frames present more of a challenge when compared to bases, as there may exist other choices for the coefficient sequence $\{c_k(f)\}_{k\in\mathbb{Z}}$. In general, if $\{f_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis, then the family of functions $\{f_1, f_1, f_2, f_3, f_4, f_5, \dots\}$ is a frame, and each function $f \in L^2(\mathbb{R})$ has several representations of the type (**??**).

For arbitrary frames, finding the coefficients $c_k(f)$ in (**??**) is complicated: first, we must find suitable functions h_k. Our process, however, can be made much simpler if the condition (**??**) holds with $A = B = 1$. In this case, we can take $h_k = f_k$. In general, when $A = B$ for a frame, that frame is said to be *tight*. We note that the condition of $\{f_k\}_{k\in\mathbb{Z}}$ being a tight frame with $A = B = 1$ does not imply that the functions f_k are pairwise orthogonal.

Example 6.2 *If $\{f_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$, then the family*

$$f_1, \frac{1}{\sqrt{2}}f_2, \frac{1}{\sqrt{2}}f_2, \frac{1}{\sqrt{3}}f_3, \frac{1}{\sqrt{3}}f_3, \frac{1}{\sqrt{3}}f_3, \dots$$

constitutes a tight frame, but not all pairs of vectors are orthogonal.

We are now ready to show the connection between Gabor systems and frames.

Definition 6.3 *If an infinite Gabor system $\{e^{2\pi imbx}g(x-na)\}_{m,n\in\mathbb{Z}}$ is a frame, we say it is a Gabor frame.*

The construction of the simplest Gabor frame follows.

Example 6.4 *Previously, we saw that the functions $\left\{\frac{1}{\sqrt{2\pi}}e^{imx}\right\}_{m\in\mathbb{Z}}$ form an orthonormal basis for $L^2[-\pi,\pi]$, and since they are periodic with period 2π, they form an orthonormal basis for $L^2(I)$ for any interval I of length 2π. To be explicit, we are looking at the exponential functions on the interval $[-\pi+2n\pi, \pi+2n\pi)$. It follows that*

$$\frac{1}{\sqrt{2\pi}}e^{imx}\chi_{[-\pi+2n\pi,\pi+2n\pi)}(x) \qquad {}_{m\in\mathbb{Z}}$$

is an orthonormal basis for $L^2(-\pi+2n\pi, \pi+2n\pi)$. We now note that the intervals $[-\pi+2n\pi, \pi+2n\pi)$ are disjoint and cover the entire real line. Due to these facts, the family

$$\frac{1}{\sqrt{2\pi}}e^{imx}\chi_{[-\pi+2n\pi,\pi+2n\pi)}(x) \qquad {}_{m,n\in\mathbb{Z}}$$

is an orthonormal basis for $L^2(\mathbb{R})$.

Although this example is relatively simple, the process of checking whether another Gabor system is a frame can be quite complicated. Even if g is the characteristic function of an interval, there are cases where it is unknown whether the Gabor system generated by g for given parameters a and b constitutes a frame or not. Much work has been spent on this problem, resulting in at least eight types of conditions under which a conclusion can be reached. Some of these conditions follow.

Example 6.5 *It can be shown that*

$$\{e^{2\pi imbx}\chi_{[0,1)}(x-na)\}_{m,n\in\mathbb{Z}}$$

is a frame for $L^2(\mathbb{R})$ *if* $b=1$ *and* $0<a\leq 1$.
We can now consider a more general case if we replace the characteristic function $\chi_{[0,1)}$ *with* $\chi_{[0,c)}$ *for some positive number* c*, and consider the associated Gabor system* $\{e^{2\pi imbx}\chi_{[0,c)}(x-na)\}_{m,n\in\mathbb{Z}}$ *with parameters* a *and* b*. Via a change of variable, the analysis of such systems can be reduced to the case* $b=1$*. The following is known for* $G=\{e^{2\pi imbx}\chi_{[0,c)}(x-na)\}_{m,n\in\mathbb{Z}}$*:*

1. *G is a frame if* $1\geq c\geq a$*;*

2. *G is a frame if* $1<c<2, 0<a<1$*, and* a *is irrational;*

3. *G is not a frame if* $a>1$*;*

4. *G is not a frame if* $a=p/q$ *for relatively prime integers* p *and* q *such that* $2-1/q<c<2$.

From these, we see that the rationality of the parameter a plays a central role.

General frame theory tells us that every function f in $L^2(\mathbb{R})$ has an expansion of type (**??**) in terms of the functions in the frame. Further, we know there exists a function h in $L^2(\mathbb{R})$ such that each f in $L^2(\mathbb{R})$ has a series expansion

$$f(x)=\sum_{m,n\in\mathbb{Z}}c_{m,n}(f)e^{2\pi imbx}g(x-na), \tag{6.4}$$

where

$$c_{m,n}(f)=\int_{-\infty}^{\infty}f(x)e^{-2\pi imbx}\overline{h(x-na)}dx$$

and convergence is in the L^2-norm. These coefficients should be compared to those of type (**??**) for general frames. In that case, we have to find an infinite family of functions $\{h_k\}_{k\in\mathbb{Z}}$ to be able to calculate the coefficients. However, for the Gabor case, one function h is sufficient.

We can rewrite the assumption of case (3) in Example **??** as $ab > 1$. So then a Gabor system G is not a frame if $ab > 1$.

If we consider the condition $a > 1$ alone, it is true that all functions in the Gabor system $\{e^{2\pi imx}\chi_{[0,1)}(x - na)\}_{m,n\in\mathbb{Z}}$ are zero on the interval $(1, a)$ and so may not be used to expand arbitrary functions in $L^2(\mathbb{R})$. Interestingly, if $ab < 1$, then the coefficients $c_{m,n}(f)$ in (**??**) are never unique.

When we compare this with the previous result that a finite regular Gabor system is linearly independent, we find major differences. The result for finite systems tells us that an arbitrary function f has *at most* one representation as a finite sum

$$f(x) = \sum_{|m|,|n|\leq N} c_{m,n}(f)e^{2\pi imbx}g(x - na).$$

From this comparison, we note that the linear independence of a finite Gabor system does not imply that the representation in (**??**) is unique. Rather, linear independence for elements in a frame $\{f_k\}_{k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$ is given by (**??**). This concept is a much stronger condition than linear independence of each finite subset $\{f_k\}_{k=1}^n$ as it involves the entire set. However, we do know how the two ideas are related - (**??**) holds if and only if $\{f_k\}_{k=1}^n$ is linearly independent for each n and the quantity

$$\inf_n \min_f \left\{ \sum_{k=1}^n \left| \int_{-\infty}^{\infty} f(x)\overline{f_k(x)}dx \right|^2 : f \in \text{span}\{f_k\}_{k=1}^n, \int_{-\infty}^{\infty} |f(x)|^2 dx = 1 \right\}$$

is positive.

Now the reader may be wondering why bother with the more complicated frame concept when we already can construct Gabor systems that form orthonormal bases? The answer is that frames give us more flexibility. While Gabor-type orthonormal bases exist, it might be that we can not find one that satisfies certain additional constraints. This can be exemplified by the "Balian-Low Theorem," which states if a function g generates an orthonormal basis with Gabor structure, then

$$\int_{-\infty}^{\infty} |xg(x)|^2 dx \int_{-\infty}^{\infty} |\gamma\hat{g}(\gamma)|^2 d\gamma = \infty. \tag{6.5}$$

This tells us that it cannot be the case for both g and $\hat{g}$ to decay rapidly at infinity in the orthonormal setting. This presents a problem in signal analysis, where the time-behavior and frequency-behavior functions are considered together. However, Gabor frames help us overcome this inconvenience. Using Gabor frames, we can construct functions g that generate frames for which the product (**??**) is finite.

Example 6.6 *Consider the Gaussian $g(x) = e^{-x^2}$, which generates a frame for $L^2(\mathbb{R})$ if $ab < 1$. For this g, $\hat{g}(\gamma) = \sqrt{\pi}e^{-\pi^2\gamma^2}$.*
So then by (**??**), *we have*

$$\begin{aligned}\int_{-\infty}^{\infty} |xe^{-x^2}|^2 dx \int_{-\infty}^{\infty} |\gamma\sqrt{\pi}e^{-\pi^2\gamma^2}|^2 d\gamma &= \int_{0}^{\infty} 2x^2 e^{-x^2} dx \int_{0}^{\infty} 2\gamma^2 \pi e^{-2\pi^2\gamma^2} d\gamma \\ &\leq \int_{0}^{\infty} 2x^3 e^{-x^2} dx \int_{0}^{\infty} 2\gamma^3 \pi e^{-2\pi^2\gamma^2} d\gamma\end{aligned}$$

Our bound on the right gives us

$$\lim_{t\to\infty} e^{-2t^2} -\frac{1}{4}t^2 - \frac{1}{8} \quad e^{-2\pi^2 t^2} -\frac{1}{4\pi^2}t^2 - \frac{1}{8\pi^4} < \infty$$

Therefore

$$\int_{-\infty}^{\infty} |xe^{-x^2}|^2 dx \int_{-\infty}^{\infty} |\gamma\sqrt{\pi}e^{-\pi^2\gamma^2}|^2 d\gamma < \infty$$

Thus the product (**??**) *is finite.*

We note that the condition $ab < 1$ is a determining factor here. If $ab = 1$, the Gaussian does not generate a frame. If it were the case that a Gabor system with $ab = 1$ is a frame, then it is actually a basis. If g is the Gaussian in this case, it would contradict the Balian-Low Theorem.

Certain infinite wavelet systems also yield representations of all functions in $L^2(\mathbb{R})$. There exist functions $\psi \in L^2(\mathbb{R})$ such that each $f \in L^2(\mathbb{R})$ has a representation (convergent in the L^2-norm)

$$f(x) = \sum_{j,k\in\mathbb{Z}} c_{j,k} 2^{j/2} \psi(2^j x - k). \tag{6.6}$$

One of the main focuses of wavelet theory is the construction of functions ψ such that $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$. This was first proposed by Mallat and Meyer in 1989, who developed multiresolution analysis. Further, Daubechies found how to construct orthonormal bases $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ for which ψ has compact support, meaning that the support of ψ must be a complete and totally bounded subset of $L^2(\mathbb{R})$.

Prior to the presentation of these methods however, frame constructions with wavelet structure appeared in a 1985 paper by Daubechies, Grossman, and Meyer. In the context of wavelets, frames are again helpful as they can satisfy properties that orthonormal bases do not. For example, we may find a frame $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ for which the function ψ is infinitely differentiable and decays exponentially, whereas no basis $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ with these properties exist.

Recent progress in wavelet theory has been centered around the fact that linearly dependent wavelet systems exist. Going back to prior work, most constructions of functions ψ for which representations of the type (**??**) are possible are based on scaling functions ϕ that satisfy equations of the sort

$$\phi(x) = \sum_{k\in\mathbb{Z}} c_k \phi(2x-k). \tag{6.7}$$

In classical wavelet analysis though, it is possible that infinitely many coefficients c_k are nonzero, so (**??**) is not yet a statement about linear dependence. However, tight frames of the form $\{\psi^1_{j,k}\}_{j,k\in\mathbb{Z}} \cup \{\psi^2_{j,k}\}_{j,k\in\mathbb{Z}}$ have recently been constructed by combining the wavelet systems associated with two functions ψ^1 and ψ^2. These functions are typically constructed by means of expressions of the form

$$\psi^i(x) = \sum d^i_k \phi(2x-k),$$

where ϕ satisfies an equation of the type (**??**) for a *finite* sequence $\{c_k\}$.

The sequences $\{d^i_k\}$ used to find ψ^1 and ψ^2 are usually finite as well. If ϕ is chosen to be a B-spline of even order, this approach leads to explicitly given functions ψ^i. If we examine figures of these functions as compared to functions from classical wavelets, we may quite easily see that we are handling much simpler functions. In fact, the aforementioned work on tight frames shows that tight frames can be found such that the generators ψ^1 and ψ^2 are splines of any desired order and with compact support.

We now end our paper with a short delving into the decomposition of frames into linearly independent subsets. Since we have already seen the possibility of elements in a frame being linearly dependent, we should naturally wonder whether we may decompose a frame $\{f_k\}_{k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$ into a finite number of linearly independent subsets. The fact is, we can do so provided the following holds:

$$\inf_k \int_{-\infty}^{\infty} |f_k(x)|^2 dx > 0. \tag{6.8}$$

Theorem 6.7 *If $\{f_k\}_{k\in\mathbb{Z}}$ is a frame for which (**??**) is satisfied, then there exists a finite partition*

$$\{f_k\}_{k\in\mathbb{Z}} = \bigcup_{j=1}^{N} \{f_k\}_{k\in I_j} \tag{6.9}$$

such that each set $\{f_k\}_{k\in I_j}$ is linearly independent (in the sense that each finite subset of $\{f_k\}_{k\in I_j}$ is linearly independent).

If we want to decompose a frame into a finite set of families satisfying (**??**), we have a more subtle problem. The following conjecture by Feichtinger addresses this concept.

Conjecture 6.8 *Every frame $\{f_k\}_{k\in\mathbb{Z}}$ satisfying (**??**) can be partitioned into a finite union as in (**??**), where each sequence $\{f_k\}_{k\in I_j}$ satisfies (**??**).*

Feichtinger conjectured further that the sequences $\{f_k\}_{k\in I_j}$ can be chosen to frames. While this conjecture is true under slightly stronger assumptions, the conjecture itself remains to be proven. It is important to note that the condition (**??**) is necessary for Theorem **??** as well as for the conjecture. For example, if $\{f_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$, then

$$f_1, \frac{1}{\sqrt{2}}f_2, \frac{1}{\sqrt{2}}f_2, \frac{1}{\sqrt{3}}f_3, \frac{1}{\sqrt{3}}f_3, \frac{1}{\sqrt{3}}f_3, \ldots$$

is a frame that cannot be decomposed into a finite union of linearly independent sets. The condition (**??**) is automatically satisfied in the important wavelet case.

7. Applications

While we have compared wavelets and frames to more traditional methods such as orthonormal bases, we have yet to address the convenience in applications. Wavelets and frames are in fact very practical across the board, and so we will highlight a few specific cases for which this is true.

The main use for wavelet systems is efficient compression methods. When we take a large class of signals, a wavelet representation contains a large number of small coefficients. Then, by replacing small coefficients with zeroes, we can find a close approximation of the signal that takes much less capacity to store or transmit. Thus, wavelets are employed in many important cases, such as the storing of fingerprint images by the FBI. While an original fingerprint image would take 13Mb of storage space, wavelets allow compression down to 1Mb capacity, while still allowing for complete recognition of traits. There are other methods of storing fingerprints, such as the use of Fourier analysis, however, those methods do not offer efficient compression.

Throughout this paper we have already shown the direct benefits of frames over the traditional orthonormal bases. In applications, frames prove to be the method of choice as well, specifically in signal processing. Here, frames are extremely efficient in suppressing noise. When a signal is transmitted from one place to the next, it is always accompanied by unnecessary noise. This noise is retained during the transmission if we represent the signal using orthonormal bases. However, if we form a representation of the signal via frames, we conveniently lose the extra noise without affecting the signal itself.

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