Visual
Abstract

A nonlinear elasto-plastic analysis of Reissner-Mindlin plates by finite element method

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ABSTRACT. In this paper, a finite element simulation of nonlinear elasto-plastic deformations of Reissner-Mindlin bending plates is described. The previously proposed four-node Q4γ element with transverse energy of shearing for thick bending plates is extended to account for isotropic material nonlinearities. An incremental finite element procedure has been used for the elasto-plastic analysis of the thick bending plate. Modified Newton-Raphson method has been used to solve the nonlinear equations. Von-Mises yield criteria have been applied for yielding of the materials along with the associated flow rule. To verify the present element, simple tests are demonstrated and various elasto-plastic problems in which the development of the plastic zone are solved.

KEYWORDS. Plate element; Reissner-Mindlin plates; Elasto-plasticity; Nonlinear analysis.

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INTRODUCTION

In recent years, considerable work has been devoted to the study of nonlinear elasto-plastic responses of Reissner-Mindlin bending plates, since plates are very important parts of engineering structures. The nonlinear elasto-plastic bending plates are analyzed by the finite element methods, the finite difference methods, the discrete element methods and the direct numerical methods. However, the finite element method is a suitable approach and has been successfully used in nonlinear elasto-plastic analysis of plates. For example, a finite element analysis of Reissner-Mindlin bending plates has been investigated by Owen and Hinton [1], Meftah [2], Rezaiee-Pajand and Sadeghi [3] and Kanber and Bozkurt [4]. The finite volume formulation is also adopted for the elasto-plastic analysis of Reissner-Mindlin plates by adapting the layered approach [5] and non-layered model [6]. The elasto-plastic analysis using the Element Free Galerkin

method (EFGM) was initially applied to fracture problems and subsequently applied to 2D problems and to 3D problems [7, 8]. On the other hand, the incremental cyclic plasticity theory is recently used firstly to determine the actual stress and strain state arising in two-dimensional or axi-symmetric notched components and later is extended by Marangon et al. [9] and Campagnolo et al. [10] to study the three-dimensional effects at the tip of rounded notches in plates of finite thickness.

The first order shear deformation theories (FSDTs), which include transverse shear deformation, for bending plates have been initially proposed by Reissner [11] and further developed by Mindlin [12]. These theories are widely employed in the nonlinear elasto-plastic behavior.

In this study, a finite element method for analyzing the elasto-plastic plate bending problems is presented. The previous Q4γ [13] plate element with transverse energy of shearing is extended to account for nonlinear elasto-plastic. The goal of this work is to present a transverse energy four-node element Q4γ with only four corner nodes which is significantly superior to the classical four-node element and is not computationally as expensive as a quadratic quadrangle 9-noded Hétérosis element. Having a finite element method for linear elastic analysis of Reissner-Mindlin bending plates, we further develop the model to investigate the elasto-plastic behavior and plastic zone of the structures under consideration. A modified Newton-Raphson method has been used to solve the non-linear equations. Von-Mises yield criterion has been adopted to deal with yielding of the materials along with the isotropic hardening. A computer program has been developed and a number of plate-bending problems have been solved. As the applications of the present element, the square plates with the various boundary conditions are calculated. The results have been compared with existing benchmark solutions. Results obtained with the Q4γ plate element, with the adopted constitutive laws, are compared with those provided by the quadratic quadrangle 9-noded Hétérosis element presented in References [1, 14] with the provision of selective integration and reduced integration. All the computations were carried out in FORTRAN Finite Element code developed by Owen and Hinton [1] and Hinton and Owen [14].

REISSNER-MINDLIN PLATE THEORY

The Reissner-Mindlin plate theory (also designated first order shear deformation theory, FSDT) is more adequate for the analysis of moderately thick plates. The sign convention for stress resultants, the displacement field and the coordinate system are indicated in Fig. 1. General notation is M_x, M_y bending moments, M_{xy} twisting moments, V_x, V_y transverse shear forces, w deflection in z-direction and θ_x, θ_y rotations of the xz- and yz-planes, respectively.
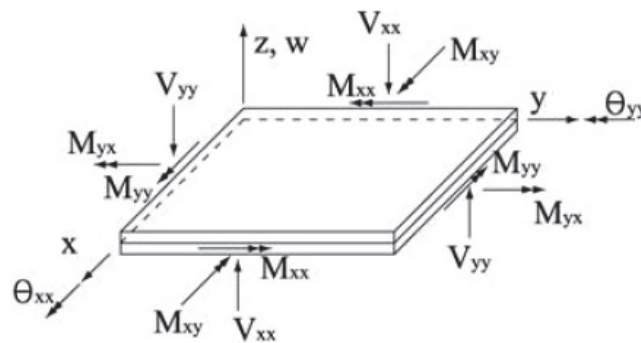

Figure 1: Schematic of the Reissner-Mindlin plate indicating the sign convention chosen for forces and moments.

The displacement field at any point within the element is given by:

$$\begin{cases} u = -z\theta_x(x, y) \\ v = -z\theta_y(x, y) \\ w = w(x, y) \end{cases} \tag{1}$$

The flexural and transverse shear strains in the plate for isotropic homogeneous linear behavior elastic can be written in the concise matrix form as:

$$\{\varepsilon_f\} = \{\varepsilon_x \quad \varepsilon_y \quad \gamma_{xy}\}^T \; ; \; \{\varepsilon_s\} = \{\gamma_{xz} \quad \gamma_{yz}\}^T \tag{2}$$

The linear relationships between the displacements and strains can be obtained by using the definitions of strains from the theory of elasticity:

$$\varepsilon_x = \frac{\partial u}{\partial x} = -z\frac{\partial \theta_x}{\partial x} \; ; \qquad \varepsilon_y = \frac{\partial v}{\partial y} = -z\frac{\partial \theta_y}{\partial y} \tag{3}$$

$$\gamma_{xy} = 2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -z\left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right) \tag{4}$$

$$\gamma_{xz} = 2\varepsilon_{xz} = \frac{\partial w}{\partial x} - \theta_x \; ; \; \gamma_{yz} = 2\varepsilon_{yz} = \frac{\partial w}{\partial y} - \theta_y \tag{5}$$

Assuming normal stress σ_{zz} to be negligibly small compared to other normal stresses, the stress-strain relationship in the matrix takes the form:

$$\sigma = [D]\varepsilon \tag{6}$$

where $\{\sigma\} = \{\sigma_{xx} \; \sigma_{yy} \; \tau_{xy} \; \tau_{yz} \; \tau_{zx}\}^T$ and the matrix $[D]$ for isotropic materials is defined:

$$[D] = \begin{bmatrix} D_f & 0 \\ 0 & D_s \end{bmatrix} \tag{7}$$

with

$$[D_s] = G.k.h\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ; [D_f] = \frac{E.h^3}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \tag{8}$$

where h is the thickness of the plate, k is a shear correction coefficient, E is the Young's modulus and υ is the Poisson's ratio.

The generalized forces per unit of length of the plate side can be obtained using the stress field; these forces are the bending moments (M) and the shear forces (V):

$$\{M\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} -z\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} dz \; ; \; \{V\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} dz \tag{9}$$

CONSTITUTIVE EQUATION FOR RATE INDEPENDENT ELASTOPLASTICITY

After initial yielding the material behavior will be partly elastic and partly plastic. During any increment of stress, the changes of strain are assumed to be divisible into elastic and plastic components, so that:

$$d\varepsilon = d\varepsilon_e + d\varepsilon_p \tag{10}$$

The elastic-plastic strain increment is given by the incremental form of:

$$d\varepsilon = [D]^{-1} d\sigma + d\lambda \frac{\partial F}{\partial \sigma} \tag{11}$$

where $d\lambda$ is a proportional constant called plastic multiplier and $\dfrac{\partial F}{\partial \sigma}$ is the flow vector, which is normal to the adopted yield function, presented in Eq. (12), considering that the yield surface only depends on the magnitude of the applied principal stresses and of a hardening parameter H:

$$F(\sigma, H) = f(\sigma, H) - \sigma_Y(H) = 0 \tag{12}$$

where σ_Y is the yield stress and $f(\sigma, H)$ is the yield criterion. In this paper, the elasto-plastic constitutive model based on the Von-Mises associated yield criterion is adopted for the Reissner-Mindlin plate theory [15]:

$$f(\sigma) = \bar{\sigma} = \frac{1}{\sqrt{2}} \left[\left(\sigma_{xx} + \sigma_{yy} \right)^2 + \sigma_{xx}^2 + \sigma_{yy}^2 + 6 \left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 \right) \right]^{\frac{1}{2}} \leq \sigma_Y \tag{13}$$

where σ_e is the Von-Mises effective stress.
The elastic-plastic incremental stress strain relationship:

$$d\sigma = \left[D^{ep} \right] \left(d\varepsilon_e + d\varepsilon_p \right) \tag{14}$$

$$d\sigma = \left[D^{ep} \right] \left(d\varepsilon_e + d\lambda \frac{\partial F}{\partial \sigma} \right) \tag{15}$$

The differential form of Eq. (12) is:

$$dF = \frac{\partial f}{\partial \sigma} d\sigma - \frac{\partial \sigma_Y}{\partial H} H = 0 \tag{16}$$

or

$$dF = a^T d\sigma - A d\lambda = 0 \tag{17}$$

in which the flow vector a^T is define as:

$$a^T = \frac{\partial f}{\partial \sigma} = \left[\frac{\partial f}{\partial \sigma_x} \quad \frac{\partial f}{\partial \sigma_y} \quad \frac{\partial f}{\partial \tau_{xy}} \quad \frac{\partial f}{\partial \tau_{yz}} \quad \frac{\partial f}{\partial \tau_{zx}} \right] \tag{18}$$

Eqns. (16) and (17) can be reduced to get the hardening parameter A as:

$$A = d\lambda^{-1} \frac{\partial \sigma_Y}{\partial H} H \tag{19}$$

The plastic rate multiplier can be obtained as:

$$d\lambda = \frac{a^T D}{A + a^T D a} d\varepsilon \tag{20}$$

By substituting the expression of the plastic multiplier $d\lambda$ into Eq. (15), the elasto-plastic tangent modulus is derived as:

$$D_{ep} = D - \frac{D a\, a^T D}{A + a^T D a} \tag{21}$$

The incremental stress-strain relationship is given as:

$$\left\{ \begin{matrix} d\sigma_f \\ d\sigma_s \end{matrix} \right\} = \begin{bmatrix} \left(D_{ep}\right)_f & 0 \\ 0 & D_s \end{bmatrix} \left\{ \begin{matrix} d\varepsilon_f \\ d\varepsilon_s \end{matrix} \right\} \tag{22}$$

For Mindlin plate, yield function F is assumed to be function of σ_f but not of the transverse shear stresses σ_s, the direct stresses associated with flexure only hence D_s always remain elastic [1, 16, 17].

FINITE ELEMENT FORMULATION

The Mindlin-Reissner theory takes the shear deformation into account by decoupling the rotation of the plate cross-section from the slope of the deformed mid-surface and the displacement field requires C^0 continuity only. Then the displacement fields (the transverse displacement w and two rotations θ_x, θ_y) are described by the same order of shape functions as follows:

$$\{d\} = \left\{ \begin{matrix} w \\ \theta_x \\ \theta_y \end{matrix} \right\} = \sum_{i=1}^{n} \begin{bmatrix} N_i & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & N_i \end{bmatrix} \left\{ \begin{matrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{matrix} \right\} \tag{23}$$

The bending and shear strain-displacement relationships are given as:

$$\delta\varepsilon_f = \sum_{i=1}^{n} B_{fi}.\delta d_i \,;\; \delta\varepsilon_s = \sum_{i=1}^{n} B_{si}.\delta d_i \tag{24}$$

with

$$B_{fi} = \begin{bmatrix} 0 & -\dfrac{\partial N_i}{\partial x} & 0 \\ 0 & 0 & -\dfrac{\partial N_i}{\partial y} \\ 0 & -\dfrac{\partial N_i}{\partial y} & -\dfrac{\partial N_i}{\partial x} \end{bmatrix} \,;\; B_{si} = \begin{bmatrix} \dfrac{\partial N_i}{\partial x} & -N_i & 0 \\ \dfrac{\partial N_i}{\partial y} & 0 & -N_i \end{bmatrix} \tag{25}$$

The tangential stiffness matrix can be written as follows:

$$K_T = \int_A \left(\left[B_f \right]^T \left[D_{ep} \right] \left[B_f \right] + \left[B_s \right]^T \left[D \right] \left[B_s \right] \right) dA \tag{26}$$

In this study the solution is obtained resorting to the modified Newton-Raphson method [1]. In this algorithm the modification consists of computing the tangent stiffness matrix only once in the beginning of each load increment than in each iteration.

Q4γ finite element

The Q4γ element shown in Fig. 2 is adopted in the present study. This element contains four nodes at the corners and the associated classical interpolation functions given by [13]:

$$N_i = \frac{1}{4}\left(1 + \xi\xi_i\right)\left(1 + \eta\eta_i\right); i = 1, 2, 3 \text{ and } 4 \tag{27}$$

with $\left(\xi_1, \xi_2, \xi_3, \xi_4\right) = \left(-1, 1, 1, -1\right)$ and $\left(\eta_1, \eta_2, \eta_3, \eta_4\right) = \left(-1, -1, 1, 1\right)$.
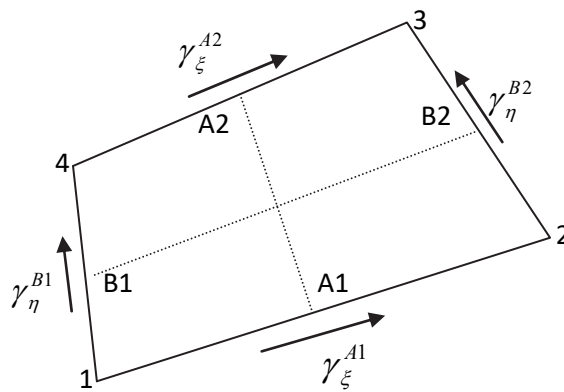

Figure 2: Q4γ quadrilateral isoparametric element [13].

For the Q4γ element the transverse field of distortion $\overline{\gamma}$ is linearly discretizes in the element of reference by side so that:

$$\gamma = \left\{\begin{matrix} \overline{\gamma}_\xi \\ \overline{\gamma}_\eta \end{matrix}\right\} = \left\{\begin{matrix} \dfrac{1-\eta}{2}\gamma_\xi^{A1} + \dfrac{1+\eta}{2}\gamma_\xi^{A2} \\ \dfrac{1-\xi}{2}\gamma_\eta^{B1} + \dfrac{1+\xi}{2}\gamma_\eta^{B2} \end{matrix}\right\} \tag{28}$$

By means of then the relations:

$$\int_{-1}^{+1}\left(\overline{\gamma}_\xi - \left(w_{,\xi} + \theta_\xi\right)\right)d\xi = 0 \; ; \; \int_{-1}^{+1}\left(\overline{\gamma}_\eta - \left(w_{,\eta} + \theta_\eta\right)\right)d\eta = 0 \; ; \text{ for } \xi = \pm 1 \text{ and } \eta = \pm 1 \tag{29}$$

One establishes that:

$$\gamma_\xi^{A1} = \frac{1}{2}\left(w_2 - w_1 + \theta_{\xi 1} + \theta_{\xi 2}\right); \; \gamma_\xi^{A2} = \frac{1}{2}\left(w_4 - w_3 + \theta_{\xi 3} + \theta_{\xi 4}\right) \tag{30.a}$$

$$\gamma_\xi^{B1} = \frac{1}{2}\left(w_4 - w_1 + \theta_{\xi 1} + \theta_{\xi 4}\right); \; \gamma_\xi^{B2} = \frac{1}{2}\left(w_3 - w_2 + \theta_{\xi 2} + \theta_{\xi 3}\right) \tag{30.b}$$

By deferring the two results above in the statement of $\overline{\gamma}$, one from of deduced that:

$$\gamma = \left\{ \begin{matrix} \overline{\gamma}_\xi \\ \overline{\gamma}_\eta \end{matrix} \right\} = \begin{bmatrix} & \dfrac{\partial N_i}{\partial \xi} & \xi_i \dfrac{\partial N_i}{\partial \xi} & 0 & \\ \cdots & & & & \cdots \\ & \dfrac{\partial N_i}{\partial \eta} & 0 & \eta_i \dfrac{\partial N_i}{\partial \eta} & \end{bmatrix} \left\{ \begin{matrix} \vdots \\ w_i \\ \theta_{\xi i} \\ \theta_{\eta i} \\ \vdots \end{matrix} \right\} \tag{31}$$

It is now necessary to express the rotations given here in the element of reference according to rotations in the local coordinate system:

$$\left\{ \begin{matrix} \theta_{\xi k} \\ \theta_{\eta k} \end{matrix} \right\} = \begin{bmatrix} J_k \end{bmatrix} \left\{ \begin{matrix} \theta_{xk} \\ \theta_{yk} \end{matrix} \right\} \tag{32}$$

where J_k is the inverse Jacobian matrix components.

NUMERICAL EXAMPLES

The resulting mathematical model of the proposed Q4γ element and the classical associative plasticity model are implemented into a FORTRAN calculation code to account for small strain elasto-plastic problems. A nonlinear elasto-plastic behavior of bending plates under mechanical loading with different boundary conditions and different aspect ratios were studied. Both problems involve square plates subjected to a uniformly distributed load of magnitude 1 KN/m^2. The material is considered as elasto-plastic (where the material is considered elastic perfectly plastic and the von Mises model is adopted) with: $L = 1.0$; $h = 0.5$ and 0.01 ; $E = 10.92$ GPa; $\nu = 0.3$ and Yield stress $\sigma_Y = 1600$ MPa. Because of the symmetry, only a quarter of the plate is modeled using 4×4 mesh as shown in Fig. 3.
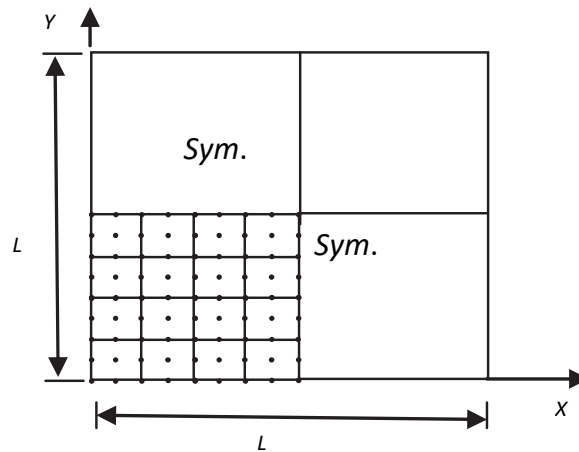

Figure 3: Square plate, its finite element models.

Simply supported square plate

In the first elasto-plastic example, a simply supported square plate subjected to uniformly distributed load is considered. The results are presented in Figs. 4 and 5 and shows the load-deflection curves with respect to the maximum deflection when nondimensional incremental load intensity is $q.L^2/M_0$ (M_0 fully plastic moment $M_0 = \sigma_Y.L.h^2/4$). This figure also shows a comparison among the Hétérosis finite element solution obtained by Owen and Hinton [1]. The central displacements, for comparison of plates with different thicknesses ($h = 0.01$ and 0.5), have been normalized as: $W.D/M_0.L^2$ where $D = \left(E.h^3\right)/\left(12(1-\nu^2)\right)$ is the flexural rigidity of the plate. The results obtained from the present 4-node element Q4γ agree well with those obtained from the 9-node Hétérosis element reported in Reference [1]. From the observation of the figures, it is possible to conclude that the accuracies of the present 4-node new element globally close

to that of the 9-node Hétérosis element. Fig. 6 shows the progression of the plastic regions at different levels of loading. From this figure, first yielding is observed at the four corners of the plate and then at the center, and the plastic regions extend along the diagonals.
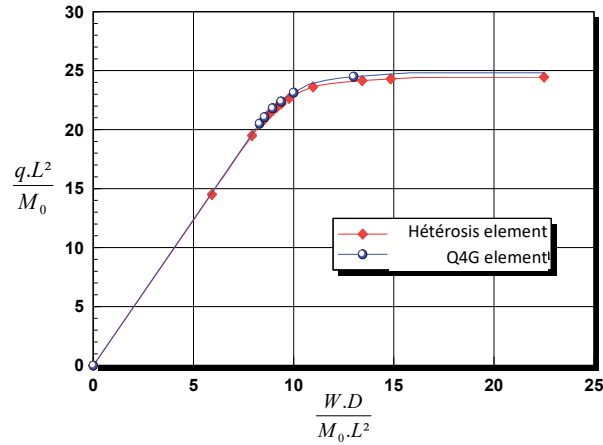

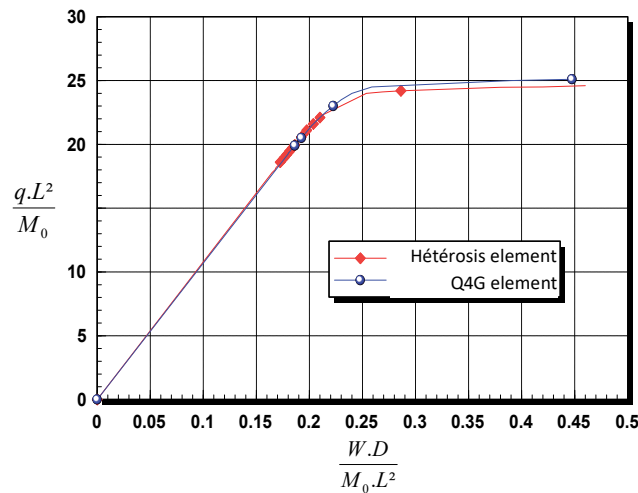Figure 4: Load-deflection curves for simply supported square plate ($h = 0.01$).

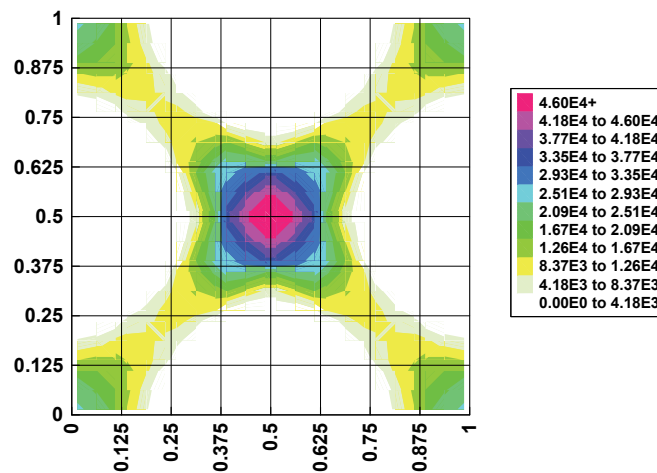Figure 5: Load-deflection curves for simply supported square plate ($h = 0.5$).

Figure 6: Progression of Yield regions for simply supported square plate ($h = 0.01$).

Clamped square plate

A clamped square plate subjected to uniform load is analyzed. Fig. 7 shows the load-deflection curve with respect to maximum deflection with ($h = 0.01$). The present 4-node element Q4γ results are compared with those of Hétérosis finite element reported by Owen and Hinton [1]. From this figure, it is observed that the results provided by the present four-node Q4γ element are in a good agreement with those of the converged results of the 9-node Hétérosis element given in [1]. The progression of the yield regions at different levels of loading is summarized in Fig. 8. In this case, the first yielding of the plate occurs at the middle of the four edges, and the plastic regions extend along these edges until the center of the plate yields.
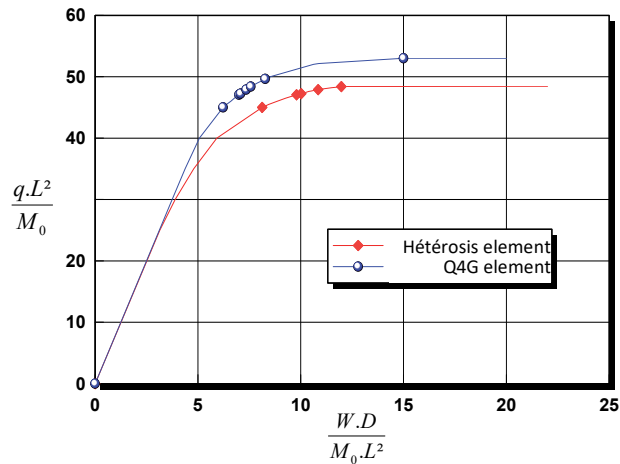

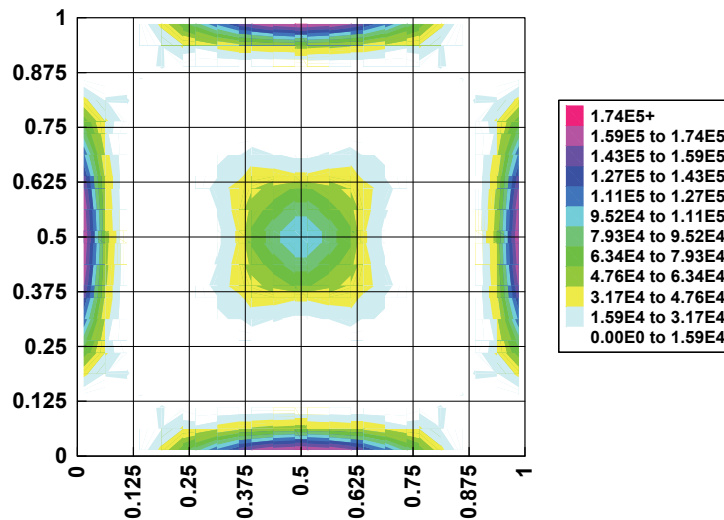Figure 7: Load-deflection curves for clamped square plate ($h = 0.01$).

Figure 8: Progression of Yield regions for clamped square plate ($h = 0.01$).

CONCLUSIONS

In this work, a finite element method for analyzing the problem of elasto-plastic bending of a square plate is presented. The previously proposed four-node Q4γ element with transverse energy of shearing for thick bending plates is extended to account for nonlinear elasto-plastic problems. The adopted constitutive model is the classical associative Von-Mises plasticity model where the modified Newton-Raphson scheme has been implemented for solving the nonlinear numerical system. To perform numerical tests, various examples within the nonlinear context are used to assess the accuracy against currently existing well-performed elements. The present element shows reliability and

robustness when compared with some reference elements from the literature. The elasto-plastic results obtained by four-node Q4γ element for the square bending plates with various boundary conditions can be treated with acceptable accuracy compared with those obtained by 9-node Hétérosis element. For the displacement field and for the plastic zone the Q4γ solutions are very similar to the Hétérosis solutions.

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