Crack paths and the linear elastic analysis of cracked bodies

L. P. Pook
21 Woodside Road, Sevenoaks TN13 3HF, United Kingdom

ABSTRACT. The linear elastic analysis of cracked bodies is a Twentieth Century development, with the first papers appearing in 1907, but it was not until the introduction of the stress intensity factor concept in 1957 that widespread application to practical engineering problems became possible. Linear elastic fracture mechanics (LEFM) developed rapidly in the 1960s, with application to brittle fracture and fatigue crack growth. The first application of finite elements to the calculation of stress intensity factors for two dimensional cases was in 1969. Finite element analysis had a significant influence on the development of LEFM. Corner point singularities were investigated in the late 1970s. It was soon found that the existence of corner point effects made interpretation of calculated stress intensity factors difficult and their validity questionable. In 1998 it was shown that the assumption that crack growth is in mode I leads to geometric constraints on permissible fatigue crack paths. Current open questions are. The need for a new field parameter, probably a singularity, to describe the stresses at surfaces. How best to allow for the influence of corner point singularities in three dimensional numerical predictions of fatigue crack paths. Adequate description of fatigue crack path stability.

KEYWORDS. Linear elastic analysis; Stress intensity factors; Corner point singularities; Crack paths; Finite element analysis.

INTRODUCTION

The complete solution of a crack growth problem includes determination of the crack path. This review is a brief survey of the development of ideas on the linear elastic analyis of cracked bodies that are relevant to crack path determination. It is based on the author's personal involvement over more than 50 years. The review is restricted to linear elastic, homogeneous, isotropic materials, with any yielding confined to a small region at a crack tip. The first relevant papers had been published 50 years earlier, but in the late 1950s theoretical understanding of crack growth due to fatigue and static loadings was limited. The situation changed dramatically in the 1960s with the development of fracture mechanics, which is the applied mechanics of crack growth [1]. It was realised that linear elastic fracture mechanics, based on linear elastic analyses, sufficed for the solution of many practical engineering problems. By the mid 1970s practical applications of fracture mechanics were well established. In considering practical aspects of linear elastic fracture mechanics, scales of observation need to be taken into account since the scale chosen can make a considerable difference to the appearance of an object in general, and a crack in particular [2]. Scales of observation of 0.1 mm and above are usually described as macroscopic. The linear elastic concept of stress intensity factor describes the linear elastic stress field in the vicinity of a crack tip, and is a singularity. Stress intensity factors may be used to characterise the mechanical properties of cracked test pieces in just the same way that stresses are used to characterise the mechanical properties of uncracked test pieces. The conventional notation for the position of a point relative to the crack tip, and for the stresses at this point, is shown in Fig. 1. A point on the crack tip is the origin of the Cartesian coordinate system and the z axis lies along the crack tip. Displacements of points within the cracked body when the body is loaded are u, v, w in the x, y, z directions. A fundamental fracture mechanics concept is that of crack tip surface displacement [3]. There are three possible modes of crack tip surface displacement, as shown in Fig. 2. These are: mode I where opposing crack surfaces move directly apart in directions parallel to the y axis; mode II where crack surfaces move over each other in the xz plane in directions parallel to the x axis, that is perpendicular to the crack tip; and mode III where crack surfaces move over each other in the xz plane in directions parallel to the z axis, that is parallel to the crack tip. By superimposing the three modes, it is possible to describe the most general case of crack tip surface displacement. Where more precise description is needed Volterra distorsioni can be used [1]. The term mixed mode means that at least one mode, other than mode I, is present. It is matter of observation that, when viewed on a macroscopic scale, and under essentially elastic conditions, cracks in metals tend to grow in mode I, so attention is largely confined to this mode. Crack surfaces are assumed to be smooth, although on a microscopic scale they are generally very irregular.

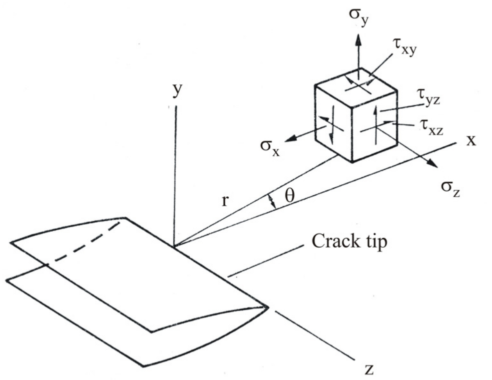

Figure 1: Notation for crack tip stress field.

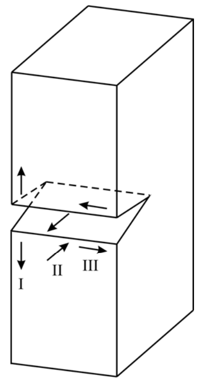

Figure 2: Notation for modes of crack tip surface displacement.

STRESS ANALYSIS OF CRACKS

The philosophical basis for a fracture mechanics analysis is that for crack growth to take place two conditions need to be satisfied [4]. Firstly, sufficient energy needs to be available to operate a crack growth mechanism (thermodynamic criterion). secondly crack tip stresses must be high enough to operate the mechanism (stress criterion).

Stress intensity factors

The key concept of stress intensity factor, for mode I and for mode II, arises from a two dimensional linear elastic analysis for a straight crack. Mode III is not possible in two dimensions, so for this mode a quasi two dimensional anti plane analysis is used. In 1957 it was shown by Williams [5] that the stress field in the vicinity of a crack tip is dominated by the leading term of a series expansion of the stress field. This leading term is the stress intensity factor, *K*, which is a singularity. A particular type of elastic crack tip stress field is associated with each mode of crack tip surface displacement, and subscripts I, II and III are used to denote mode. Other terms are non singular. Individual stress components are proportional to $K/\sqrt{r}$ where *r* is the distance from the crack tip. Displacements are non singular and proportional to $K\sqrt{r}$. Once *K* is known, stress and displacement fields in the vicinity of the crack tip are given by standard equations. For a mode I crack, the second term is a stress parallel to the crack, usually called the *T*-stress [6]. The *T*-stress is used in some linear elastic fracture mechanics analyses. The thermodynamic criterion implies that the energy needed to create new crack surfaces must be considered. Stress intensity factors automatically satisfy the stress criterion. In 1957 Irwin [7] showed that they also satisfy the thermodynamic criterion. In 1967 finite element stress analysis software was becoming generally available [8]. Finite elements were first used for the calculation of mode I stress intensity factors in two dimensional geometries in 1969 [9].
A stress intensity factor provides a reasonable description of the crack tip stress field in a *K* – dominated region at the crack tip, radius $r \approx a/10$ where *a* is crack length, as shown in Fig. 3. An apparent objection to the use of the stress intensity factor approach is the violation, in the immediate vicinity of the crack tip, of the initial linear elastic assumptions, in that strains and displacements are not small. However, as noted by Williams in 1962 [10], if the assumptions are

violated only in a small core region, radius << *r*, than the general character of the *K*-dominated region is, to a reasonable approximation, unaffected. Similarly, by this small scale argument, small scale non-linear effects due to crack tip yielding, microstructural irregularities, internal stresses, irregularities in the crack surface, the actual fracture process, etc, may be regarded as within the core region. In the early 1960s there was scepticism about the validity and utility of stress intensity factors [8]. However, collaborative theoretical and experimental work in the late 1960s helped to establish confidence [11, 12]) and by 1974 their use for the solution of practical engineering problems was well established [13], nearly two decades after Williams' discovery of stress intensity factors.

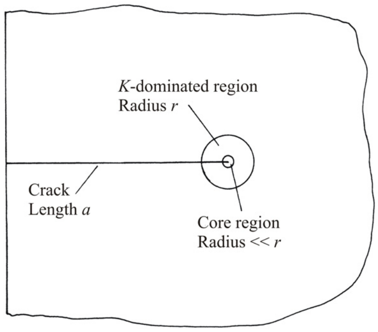

Figure 3: *K*-dominated and core regions at a crack tip.

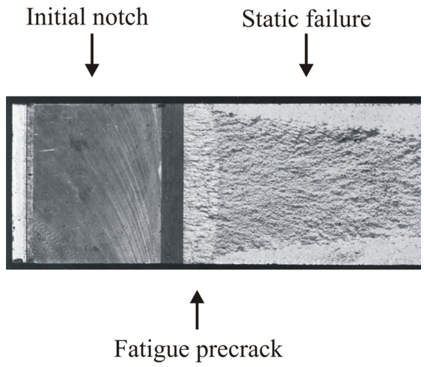

Figure 4: Fracture surface of 19 mm thick aluminium alloy fracture toughness test piece.

Corner point singularities

In three dimensional geometries, the derivation of stress intensity factors makes the implicit assumption that a crack front is continuous. This is correct for cracks growing from internal defects. It is not correct in the vicinity of a corner point where a crack front intersects a free surface. The nature of the crack tip singularity changes in the vicinity of a corner point, and these corner point singularities, sometimes called vertex singularities, are an important source of three dimensional effects [1]. Many of the cracks observed in service, and in laboratory tests, have corner points, for example Fig. 4. Kinematic considerations for a crack surface intersection angle, γ, (Fig. 5) of 90°, and a crack front intersection angle, β, (Fig. 6) of 90° show immediately that mode II and mode III crack tip surface displacements cannot exist in isolation in the vicinity of a corner point. The presence of one of these modes always induces the other, sometimes called a coupled mode, and indicated by a superscript c. Thus mode II induces mode III^c and mode III induces mode II^c. This has been demonstrated experimentally by foam plastic models [14]. Hence, in the vicinity of a corner point there are two modes of crack tip surface displacement. One is the symmetric mode, which is mode I. The other is the antisymmetric mode, a combination of modes II and III. For corner point singularities, the polar coordinates in Fig. 1 are replaced by spherical coordinates (r, θ, ϕ) with origin at the corner point. The angle ϕ is measured from the crack front.

There do not appear to be any exact analytic solutions for corner point singularities. An approximate solution was obtained in 1977 by Benthem [15] for the restricted case of a quarter infinite crack in a half space. A more general approximate solution, using essentially the same approach, was obtained in 1979 by Bažant and Estenssoro [16]. In their analysis they assumed that all three modes of crack tip surface displacement are proportional to $r^{\lambda}F(\theta, \phi)$. They then calculated λ numerically for a range of situations. The stress intensity measure, K_λ, may be used to characterise corner point singularities, where λ can be regarded as a parameter defining the corner point singularity. It follows from the initial assumption that stresses are proportional to K_λ/r^λ and displacements to $K_\lambda r^{1-\lambda}$, where r is measured from the crack front. Hence, stress and displacement plots are straight lines when plotted using logarithmic scales, and such plots obtained from finite element analyses can be used to determine values of λ. For a crack surface intersection angle, γ, of 90° and a crack front intersection angle, β, of 90° there are two modes of stress intensity measure corresponding to the modes of crack tip surface displacement. For the symmetric mode the stress intensity measure is $K_{\lambda S}$, and for the antisymmetric mode it is $K_{\lambda A}$. For Poisson's ratio, $\nu = 0.3$ $\lambda = 0.452$ for the symmetric mode and 0.598 for the antisymmetric mode. Recent highly accurate finite element results for discs and plates under anti plane loading [17, 18, 19] do not confirm the latter value. Bažant and Estenssoro's analysis shows that, for a crack surface intersection angle, γ, of 90°, λ is a function of Poisson's ratio, ν, and the crack front intersection angle. At a critical crack front intersection angle, β_c, $\lambda = 0.5$ and stress intensity factors are recovered. $K_{\lambda S}$ becomes K_I, and $K_{\lambda A}$ a combination of K_{II} and K_{III}. For a growing crack the crack front shape adjusts itself such that the crack front intersection angle tends to β_c. For the symmetric mode and $\nu = 0.3$, $\beta_c = 100.4°$ which, as predicted, is approximately the crack front intersection angle in Fig. 4. For the antisymmetric mode and $\nu = 0.3$, $\beta_c = 67.0°$. Crack front intersection angles of about this value have been observed [20].

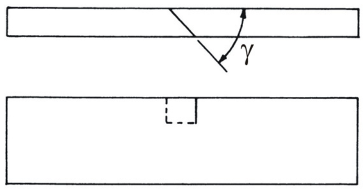

Figure 5: Angle crack test piece, definition of crack surface intersection angle, γ.

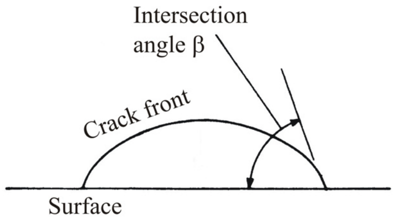

Figure 6: Definition of crack front intersection angle, β.

CRACK PATHS

Crack front line tension

A crack has some analogies with a crystal dislocation. In particular, the elastic stress fields associated with a crack front and with a dislocation are both singularities. The associated energy means that a dislocation has a line tension, which controls its shape under an applied stress field. Similarly [21], a crack front has a line tension which controls its shape, but with the important difference that a crack can grow, but in general cannot contract. At a corner point the corner point singularity provides a point force which balances the line tension in a direction corresponding to the crack front intersection angle. In consequence on a macroscopic scale, a crack front is smooth, and is usually curved as in Fig. 4. Any initial sharp corners rapidly disappear [22]. Further in some circumstances, a growing fatigue crack tends to a particular stable shape [23, 24].

Initial direction of crack growth

In general, criteria are needed for the formation of a branch crack, its initial direction, and once formed, whether it will grow [25]. It is important to distinguish between criteria for formation of a branch crack and criteria for its growth. Depending on circumstances, either can dominate behaviour [26, 27]. In two dimensions only modes I and II are possible, the crack front is a point and the crack is a line. The direction of a mode I branch crack is given by the angle θ (Fig. 7). Related experimental work is usually carried out on plates or sheets of constant thickness, which are regarded as quasi two-dimensional. Numerous criteria have been proposed for the initial direction of crack growth [28]. Predicted initial branch crack directions are not sharply defined, so minor deviations from isotropy may have a significant influence on initial branch crack direction. In mixed modes I and II fatigue testing measurement of the initial branch crack direction is of interest. However, the irregularity of actual fracture surfaces and, for curved branch crack paths, the need to estimate a tangent at the start of the branch crack make it difficult to measure crack directions accurately. The directions given in Reference [29] are only repeatable to within 2 or 3 degrees. Experimental results show that under fatigue loading initial crack directions vary widely [30].

The most widely used criterion for the initial crack growth direction, θ, is the Maximum Tensile Stress (MTS) criterion proposed by Erdogan and Sih in 1963 [31]. They considered a circle around the crack tip and took the direction of crack growth to be in the direction of the maximum tangential stress. This is a principal stress so the shear stress is zero, and leads to $K_I \sin\theta = K_{II}(3\cos\theta - 1)$ $(70.5\ldots° \leq \theta \leq -70.5\ldots°)$ where K_I and K_{II} are the modes I and II initial crack stress intensity factors. An alternative approach is to find the value of θ for which the mode I branch crack stress intensity factor, k_I, has its maximum value and the mode II stress intensity, k_{II}, is zero. Calculation of k_I and k_{II} by comparison of stress field components leads to the same result [26, 27], so the two approaches are equivalent.

Several criteria have been proposed for the initial direction of crack growth in the general three dimensional case of mixed modes I, II and III loading [28]. In 2001 Schöllmann et al [32] proposed a new criterion, which assumes that crack growth from a point on the crack front is perpendicular to the maximum principal stress. In two dimensions this is equivalent to the MTS criterion. A complication is that, in the presence of mode III on the initial crack, a mode I branch crack only intersects the initial crack front at one point [26, 27, 28]. What happens when a fatigue loading is applied in the presence of mode III on the initial crack is illustrated by the fracture surface of the mild steel angle notch test piece shown in Fig. 8 [33] (cf Fig. 5). The rough area is static fracture where the test piece was broken open for examination. The expected tendency to mode I fatigue crack growth was observed on two distinct scales. On a scale of 1 mm crack the crack growth surface was smooth, crack fronts were approximately straight, and initially crack growth was mixed mode. As the fatigue crack grew the crack front rotated and crack growth eventually became mode I. Rotation continued until the crack surface intersection angle was 90°, and the crack front was curved. On a smaller scale of 0.1 mm the tendency to mode I fatigue crack growth results in the production of what is known as a twist crack [34]. A twist crack consists of narrow mode I facets usually connected by irregular, predominantly mode III cliffs. The mode I facets gradually merged and the crack growth surface became smooth on this scale.

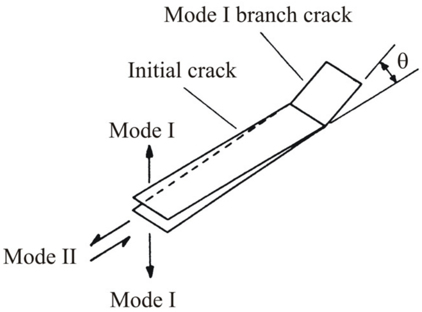

Figure 7: Quasi two dimensional mixed modes I and II initial crack with a small mode I branch crack, crack growth angle, *θ*.

Figure 8: Twist crack fracture surface of mild steel angle notch test piece, fatigue loading.

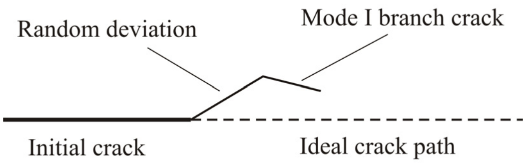

Figure 9: Directionally stable mode 1 fatigue crack growth.

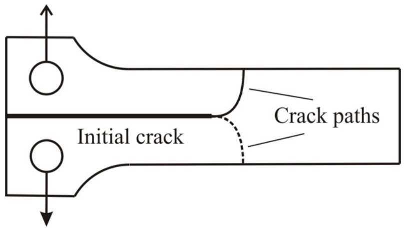

Figure 10: Mode I fatigue crack paths in a double cantilever beam specimen.

Crack path stability

A fatigue crack growing in mode I is not necessarily directionally stable [28]. Two dimensional linear elastic analyses are normally used in the consideration of crack path stability, with related experimental work on sheets or plates of constant thickness. A fatigue crack growing in mode I may be regarded as directionally stable if, after a small random deviation, perhaps due to microstructural irregularity, it returns to its expected ideal crack path, as shown in Fig. 9. A directionally unstable crack does not return to the ideal path following a small random deviation; its path is a random walk. These ideas are not easily given rigorous mathematical form. For example, arbitrary limits have to be placed on what is regarded as returning to the ideal crack path. The direction stability of a mode I crack was analysed by Cotterell [35] in 1966. He found that if the *T*-stress is compressive and there is a small random crack deviation, then the direction of mode I crack growth is towards the initial crack line. Repeated random deviations mean that the crack follows a zigzag path about the ideal crack path, which is an attractor [28]. When the *T*-stress is tensile a crack is directionally unstable, and following a small random deviation, it does not return to the ideal crack path. A fatigue crack growing in a double cantilever beam specimen is directionally unstable in this sense. Typical crack path behaviour is shown schematically in Fig. 10. An initial random deviation can be either above or below the centre line, so there are two possible crack paths. These are shown as solid and dashed lines in the figure. The directional stability of a crack may change as it propagates, and a stable mode I crack may follow a curved path. Cracks tend to be attracted by boundaries and are increasingly stable as a boundary is approached.

The biaxiality ratio, *B*, is a nondimensional function of the *T*-stress. It is given by [6]:

$$B = \frac{T\sqrt{\pi a}}{K_{\mathrm{I}}} \tag{1}$$

where a is the crack length (half crack length for an internal crack), and K_I is the mode I stress intensity factor. It is sometimes found that cracks are directionally stable even when the T-stress is tensile (or B is positive). In particular, the T-stress is tensile for the widely used compact tension test piece (Fig. 11). This is specified in some fracture mechanics based mode I testing standards [12], and in practice cracks are usually directionally stable. An alternative approach, proposed by Pook [36] in 1998, is to consider the direct stresses, on the crack line, and near the crack tip. That due to the T-stress is simply T. The stress, σ_x, due to the Mode I stress intensity factor, on the crack line and ahead of the crack, is given by:

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}} \tag{2}$$

where r is the distance from the crack tip. The T-stress ratio, T_R, may now be defined as the ratio of the T-stress to σ_x, given by Eq. (2), at some characteristic value of r, r_{ch}. Provided that r_{ch} is small T_R is a point criterion which is within the K-dominated region. Since the T-stress criterion is based on the idea of random crack path perturbations due to microstructural irregularities, r_{ch} should be of the same order of size as microstructural features. Taking r_{ch} = 0.0159... mm leads, using MN-m units, to the convenient expression

$$T_R = \frac{0.01B}{\sqrt{\pi a}} \tag{3}$$

which implies that there is a size effect. For a particular material, there is a critical value of T_R, T_{Rc}, below which a fatigue crack path is directionally stable [28]. For fatigue tests on biaxially loaded Waspaloy sheets T_{Rc} = 0.022 and for static tests on biaxially loaded PMMA sheets T_{Rc} = 0.013. For a compact tension test piece with W = 50 mm T_{Rc} < 0.22. It is unusual to carry out tests using compact tension test pieces with W < 50 mm so this value is consistent with the Waspaloy result.

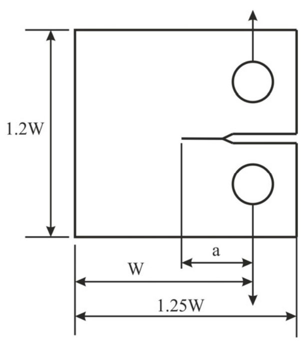

Figure 11: Compact tension test piece.

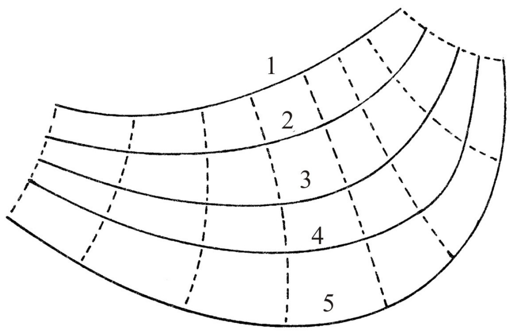

Figure 12: Crack fronts (solid lines) and trajectories (dashed lines) on a smoothly curved crack growth surface.

Geometric constraints on mode I crack paths

In 1998 it was shown by Pook [37] that the assumption that crack growth is in mode I leads to geometric constraints on permissible fatigue crack paths. Consider a general, smoothly curved, crack growth surface, that is one that is differentiable at least three times. Assume that on this surface there is an ordered family of smooth curves representing successive positions of the crack front, as shown by the numbered solid lines in Fig. 12. Crack growth must be in mode I for the surface to remain smooth. Crack front line tension ensures that crack fronts are smooth curves. Corresponding crack growth directions are given by the family of orthogonal trajectories. These are also smooth curves, and are shown by the dashed lines on the figure. In differential geometry terms these two families are an orthogonal net [38]. In general, there are two principal directions on the surface where the curvature is either a maximum or a minimum. For a given smooth surface the trajectories of these principal directions have zero torsion (twist in space), and are a unique orthogonal net [38]. Geometric considerations show that, for mode I crack growth this orthogonal net defines permissible families of crack fronts and crack trajectories. On a crack growth surface either family of principal curvature trajectories can define a family of crack front positions. Further, since a family of crack front positions is ordered, crack growth can be in either direction along the trajectories. Hence, for a given crack growth surface there are four possible crack front families. There is an important exception to the geometric constraint on permissible mode I crack front families. On a plane the curvature is zero in all directions, so principal directions cannot be defined, and there is no orthogonal net of principal curvature directions. Therefore, for a flat mode I crack there is no geometric constraint on permissible crack front families, and a wide range of crack front families is possible and, indeed, observed [39, 40].

Fatigue crack path prediction

In general, fatigue crack paths are difficult to predict. In practice, fatigue crack paths in structures are often determined by large scale structural tests [23, 41]. Theoretical predictions are carried out numerically using finite element analysis or boundary element analysis. The validity of the algorithms used is checked by comparison of predictions with experimental data. Two dimensional mode I fatigue crack path predictions have been carried out by a number of authors, using the same general scheme including Portela [42] in 1993. Calculations are carried out using small increments of straight crack growth. The size of the increment is calculated using an appropriate fatigue crack growth equation. The direction taken by each increment is selected using the criterion that the increment is pure mode I. Agreement between numerical predictions and experimental data obtained using thin sheets is variable [21]. The availability of increasingly powerful computers mean that two dimensional predictions have now largely been superseded by three dimensional predictions.

For the special case of a flat mode I initial crack in a symmetrical configuration crack mode I fatigue crack growth is in the same plane as the initial crack, and in a direction perpendicular to the crack front. By definition the crack growth angle, θ, (Fig. 7) is zero. It is implicitly assumed that the crack is directionally stable. Increments of straight fatigue crack growth are calculated for a set of points along the crack front. The tips of these increments, connected by straight lines or a fitted curve, define a new crack front. In 1989 Smith and Cooper [22] carried out some finite element calculations for a flat irregular mode I crack and showed that irregularities rapidly disappeared. In 2005 Dhondt [43] carried out some finite element calculations for a flat mode I fatigue crack path in a model containing geometric discontinuities. The results showed that the crack front intersection angle has a trend towards 100°, which is close to the theoretical critical crack front intersection angle. When the crack crosses a geometric discontinuity there is a large, abrupt change in the crack front intersection angle, but it then converges to the critical crack front intersection angle.

Schöllmann et al's criterion [32] for the initial direction of crack growth in the general three dimensional case of mixed modes I, II and III loading is evaluated on a curved cylinder with centre line along the crack front. The ends of the cylinder are perpendicular to its axis to avoid meshing difficulties. The resulting equations have to be evaluated numerically. The criterion also includes a method of calculating an equivalent mode I stress intensity factor for use in the calculation of fatigue crack growth increments. In the presence of mode III the overall crack path increment is mixed mode. This is compatible with the idea of a twist crack where, on a scale of 1 mm, the crack path is mixed mode (Fig. 8). A fully automatic finite element program incorporating the criterion was used to analyse fatigue crack growth in the frame of a hydraulic press that had failed in service [44]. The results showed good agreement with the observed crack path in the structure. However, there was considerable scatter in the predicted crack front intersection angles. A detailed description of a fully automatic finite element program for the calculation of fatigue crack paths in the general three dimensional case was given by Dhondt [45] in 2014. A particular difficulty is meshing in the vicinity of a corner point when the crack front is not perpendicular to the surface. This is avoided by an approximation in which the surface is locally perpendicular to the crack front. The crack growth direction is determined iteratively using a principal stress criterion. The results obtained are justified by comparison with experimental data. Differences arise where mode III crack surface displacements lead to friction between opposing crack surfaces and to the development of twist cracks. Neglecting friction leads to a conservative result.

PLANE STRAIN FRACTURE TOUGHNESS TESTING

Interest in assessing the fracture toughness, or resistance to brittle fracture, of metals goes back to 1822 when Tredgold commented on the assessment of cast iron [1]. By 1962 numerous empirical tests had been developed in order to determine whether a steel was brittle or ductile [1]. The best known test is the Charpy impact test using notched test pieces. It is still in use, but it does not provide quantitative fracture toughness data that can be used in design [13]. The situation changed dramatically in the 1960s with the development of standards for plane strain fracture toughness (K_{Ic}) testing. This was based on two empirical observations [1, 46]. First, that the sharpest possible machined notch may not adequately represent a crack. Second, that in a test piece of constant thickness, the fracture toughness is a function of test piece thickness. However, if the test piece thickness is above a minimum value, which depends on the material, then the fracture toughness is a minimum, and it is a material constant [46]. This minimum value became known as K_{Ic}. The minimum thickness depends on the yield stress, σ_Y, of the material, and is given by $2.5(K_I/\sigma_Y)^2$. The factor 2.5 is empirical and is a compromise. It was also found that consistent values are obtained for K_{Ic} if it is calculated from the load necessary to cause a significant amount of crack growth, defined as 2 per cent of the initial crack length. Only two dimensional stress intensity factor solutions were available for test pieces of constant thickness, such as the compact tension test piece, shown in Fig. 11 so a limit had to be based on permissible crack front curvature, and an appropriate through thickness average crack length defined. The empirical basis of some aspects of K_{Ic} testing meant that extensive development work was needed to develop workable standards [8, 12]. It also means that standards contain very detailed requirements to ensure reproducibility between laboratories.

DISCUSSION AND CONCLUSIONS

The linear elastic analysis of cracked bodies, usually known as linear elastic fracture mechanics (LEFM), is a Twentieth Century development. The first theoretical analysis appeared in 1907, but it was not until the introduction of the stress intensity factor concept in 1957 that widespread application to practical engineering problems became possible. LEFM developed rapidly in the 1960s, with application to brittle fracture and fatigue crack growth, and the development of a standard for the plane strain fracture toughness testing of metals. The first application of finite elements to the calculation of stress intensity factors for two dimensional cases was in 1969. The 1970s were a period of consolidation. LEFM was increasingly used in failure analysis. Analyses were assisted by the publication of stress intensity factor handbooks.

Corner point singularities were investigated in the late 1970s. A key finding was that a corner point modes II and III cannot exist in isolation. Hence, the presence of one of these modes always induces the other, and is sometimes called a coupled mode. By 1986 the increasing power of mainframe computers meant that three dimensional finite element analysis of cracked bodies became feasible. Finite element analysis of cracked three dimensional configurations, which started in the late 1980s, confirmed the existence of coupled modes. It was soon found that the existence of corner point effects made interpretation of calculated stress intensity factors difficult, and their validity questionable. In a recent investigation a coupled mode generated by anti-plane loading of a straight through-the-thickness crack in linear elastic discs and plates was studied using accurate three dimensional finite element models. The results make it clear that Bažant and Estenssoro's analysis of corner point singularities is incomplete. An open question is the need for a new field parameter, probably a singularity, to describe the stresses at surfaces. Finite element analysis had a significant influence on this aspect of the development of LEFM.

Numerical two dimensional mode I crack path predictions were carried out in the early 1990s. Agreement between theoretical predictions and experimental data obtained using thin sheets is variable. The availability of increasingly powerful computers means that two dimensional predictions have now largely been superseded by three dimensional predictions. Numerical three dimensional predictions of fatigue crack paths between 2003 and 2014 showed good agreement with observed fatigue crack paths. However, how best to allow for the influence of corner point singularities in three dimensional numerical predictions of fatigue crack paths is an open question. Crack path prediction would not have been possible without the use of finite element analysis or boundary element analysis.

In 1998 it was shown that the assumption that crack growth is in mode I leads to geometric constraints on permissible fatigue crack paths. Line tension ensures that crack fronts are smooth curves. A necessary condition for crack growth to be in mode I is that the crack growth surface must be smooth. On a smoothly curved crack growth surface there is an ordered family of smooth curves representing successive positions of the crack front and a family of crack growth trajectories. These are an orthogonal net passing along directions of maximum and minimum curvature. On a plane the curvature is equal in all directions, there is no orthogonal net, and therefore no geometric constraint on permissible crack front families.

The biaxiality ratio, a nondimensional function of the T-stress, is sometimes used as a crack path stability criterion, but it is not satisfactory. An alternative approach suggested in 1998, is to use the T-stress ratio, T_R, which is a point criterion based on the T-stress. For a particular material, there appears to be a critical value of T_R, T_{Rc}, below which a fatigue crack path is directionally stable. However, adequate description of fatigue crack path stability is an open question.

Requirements for a valid K_{Ic} fracture toughness test in the latest standard [47] and are essentially unchanged from those in the first standard published in 1970 [48]. In other words, 1960s technology is still being used. This would not matter if the latest standards were satisfactory in practice. However, in 2012 Schijve pointed out that the transferability of fracture toughness test data to practical situations was restricted so structural testing was sometimes needed [49]. Also in 2012

Kotousov et al [50] remarked that current standards ignore three dimensional effects. Development of an adequate plane strain fracture toughness testing standard is an open question.

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