Graph Nim

1. Introduction

Nim is a well-known two-player impartial combinatorial game. Various versions of playing Nim on graphs have been investigated. We investigate a new version of Nim called Graph Nim. Given a graph with n vertices and multiple edges, players take turns removing edges until there are no edges left. Players have to choose a vertex and remove at least one edge incident to the chosen vertex. The player that removes the last edge or edges wins the game. In this paper, we give the solution for certain game boards of Graph Nim, compare the game of Graph Nim to another impartial combinatorial game, and discuss open problems.

2. Impartial Combinatorial Games.

An *impartial combinatorial game* has several features that set it apart from other games, specifically [5]:

1. There are two players that alternate moves;
2. There are no elements of chance - for example, no rolling dice or distributing cards;
3. There is perfect information - all possible moves are known to both players;
4. The game must end and there are no draws;
5. The last move determines the winner - in normal play, the last player to move wins the game.

Examples of games that are not impartial combinatorial games are Go, since the last person to move is not necessarily the winner, Backgammon, since there is an element of chance (rolling the dice), Tic-Tac-Toe, since it can end in a draw, and Rock-Paper-Scissors, since the players do not alternate moves. Impartial combinatorial games are purely about strategy. In 1912 it was proven that in an impartial combinatorial game one player has a strategy to win the game [7].

Nim is one of the most common impartial combinatorial games. It is played with n piles of tokens with $k_1, k_2, ..., k_n$ tokens in each pile. The two players take turns removing at least one token from one selected pile. The player that removes the last token or tokens wins the game. Although the exact origin of Nim is unknown, it is reported to date back to ancient times. Charles Bouton found the solution to Nim in 1902 [1], and that result is considered to have given rise to combinatorial game theory. The solution to Nim uses binary numbers and is very interesting. We will not go over the solution here, since it is not the solution to the game we investigated. In [2] and [4], there is a comprehensive solution to Nim.

Study of combinatorial games consists of finding the winning and losing possibilities of players from a given game position or game board. For the purpose of this paper, we define a W-position as a position in which the next player has a strategy to win the game (so a winning position), an L-position as a position in which the next player will lose the game if the opponent plays optimally (so a losing position) and a terminal position is a position from which there are no more moves available. Notice that all game positions are either an L-position or a W-position, so L-positions and W-positions partition the set of all game positions for a given game.

There are three characteristic properties of L-positions and W-positions that are valid for all impartial combinatorial games [4]. Proving that these three properties hold is finding a solution to the game. The three properties are:

1. All terminal positions are L-positions.
2. From every W-position, there is at least one move to an L-position.
3. From every L-position, every move is to a W-position.

Another way to think about these three properties is:

1. If it is your turn and there are no more moves to make, you just lost the game.
2. If you are in a winning position (W-position), there is at least one move you can make to hand your opponent a losing position (L-position).
3. If you are in a losing position (L-position), every move you make hands your opponent a winning position (W-position).

3. Graph Nim

Here we introduce some necessary graph theory definitions. For a comprehensive treatment of graph theory see [6]. A *graph* G is a multi-triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates each edge with two vertices called its endpoints. When u and v are the endpoints of an edge, they are *adjacent*. If vertex v is an endpoint of edge e, then v and e are called *incident*. *Multiple edges* are edges having the same pair of endpoints, and a *loop* is an edge whose endpoints are equal. A *cycle* is a graph whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. Notationally, C_3 and C_4 are cycles of 3 vertices and 4 vertices respectively.

A variation of Nim, called *Graph Nim*, can be played on graphs with multiple edges, but no loops. This game is played by first choosing a vertex, then removing at least one edge that is incident to the chosen vertex. The players take turns until all the edges have been removed, and the player that removes the last edge or edges wins the game.

This version of Graph Nim was introduced at a Research Experience for Teachers lead by Dr. Michael Ferrara and Dr. Breeann Flesch at University of Colorado Denver in 2010. The teachers proved Theorem 3.1, but the result never appeared in print. Here we independently prove Theorem 3.1 and then prove other results about this game.

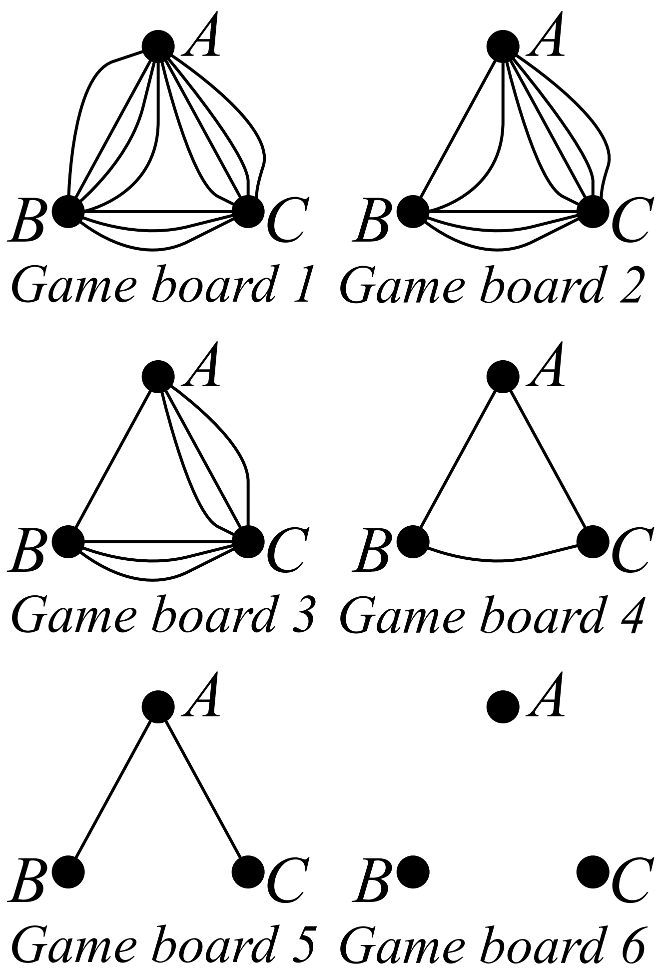

Figure 1: An example of a game played on C_3.

To understand the game better, let's consider an example of Graph Nim on a C_3 with multiple edges. This game is illustrated in Figure 1, and the game starts with game board 1. Player 1 chooses vertex A and removes two edges between A and B, which leaves Player 2 with game board 2. Then Player 2 chooses vertex A and removes one edge between A and B and one edge between A and C. Now Player 1 is working off of game board 3; player

1 chooses vertex C and removes two edges between A and C and two edges between B and C. This leaves Player 2 with game board 4. Next Player 2 chooses vertex B and removes one edge between B and C. Examining game board 5, Player 1 sees an opportunity to win the game. Player 1 chooses vertex A and removes all of the remaining edges. Thus Player 1 wins the game. Notice that Player 2 was left with a graph with no edges, game board 6, and that is the terminal position for this game.

We now present the solution to Graph Nim on C_3 with multiple edges.

3.1. Graph Nim on C_3

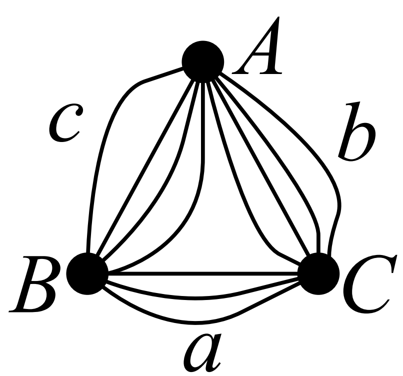

Figure 2: A C_3 with multiple edges, where a, b, and c are the number of edges as shown in the picture above.

Theorem 3.1 *In Graph Nim on C_3 with multiple edges, a position is an L-position if and only if $a = b = c$ such that a, b and c are the number of edges as shown in Figure 2.*

Proof: To prove this, we will show that all three of the characteristic properties of L- and W-positions hold. In Graph Nim a terminal position is where $a = b = c = 0$, which implies that the number of edges in a terminal position are equal. Therefore, it is an L-position. This satisfies the first characteristic property of L-positions and W-positions.

Assume it is not true that $a = b = c$. Without loss of generality, let a be the smallest. Then either $c - a$ or $b - a$ is not zero, because either $a \neq c$ or $b \neq a$. Thus the move is to choose vertex A and remove $c - a$ edges between vertices A and B and $b - a$ edges between vertices A and C. Now we have $c - (c - a) = a$ edges between A and B, and $b - (b - a) = a$ edges between A and C. Therefore, $a = b = c$, which is an L-position. This satisfies the second characteristic property of L-positions and W-positions.

Assume $a = b = c$. By the rules of the game, it is necessary for the player to remove at least one edge. Without loss of generality, choose vertex A. If we remove an edge incident to A, either $b \neq a$ or $c \neq a$, which is an W-position. Thus satisfying the third characteristic property of L-positions and W-positions. ❑

Now let us look back at the game that was played in Figure 1. On game board 1, $a = 3, b = 4$ and $c = 4$, so Player 1 was in a winning position (W-position). However, he/she did not know the correct move to make to hand Player 2 a losing position (L-position). To figure out the correct move, we look to the proof of Theorem 3.1. The smallest of a, b and c is $a = 3$, so we choose vertex A and remove $b - a = 4 - 3 = 1$ edge between A and B and $c - a = 4 - 3 = 1$ edge between A and C. After removing these edges, it would be that $a = b = c = 3$, which is an L-position.

To illustrate the last property, we can look at game board 4 in Figure 1. On this game board $a = b = c = 1$, so by Theorem 3.1 it is an L-position. When Player 2 moves, he/she must remove at least one edge and can remove at most two edges, regardless of which vertex is chosen. Either way, this will leave the opponent with a way to win the game by moving all of the remaining edges.

Similar to Graph Nim on C_3, this game can be played with larger graphs with more vertices. Next, we will see the solution to Graph Nim played on C_4 with multiple edges.

3.2. Graph Nim on C_4

Theorem 3.2 *In a graph Nim on C_4, a position is an L-position if and only if $a = c$ and $b = d$ such*

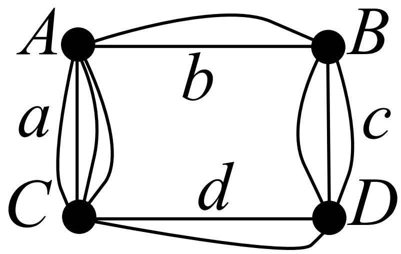

Figure 3: A C_4 with multiple edges, where $a, b, c,$ and d are the number of edges as shown in the picture above.

that a, b, c and d are the number of edges as shown in Figure 3.

Proof: In Graph Nim a terminal position is where $a = b = c = d = 0$, which implies that $a = c$ and $b = d$. Thus it is an L-position, satisfying the first characteristic property of L-positions and W-positions.

Assume $a \neq c$ or $b \neq d$. Without loss of generality, let $b > d$ and $a \geq c$. Now we choose vertex A then remove $b - d$ edges between vertices A and B and remove $a - c$ edges between vertices A and C. Now we get $a = c$ and $b = d$, implying it is an L-position. This satisfies the second characteristic property of L-positions and W-positions.

Assume $a = c$ and $b = d$. The players are required to at least one edge. Without loss of generality, choose vertex A and remove an edge or edges incident to A. Thus we get $a < c$ or $b < d$. Therefore, it is an W-position, satisfying the third characteristic property of L-positions and W-positions. ❑

After proving these two results, we started to look at Graph Nim on other game boards, for example on C_5. However, the results were not forthcoming, so we decided to investigate other impartial combinatorial games to try to inform our investigation. There are many other impartial combinatorial games that are variations of Nim. One such game that we will now consider is called Circular Nim.

4. Circular Nim.

Circular Nim was introduced by Matthieu Dufour and Silvia Heubach in 2013 [3]. In this alteration of Nim, n stacks of tokens are arranged in a circle. The two players take turns removing at least one token from one or more of k consecutive stacks. The game is denoted by $CN(n, k)$ where n is the number of stacks of tokens and k is the number of consecutive stacks from which the players can remove tokens. The player that removes the last token or tokens wins the game. When $k = 1$, the game is just Nim, but when $k > 1$ the solution to Nim does not apply.

A position in Circular Nim can be denoted by a vector $\mathbf{p} = (p_1, p_2, p_3, ..., p_n)$ where p_i represents the number of tokens in stack i . With the use of legal moves, if the position $\mathbf{p}$ is moved to $\mathbf{p'}= (p'_1, p'_2, ..., p'_n)$, we call this $\mathbf{p'}$ position to be the option of $\mathbf{p}$. This change in position can be represented by the notation $\mathbf{p} \to \mathbf{p'}$ [3].

Figure 4 shows an example of a $CN(5, 2)$, where there are 5 stacks and you can remove from up to 2 consecutive stacks. The game starts in position $\mathbf{p} = (5, 1, 6, 3, 2)$. Player 1 starts the game by choosing stack 1 and 5 and removing 2 and 1 token, respectively. This results in game board 2 in Figure 4, which is $\mathbf{p} = (3, 1, 6, 3, 1)$. Now player 2 removes 2 tokens from stack 4 and 1 token from stack 5, resulting is position $\mathbf{p} = (3, 1, 6, 1, 0)$ depicted in game board 3. Removing the token from stack 4 and 4 tokens from stack 3, Player 1 makes position $\mathbf{p} = (3, 1, 2, 0, 0)$. Now Player 2 is faced with game board 4 and chooses to remove the one token in stack 2, resulting in position $\mathbf{p} = (3, 0, 2, 0, 0)$. Player 1 removes all the tokens from stack 3, giving Player 2 game board 6 and a chance to win. From here Player 2 will remove all three tokens from stack 1 and win the game.

If $k = 2$, then Circular Nim is equivalent to Graph Nim on C_n with multiple edges. We prove this now.

Theorem 4.1 *The game $CN(n, 2)$ is equivalent to Graph Nim on C_n with multiple edges.*

Proof: In case of $CN(n, 2)$, there are n stacks,

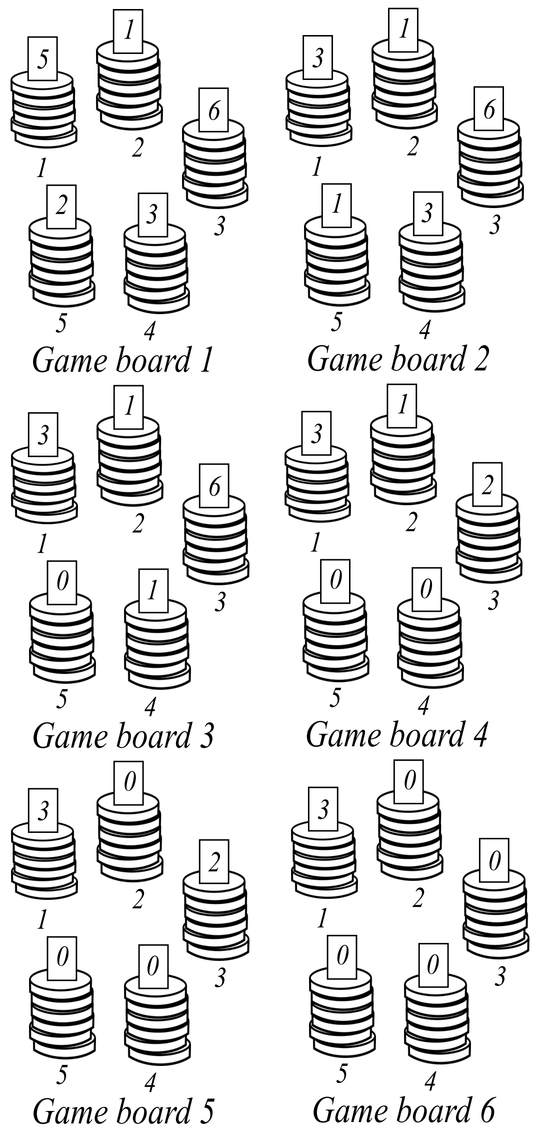

Figure 4: An example of a game of $CN(5,2)$

and the players can choose up to two consecutive stacks from which they remove at least one token. Let's assume there are k_1,k_2,k_3,...,k_n tokens in stacks 1, 2, 3,...n respectively, so $\mathbf{p} = (k_1, k_2, k_3, ..., k_n)$. For a Graph Nim on C_n, we have n vertices that are connected to each other by a number of edges. Let's assume that there are k_1 edges connecting vertices N_1 and N_2, k_2 edges connecting vertices N_2 and N_3, k_3 edges connecting N_3 and N_4, continuing in this way until we have k_n edges connecting vertices N_n and N_1. The number of stacks in $CN(n,2)$ is equivalent to the number of sets of edges in C_n. The number of tokens in each stack in $CN(n,2)$ is equivalent to the size of the set of edges connecting the respective vertices in the graph.

In $CN(n,2)$, without loss of generality, let's assume the player removes i tokens from stack 1 and j tokens from stack 2, where $i \leq k_1$ and $j \leq k_2$. This implies we reach to a position $\mathbf{p'} = (k_1 - i, k_2 - j, k_3, ..., k_n)$. The equivalent move in Graph Nim on C_n is where the player chooses vertex N_2, then removes i edges between vertices N_1 and N_2, and j edges between N_2 and N_3. This leads to a position in Graph Nim on C_n where there are $k_1 - i$ edges connecting N_1 and N_2, $k_2 - j$ edges connecting vertices N_2 and N_3, k_3 edges connecting N_3 and N_4, continuing in this way until we have k_n edges connecting vertices N_n and N_1.

Thus the game boards and moves in $CN(n,2)$ are equivalent to those in Graph Nim on C_n with multiple edges, and the games are equivalent. ❏

In [3] there are three results for Circular Nim when $k = 2$, which are listed below. Notice that the first two results are the same as Theorem 3.1 and 3.2.

Theorem 4.2 *[3] For the game $CN(3,2)$, the set of losing positions is $L = \{(a,a,a)|a \geq 0\}$.*

Theorem 4.3 *[3] For the game $CN(4,2)$, the set of losing positions is $L = \{(a,b,a,b)|a,b \geq 0\}$.*

Theorem 4.4 *[3] The game $CN(5,2)$ has losing positions $L = \{(a,b,c,d,b)|a+b = c+d$ and a is the max$(\mathbf{p})\}$.*

The third result is equivalent to a result for Graph Nim on C_5. We use the proof in [3] to inform the following result for Graph Nim on C_5.

Theorem 4.5 *In a Graph Nim on a C_5, a position is an L-position if and only if we can assign a, b, c, d, e consecutively to the size of the edge sets such that $b = e$, $a + b = c + d$ and a is maximum of $\{a, b, c, d, e\}$.*

Proof: Unlike our other proofs $a, b, c, d,$ and e may move around in this proof, since a must be a

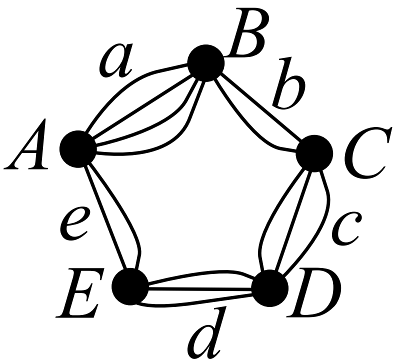

Figure 5: An example of Graph Nim on C_5, where a, b, c, d and e are the number of edges as shown in the picture above.

maximum of a, b, c, d and e. Notice that a may be one of many maximum values.

In Graph Nim a terminal position is where $a = b = c = d = e = 0$. This implies that $b = e$, $a + b = c + d = 0$ and a is the maximum of a, b, c, d and e. Therefore, it is an L-position, and this satisfies the first characteristic property of L-positions and W-positions.

Note that a position is an L-position if and only if adding or removing one from every edge set is also an L-position. For instance, the graph in Figure 5 is an L-position. If we add an edge to every edge set in the graph, we get $a = 5$, $b = 3$, $c = 4$, $d = 4$, and $e = 3$. We still have $b = e = 3$, $a + b = c + d = 8$ and a is a maximum. Similarly, if we remove three edges from every edge set in the graph, we get $b = e = 0$, $a + b = c + d = 2$ and $a = 2$ which is still a maximum of a, b, c, d and e. Therefore, it can be generalized that a position is an L-position if and only if removing or adding the same amount of edges to each edge set is also an L-position. This allows us to assume that the minimum number of edges is zero.

Assume $b \neq e$ or $a + b \neq c + d$ or a is not a maximum. The two cases that we will look into is either the 0 is next to the largest set of edges or it is not.

Case 1: Assume 0 is next to a maximum size set of edges. Let the number of edges be 0

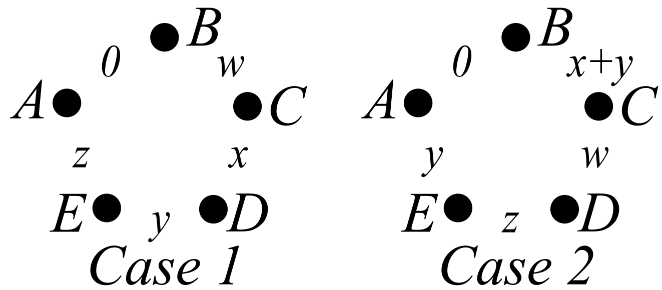

Figure 6: A picture for the two cases in the proof for Theorem 4.5

between A and B, w between B and C, which is a maximum, x between C and D, y between D and E and z between E and A (see Figure 6 Case 1). If $w \geq z + y$, choose vertex C and remove x edges between C and D and $w - (z + y)$ edges between B and C. Now, we have $w = z + y$ corresponds to a since w is the maximum. The edge sets with 0 edges corresponds to b and e, y and z correspond to c and d respectively. Thus $b = e$, $a + b = c + d$ and a is the maximum, implying it is an L-position. If $w < z + y$, choose vertex D, and remove x edges between C and D and $y - (w - z)$ edges between D and E. Now, we have w corresponds to a since w is the maximum. The edge sets with 0 edges corresponds to b and e, $w - z$ and z correspond to c and d respectively. Thus $b = e$, $a + b = c + d$ and a is the maximum of a, b, c, d and e, implying it is an L-position.

Case 2: Without loss of generality, assume the number of edges between B and C is greater than or equal to the number of edges between E and A. Now assign assume there are 0 between A and B, $x + y$ between B and C, w between C and D, z between D and E and y between E and A (see Figure 6 Case 2). Also assume either w or z is a maximum number of edges. If $z \geq x$, choose vertex D and remove w edges between C and D and $z - x$ edges between D and E. This leads to a position where $x + y$ corresponds to a since $x + y$ is the biggest. The edge sets with 0 edges corresponds to b and e, x and y correspond to c and d respectively. We now have $b = e$, $a + b = c + d$ and a is the biggest implying it is an L-position.

If $z < x$, choose vertex C and remove $x - z$ edges between B and C and w edges between C and D. This results as $y + z$ corresponding to a since it is the maximum. The edge sets with 0 edges corresponds to b and e, z and y correspond to c and d respectively. We now have $b = e$, $a + b = c + d$ and a is the biggest, implying it is an L-position. Therefore the two cases satisfy the second characteristic property of L-positions and W-positions.

Assume $b = e$, $a + b = c + d$ and a is the maximum as shown in Figure 5. By the rules of the game, it is necessary for the player to remove at least one edge. If we choose vertex A and remove edges from a then $a + b \neq c + d$ or if we remove edges from e then $b \neq e$. If we choose vertex B and remove edges from b then $b \neq e$ or if we remove edges from a then $a + b \neq c + d$. If we choose vertex C and remove edges from c then $a + b \neq c + d$ or if we remove edges from b then $b \neq e$. If we choose vertex D and remove edges from c or d then $a + b \neq c + d$. If we choose vertex E and remove edges from d then $a + b \neq c + d$ or if we remove edges from e then $b \neq e$. In other words, no matter where we remove an edge or edges, we will create a W-position. This satisfies the third characteristic property of L-positions and W-positions. ❑

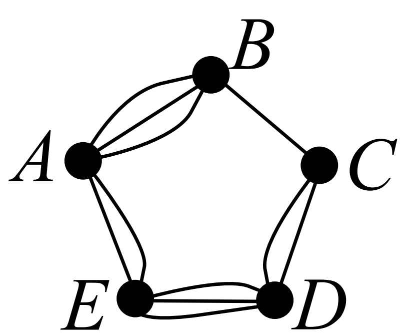

Figure 7: An example game board for Graph Nim on C_5.

Let us go over an example of how the proof of Theorem 4.5 helps us make a move when playing Graph Nim on C_5. Figure 7 is an example game board, and we must first determine whether this position is an L- or W- position. For it to be an L-position we must be able to assign a, b, c, d, e consecutively to the size of the edge sets such that $b = e$, $a + b = c + d$ and a is maximum of a, b, c, d, e. Since there are three edges between A and B and between E and D, one of these two edge sets must be assigned a. In the former case, then $b = 1$ and $e = 2$, so $b \neq e$ and it fails to meet the criteria for being an L-position. If we assign a to the edge set between E and D, then $b = e = 2$, which is a good first step. However, then $c = 3$ and $d = 1$, and $a + b \neq c + d$, again failing the criteria for being an L-position. Thus there is no way to meet the criteria of being an L-position, so the game board in Figure 7 must be a W-position.

This is good news if it is your turn, because that means you are in a winning position. However, you must know the correct move to make to give your opponent an L-position. This is where we look to the proof of Theorem 4.5. Although none of the edge sets are of size 0, like in the proof, we can think of the minimum as being the set of size 0. We can also see a minimum set of edges is next to a maximum set of edges, so we can use case 1 from the proof. Thus we choose vertex E and remove one edge between A and E and one edge between D and E. Here we have that $a = 3$ is from A to B, and $b = e = 1$, which is between B and C and between A and E. Now $c + d = 4 = a + b$, so it meets the criteria for an L-position.

Now your opponent will make a move, which result in you again having a W-position. If you can keep finding the correct move to give your opponent an L-position, then you will eventually win the game.

5. Conclusion

In this paper, we proved the solution for three different Graph Nim game boards. We also proved the equivalence of Graph Nim on C_n and certain versions of Circular Nim, which is another impartial combinatorial game. However, there are many different graphs that could be used as game boards for Graph Nim. For example, there is no known solution for Graph Nim on C_6 or K_4. Thus there are many open problems in this area. Moving forward, we would like to investigate the solution to other game boards in Graph Nim, eventually finding a general solution.

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