A unified rule to estimate multiaxial elastoplastic notch stresses and strains under in-phase proportional loadings

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ABSTRACT. Several methods can be used to estimate elastoplastic (EP) notch-tip stresses and strains from linear elastic calculations, providing EP stress and strain concentration factors. For uniaxial load histories, Neuber's and Glinka's rules are perhaps the most used. For non-proportional multiaxial histories, such corrections require incremental plasticity calculations to correlate stresses and strains at the notch root, a quite challenging task. However, for in-phase proportional multiaxial histories, where the principal directions do not change and the load path in a stress diagram follows a straight line, approximate methods can be used without requiring an incremental approach. Most of these methods are based on Neuber's rule, so they usually result in conservative predictions, especially in plane strain-dominated cases associated with sharp notches. In this work, a Unified Notch Rule (UNR) is proposed for uniaxial and in-phase proportional multiaxial histories. The UNR can reproduce Neuber's or Glinka's rules, interpolate their notch-tip behaviors, or even extrapolate them for notches with increased constraint. Moreover, the UNR also allows a non-zero normal stress perpendicular to the free-surface. The proposed method predictions are compared with elastoplastic Finite Element calculations on notched shafts.

KEYWORDS. Multiaxial fatigue; In-phase proportional loadings; Unified Notch Rule; Finite Element.

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INTRODUCTION

Direction-sensitive materials like most metallic alloys tend to initiate a single dominant microcrack under fatigue loadings. Under multiaxial loading conditions this behavior tends to be well modeled by critical-plane fatigue-damage models, which search for the material plane at the critical point where the corresponding accumulated damage parameter is maximized.

To calculate elastoplastic (EP) strains from a given multiaxial stress history, it is usually necessary to adopt an incremental plasticity formulation, which integrates non-linear differential equations to obtain the stress-strain behavior [1]. In the presence of notches, a much simpler approach is to perform a single linear elastic (LE) Finite Element (FE) calculation on the entire piece for a static unit value of each applied loading. The resulting values at the notch root are called pseudo-stresses and pseudo-strains, which are fictitious quantities calculated using the theory of elasticity at the critical point of the piece, while assuming that the material follows Hooke's law [2]. These pseudo values are represented here with a "~" symbol on top of each variable.

Under in-phase proportional loadings, approximate models to obtain the stress and the strain concentration factors K_σ and K_ε can be used to avoid computationally-intensive incremental plasticity calculations. They provide notch corrections that try to correlate pseudo and notch-tip values using a scalar parameter such as the Mises equivalent stress. The main EP notch models for in-phase proportional histories are the constant ratio [3], Hoffmann-Seeger's [4-5], and Dowling's [6] models. These models require some variable definitions, namely:

- $\tilde{\sigma}_i$ and $\tilde{\varepsilon}_i$ *:* pseudo principal stresses and strains at the notch tip, where $i = 1, 2, 3$.
- σ_i and ε_i*:* actual elastoplastic principal stresses and strains at the notch tip.
- λ_2 and λ_3*:* biaxiality ratios between the principal stresses, $\lambda_2 \equiv \sigma_2/\sigma_1$ and $\lambda_3 \equiv \sigma_3/\sigma_1$, both assumed between -1 and 1.
- ϕ_2 and ϕ_3*:* biaxiality ratios between principal strains, where $\phi_2 \equiv \varepsilon_2/\varepsilon_1$ and $\phi_3 \equiv \varepsilon_3/\varepsilon_1$, also assumed between -1 and 1; and
- $\bar{\nu}$ *:* effective Poisson ratio, with $\nu < \bar{\nu} \leq 0.5$ in the EP case, where ν is the (LE) Poisson ratio.

Dowling's model [6] assumes that the principal stresses σ_1 and σ_2 act on the free surface of the critical point (thus $\sigma_3 = 0$), but it considers that both λ_2 and ϕ_2 are constant, estimating them from the pseudo-stresses and pseudo-strains:

$$\lambda_2 = \frac{\sigma_2}{\sigma_1} \cong \frac{\tilde{\sigma}_2}{\tilde{\sigma}_1} \cong \frac{\phi_2 + \nu}{1 + \phi_2 \nu}, \quad \phi_2 = \frac{\varepsilon_2}{\varepsilon_1} \cong \frac{\tilde{\varepsilon}_2}{\tilde{\varepsilon}_1} \cong \frac{\lambda_2 - \nu}{1 - \lambda_2 \nu} \tag{1}$$

The model then directly correlates σ_1 and ε_1 using effective Ramberg-Osgood parameters E^* and H_c^**:*

$$\varepsilon_1 = \frac{\sigma_1}{E^*} + \left[\frac{\sigma_1}{H_c^*}\right]^{1/h_c} \tag{2}$$

$$E^* = E \cdot \left(\frac{1 + \phi_2 \nu}{1 - \nu^2}\right), \quad H_c^* = H_c \cdot \frac{(1 - \lambda_2 + \lambda_2^2)^{(h_c - 1)/2}}{(1 - \lambda_2 / 2)^{h_c}} \tag{3}$$

In notched components, assuming that the principal directions of the EP stresses and pseudo-stresses are equal, a reasonable supposition, then a variation of Neuber's rule [7] could be used to calculate the EP notch-tip σ_1 (and then ε_1) from the pseudo $\tilde{\sigma}_1$:

$$\tilde{\sigma}_1 \cdot \left[\frac{\tilde{\sigma}_1}{E^*}\right] = \tilde{\sigma}_1 \cdot \tilde{\varepsilon}_1 = \sigma_1 \cdot \varepsilon_1 = \sigma_1 \cdot \left[\frac{\sigma_1}{E^*} + \left(\frac{\sigma_1}{H_c^*}\right)^{1/h_c}\right] \text{ (Dowling)} \tag{4}$$

The above equation does not require a plastic term on the left hand side, because the pseudo-stresses and pseudo-strains are, by definition, LE. Finally, the other notch-tip EP principal stresses and strains are then obtained from σ_1 and ε_1:

$$\begin{cases} \sigma_2 = \lambda_2 \sigma_1, \quad \sigma_3 = 0 \\ \varepsilon_2 = \phi_2 \varepsilon_1, \quad \varepsilon_3 = -\bar{\nu} \varepsilon_1 \dfrac{1 + \lambda_2}{1 - \lambda_2 \bar{\nu}}, \quad \bar{\nu} = 0.5 - (0.5 - \nu) \dfrac{\sigma_1}{E^* \varepsilon_1} \end{cases} \tag{5}$$

THE UNIAXIAL UNIFIED NOTCH RULE (UNR)

Noting that Glinka's rule [8] usually underestimates while Neuber's rule [7] overestimates notch-tip stresses and strains, when compared to experimental results and FE analyses, a unified incremental rule (UNR) has been proposed by Ye et al. in [9], which returns values in-between them. For a monotonic uniaxial loading in the x direction, it states that

$$\sigma_x d\varepsilon_x \cdot (1+\alpha_{ED}) + \varepsilon_x d\sigma_x \cdot (1-\alpha_{ED}) = \tilde{\sigma}_x d\tilde{\varepsilon}_x + \tilde{\varepsilon}_x d\tilde{\sigma}_x \tag{6}$$

where $0 \le \alpha_{ED} \le 1$ was called the energy dissipation coefficient, assumed in [9] as a material parameter, estimated based on an energy argument as $\alpha_{ED} \cong (1-2h_c)/(1-h_c)$, where h_c is the cyclic exponent of Ramberg-Osgood's equation. However, α_{ED} might depend not only on the material but also on the notch geometry and constraint factor. This coefficient α_{ED} can also be regarded as a fitting parameter if experimental data or reliable EP FE analyses are available for its calibration.

To extend the UNR rule to multiaxial problems, a deviatoric version of Eq. 6 is proposed in this work:

$$s_x de_x \cdot (\alpha_U) + e_x ds_x \cdot (2-\alpha_U) = \tilde{s}_x d\tilde{e}_x + \tilde{e}_x d\tilde{s}_x \tag{7}$$

where $s_x \equiv (2\sigma_x - \sigma_y - \sigma_z)/3$ and $e_x \equiv (2\varepsilon_x - \varepsilon_y - \varepsilon_z)/3$ are the deviatoric stresses and strains in the x direction at the notch tip, while $\alpha_{Uc} \equiv (1+\alpha_{ED})$ is called the notch constraint factor, with values $1 \le \alpha_U \le 2$ to interpolate the Incremental Neuber rule [10-11] (for which $\alpha_U = 1$) and a similar Incremental Glinka rule (which has $\alpha_U = 2$).

As the deviatoric stresses s_x, s_y and s_z are linearly-dependent, since $s_x + s_y + s_z = 0$, it is possible to reduce the deviatoric stress and strain space dimensions using:

$$s_1 \equiv \sigma_x - \frac{\sigma_y + \sigma_z}{2} = \frac{3}{2} s_x \; , \quad s_2 \equiv \frac{\sigma_y - \sigma_z}{2}\sqrt{3} = \frac{s_y - s_z}{2}\sqrt{3} \tag{8}$$

$$e_1 \equiv \varepsilon_x - \frac{\varepsilon_y + \varepsilon_z}{2} = \frac{3}{2} e_x \; , \quad e_2 \equiv \frac{\varepsilon_y - \varepsilon_z}{2}\sqrt{3} = \frac{e_y - e_z}{2}\sqrt{3} \tag{9}$$

Assuming that Eq. 7 is valid for the transformed deviatoric stresses and strains from Eqs. 8 and 9, then

$$\begin{cases} (\alpha_U)\cdot s_1 de_1 + (2-\alpha_U)\cdot e_1 ds_1 = \tilde{s}_1 d\tilde{e}_1 + \tilde{e}_1 d\tilde{s}_1 \\ (\alpha_U)\cdot s_2 de_2 + (2-\alpha_U)\cdot e_2 ds_2 = \tilde{s}_2 d\tilde{e}_2 + \tilde{e}_2 d\tilde{s}_2 \end{cases} \tag{10}$$

where, as explained before, the symbol "~" is used for pseudo-values calculated from LE analyses.

The Unified Notch Rule (UNR) proposed in this work can then be obtained from the integration of Eq. 10, which can be used for both uniaxial and in-phase proportional histories. For uniaxial histories, this integration results in the scalar UNR:

$$\tilde{\varepsilon}^2 = \frac{\sigma}{E}\cdot\left[\frac{\sigma}{E} + \bar{\alpha}_U \cdot \left(\frac{\sigma}{H_c}\right)^{1/h_c}\right], \quad \bar{\alpha}_U \equiv \frac{\alpha_U + h_c(2-\alpha_U)}{1+h_c} \text{ (UNR)} \tag{11}$$

where $\bar{\alpha}_U$ is the effective notch constraint factor. This equation can reproduce Neuber for $\alpha_U = 1$ and thus $\bar{\alpha}_U = 1$, or Glinka's rule for $\alpha_U = 2$ and thus $\bar{\alpha}_U = 2/(1+h_c)$.

Although conceptually different, α_U shares some similarities with Newman's constraint factor α [12], varying from 1.0 under plane stress conditions (where Neuber's rule is recommended) to usually more than 3.0 under plane strain. Thus, both α_U and Newman's α reflect increased stress-state constraint and associated plasticity decrease at the critical point, however using α_U at notch tips and Newman's α at crack tips.

THE MULTIAXIAL UNIFIED NOTCH RULE

The multiaxial version of the UNR assumes in-phase proportional loading under free-surface conditions, supposing $\tau_{xz} = \tau_{yz} = 0$, but allows the presence of a surface normal $\sigma_z \neq 0$, where the z axis is assumed perpendicular to the surface, and the x and y axes are aligned with the remaining principal directions, with x in the direction of the maximum absolute principal stress. Therefore, the principal notch tip stresses $\sigma_x \equiv \sigma_1$, $\sigma_y \equiv \sigma_2$, and $\sigma_z \equiv \sigma_3$ are assumed to satisfy $|\sigma_x| \geq |\sigma_y|$ and $|\sigma_x| \geq |\sigma_z|$ during the entire load history. The involved variables are the same as the ones defined before, in addition to an elastic and plastic separation of the strain biaxiality ratios, through:

- ϕ_{2el} and ϕ_{3el}: biaxiality ratios between principal elastic strains, where $\phi_{2el} \equiv \varepsilon_{2el}/\varepsilon_{1el}$ and $\phi_{3el} \equiv \varepsilon_{3el}/\varepsilon_{1el}$ are both assumed between -1 and 1; and
- ϕ_{2pl} and ϕ_{3pl}: same definition, but for plastic strains (for pressure-insensitive materials, where $\varepsilon_{1pl} + \varepsilon_{2pl} + \varepsilon_{3pl} = 0$, it follows that $1 + \phi_{2pl} + \phi_{3pl} = 0$ and thus $\phi_{2pl} + \phi_{3pl} = -1$).

Since the multiaxial loading history is assumed here to be proportional, the deviatoric stress increment is always parallel to the plastic straining direction, so the Prandtl-Reuss plastic flow rule [1] gives, for the normal deviatoric strain components,

$$\begin{bmatrix} de_1 \\ de_2 \end{bmatrix} = \begin{bmatrix} d\varepsilon_{xpl} - (d\varepsilon_{ypl} + d\varepsilon_{zpl})/2 \\ (d\varepsilon_{ypl} - d\varepsilon_{zpl}) \cdot \sqrt{3}/2 \end{bmatrix} = \frac{1}{P} \cdot \begin{bmatrix} d\sigma_x - (d\sigma_y + d\sigma_z)/2 \\ (d\sigma_y - d\sigma_z) \cdot \sqrt{3}/2 \end{bmatrix} \tag{12}$$

where P is called the generalized plastic modulus (proportional to the slope of the stress vs. plastic strain curve at the current stress state), and all shear increments are zero since x, y, and z are defined in the principal directions. Integrating the above equation using the plastic biaxiality ratio definitions, then

$$\int_0^{\varepsilon_{xpl}} d\varepsilon_{xpl} \cdot \begin{bmatrix} 1 - (\phi_{2pl} + \phi_{3pl})/2 \\ (\phi_{2pl} - \phi_{3pl}) \cdot \sqrt{3}/2 \end{bmatrix} = \int_0^{\sigma_x} \frac{1}{P} \cdot d\sigma_x \cdot \begin{bmatrix} 1 - (\lambda_2 + \lambda_3)/2 \\ (\lambda_2 - \lambda_3) \cdot \sqrt{3}/2 \end{bmatrix} \tag{13}$$

Neglecting the isotropic hardening transient, let's assume that the material follows Ramberg-Osgood with cyclic constant Hc and exponent hc. Moreover, assuming that this proportional loading is balanced, i.e. it does not cause ratcheting or mean stress relaxation, then a Mróz multi-surface hardening model can be adopted [1] (instead of the more general non-linear kinematic hardening models). To improve accuracy, let's adopt an infinite number of hardening surfaces, as discussed in [13], see Fig. 1. From the calibration of the Mróz model, the generalized plastic modulus $P = P_i$ for the hardening surface with radius r_i becomes

$$P_i = (2/3) \cdot h_c H_c \left(r_i/H_c\right)^{1-1/h_c} \tag{14}$$

Consider a monotonic proportional loading departing from the origin of the deviatoric stress space, as shown in Fig. 1, assuming x, y and z as principal directions. In this case, the radius r_i of the current active surface from the Mróz model is equal to the norm (and thus the Mises equivalent value) of the current stress state. Replacing the values of $P = P_i$ and r_i into Eq. 13, and using the plastic strain incompressibility condition $\phi_{2pl} + \phi_{3pl} = -1$, it follows that

$$\tilde{\sigma}_1 \cdot \left[\frac{\tilde{\sigma}_1}{E^*}\right] = \sigma_1 \cdot \left[\frac{\sigma_1}{E^*} + \bar{\alpha}_U \cdot \left(\frac{\sigma_1}{H_c^*}\right)^{1/h_c}\right] \tag{15}$$

$$E^* \equiv E/[1 - \nu \cdot (\lambda_2 + \lambda_3)],\ H_c^* \equiv H_c \cdot \frac{[1 - (\lambda_2 + \lambda_3) + (\lambda_2{}^2 + \lambda_3{}^2) - \lambda_2\lambda_3]^{(h_c-1)/2}}{[1 - (\lambda_2 + \lambda_3)/2]^{h_c}} \tag{16}$$

$$\begin{cases} \varepsilon_{1el} = \sigma_1 / E^*, \ \varepsilon_{1pl} = \left(\sigma_1 / H_c^*\right)^{1/h_c}, \ \varepsilon_1 = \varepsilon_{1el} + \varepsilon_{1pl} \\ \sigma_2 = \lambda_2 \sigma_1, \ \sigma_3 = \lambda_3 \sigma_1 \\ \varepsilon_2 = \phi_{2el} \cdot \varepsilon_{1el} + \phi_{2pl} \cdot \varepsilon_{1pl}, \ \ \varepsilon_3 = \phi_{3el} \cdot \varepsilon_{1el} + \phi_{3pl} \cdot \varepsilon_{1pl} \end{cases} \tag{17}$$

$$\phi_{2pl} = \frac{\lambda_2 - 0.5 \cdot (1 + \lambda_3)}{1 - 0.5 \cdot (\lambda_2 + \lambda_3)}, \quad \phi_{3pl} = \frac{\lambda_3 - 0.5 \cdot (1 + \lambda_2)}{1 - 0.5 \cdot (\lambda_2 + \lambda_3)} \tag{18}$$

$$\phi_{2el} = \frac{\lambda_2 - \nu \cdot (1 + \lambda_3)}{1 - \nu \cdot (\lambda_2 + \lambda_3)}, \quad \phi_{3el} = \frac{\lambda_3 - \nu \cdot (1 + \lambda_2)}{1 - \nu \cdot (\lambda_2 + \lambda_3)} \tag{19}$$

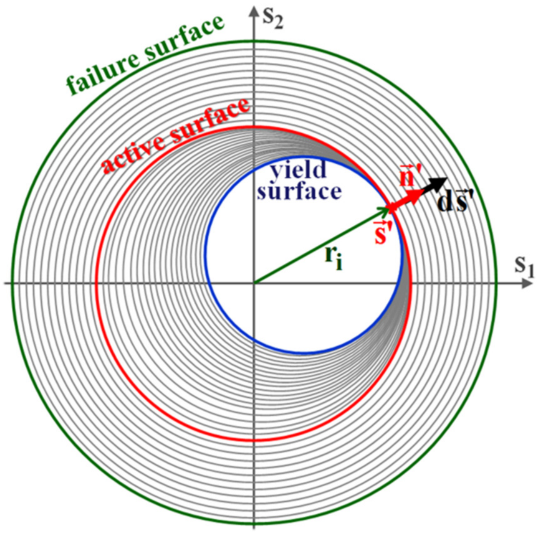

Figure 1: Mróz infinite-surface hardening model for a monotonic proportional loading.

Dowling's model for in-phase proportional loadings is a particular case of the more general in-phase proportional UNR, setting $\bar{\alpha}_U = 1$ (to reproduce Neuber's rule) and also $\lambda_3 = 0$ (free-surface with $\sigma_3 = 0$), assuming as well that $\phi_{2pl} = \phi_{2el}$ based on ν, and that $\phi_{3pl} = \phi_{3el}$ based on an effective Poisson ratio $\bar{\nu}$.

Both Dowling's and UNR models assume the nominal section (away from the notch) remains LE. In other words, they are valid even under general yielding of the net cross section, but they do not account for yielding of the gross cross section. To perform this correction, the pseudo principal stress $\tilde{\sigma}_1$ is represented as the product of a LE stress concentration factor K_t multiplied by a nominal stress σ_{n1}, i.e. $\sigma_{n1} \equiv \tilde{\sigma}_1 / K_t$, where σ_{n1} is assumed to follow Ramberg-Osgood, giving

$$K_t^2 \cdot \sigma_{n1} \cdot \left[\frac{\sigma_{n1}}{E^*} + \bar{\alpha}_U \cdot \left(\frac{\sigma_{n1}}{H_c^*} \right)^{1/h_c} \right] = \sigma_1 \cdot \left[\frac{\sigma_1}{E^*} + \bar{\alpha}_U \cdot \left(\frac{\sigma_1}{H_c^*} \right)^{1/h_c} \right] \tag{20}$$

VERIFICATION OF THE UNR WITH ELASTOPLASTIC FINITE ELEMENTS

The proposed UNR and Dowling's classic notch rule are checked against elastoplastic (EP) Finite Element (FE) calculations, for multiaxial in-phase proportional tension-torsion problems. The comparison is based on the calculation of the peak EP stresses and strains at a notched solid shaft with largest diameter *50.8* mm and a semi-circular U-notch with a sharp radius *0.254* mm. The shaft is assumed made of a heat-treated 1070 steel with Young modulus

$E = 210GPa$, Poisson ratio $\nu = 0.3$, and Ramberg-Osgood parameters $H_c = 1736$MPa and $h_c = 0.199$, using data reported in [14]. For the EP calculations all simulations were carried out using ANSYS software, using the SOLID186 3D elements with 20 nodes each and 3 degrees of freedom per node. The model used for the EP FE simulations is illustrated in Fig. 2, and its notch tip grid is depicted in Fig. 3.

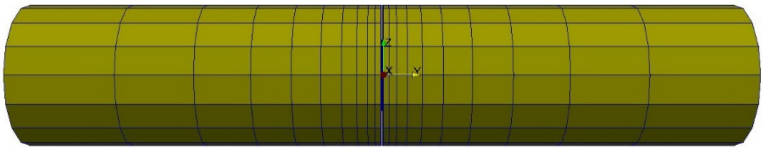

Figure 2: The FE model for the sharply notched shaft.

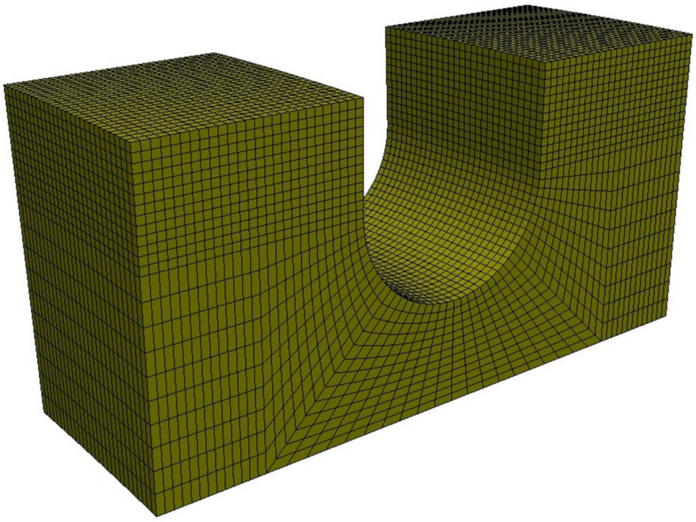

Figure 3: The grid around the notch tip.

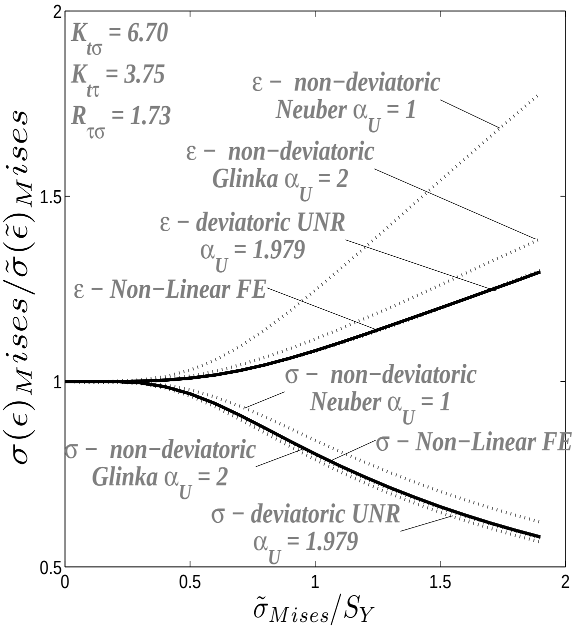

Figure 4: Predicted and FE-calculated EP strain and stress.

Fig. 4 shows the ratio between the EP and the pseudo Mises strain $\varepsilon_{Mises}/\tilde{\varepsilon}_{Mises}$ and stress $\sigma_{Mises}/\tilde{\sigma}_{Mises}$ for a particular case of a tension-torsion multiaxial loading assuming a proportionality stress ratio between equivalent shear and normal nominal stresses $R_{\tau\sigma} = \sqrt{3}$ .The LE stress concentration factors for this sharply notched shaft are $K_{t\sigma} = 6.70$ for normal stresses and $K_{t\tau} = 3.75$ for pure shear stresses. The two solid lines for both strains and stresses show the numerically obtained EP results obtained from the FE simulations, which are overestimated by both Neuber's ($\alpha_U = 1$, the rule adopted in Dowling's multiaxial model) and Glinka's rules ($\alpha_U = 2$). The third dashed lines also for both strains and stresses are the better estimates obtained from the proposed UNR, calibrated for $\alpha_U = 1.979$. These results show that the proposed rule is able to improve significantly the traditional estimates from Neuber's and Glinka's models, in particular for notched components with high transversal constraints around the notch tip, the case of the studied sharply notched shaft. Finally, as expected, all predictions tend to the $\varepsilon_{Mises}/\tilde{\varepsilon}_{Mises} = \sigma_{Mises}/\tilde{\sigma}_{Mises} = 1$ under low stresses.

CONCLUSIONS

In this work, a Unified Notch Rule (UNR) was proposed to predict elastoplastic stresses and strains at a notch roots from linear elastic calculations, for uniaxial and in-phase proportional multiaxial histories. The UNR can interpolate between Neuber's and Glinka's rules using its α_U parameter calibration to account for the magnitude of the transversal constraint around the notch tip, or even to extrapolate them to better reproduce increased constraint effects around sharp notch tips. Moreover, the proposed UNR allows biaxiality ratios $\lambda_3 \equiv \sigma_3/\sigma_1 \neq 0$, an improvement over Dowling's model, which always assume $\lambda_3 = 0$. Even though the derivation of the UNR model assumed an integration for a monotonic load, the resulting equations could be applied to cyclic loadings, as long as they are also in-phase and proportional, and the appropriate biaxiality ratios can be assumed constant.

REFERENCES

[1] Castro, J.T.P., Meggiolaro, M.A., Fatigue Design Techniques: Vol. II - Low-Cycle and Multiaxial Fatigue. CreateSpace Publishing Company, Scotts Valley, CA, USA, (2016).
[2] Köettgen, V.B., Barkey, M.E., Socie, D.F., Fatigue Fract Eng Mat Struct, 18 (1995) 981-1006.
[3] Socie, D.F., Marquis, G.B., Multiaxial Fatigue, SAE, (2000).
[4] Hoffmann, M., Seeger, T., A generalized method for estimating multiaxial elastic plastic notch stresses and strains, Part I: Theory, J Eng Mat Tech, 107 (1985) 250-254.
[5] Hoffmann, M., Seeger, T., A Generalized Method for Estimating Multiaxial Elastic-Plastic Notch Stresses and Strains. Part 2: Application and General Discussion, J Eng Mat Tech, 107 (1985) 255-260.
[6] Dowling, N.E., Brose, W.R., Wilson, W.K., In: Fatigue Under Complex Loading: Analysis and Experiments, AE-6, SAE, (1977).
[7] Neuber, H., Theory of stress concentration for shear-strained prismatic bodies with arbitrary nonlinear stress- strain law, J Appl Mech, 28 (1961) 544-551.
[8] Molski, K., Glinka, G., A method of elastic-plastic stress and strain calculation at a notch root, Mat Sci Eng, 50 (1981) 93-100.
[9] Ye, D., Hertel, O., Vormwald, M., A unified expression of elastic-plastic notch stress-strain calculation in bodies subjected to multiaxial cyclic loading, Int J Solids Struct, 45 (2009) 6177-6189.
[10] Kujawski, D., On energy interpretations of the Neuber's rule, Theor Appl Fract Mech, 73 (2014) 91-96.
[11] Ince, A., Glinka, G., A numerical method for elasto-plastic notch-root stress–strain analysis, J Strain Analysis Eng Design, 48 (2013) 229-224.
[12] Newman, J.C. Jr, Crews, J.H., Bigelow, C.A., Dawicke, D.S. ASTM STP 1244 (1995) 21-42.
[13] Chu, C.C., The analysis of multiaxial cyclic problems with an anisotropic hardening model, Int J Solids Struct, 23 (1987) 567-579.
[14] Jiang, Y.R., Sehitoglu, H., Materials Engineering – Mechanical Behavior Report #413, University of Illinois, Urbana, Illinois, (1992).