



## Numerical measures of the degree of non-proportionality of multiaxial fatigue loadings

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**ABSTRACT.** The influence of the non-proportional loadings on the fatigue life depends on the material ductility. Ductile materials react with a shortening of lifetime compared to proportional loading conditions. For a semi-ductile material there is almost no difference between proportional and non-proportional loadings with respect to the fatigue life. Brittle materials show an increase of the lifetime under non-proportional loadings. If fatigue life assessment is performed using stress-based hypotheses, it is a rather difficult task to take into account material ductility correctly, especially the fatigue life reduction as displayed by ductile materials.

Most stress-based hypotheses will compute a longer fatigue life under non-proportional loading conditions. There are also hypotheses, which already include quantitative evaluation of the non-proportionality (e.g. EESH, SSCH and MWCM). Anyway in order to improve assessment for ductile materials, some sort of numerical measure for the degree of non-proportionality of the fatigue loading is required. A number of measures of this kind (or non-proportionality factors) were proposed in the literature and are discussed here:

- the factor used in EESH is a quotient of stress amplitudes integrals,
- the factor according to Gaier, which works with a discrete stress tensor values in a scaled stress space,
- the factor according to Kanazawa, which makes use of plane-based stress values,
- the factor used in MWCM, which exploits stress values in the plane with the highest shear stress amplitude, a new non-proportionality factor, which is based on the correlation between individual stress tensor components, is proposed.

General requirements imposed on the non-proportionality factors are discussed and each of the factors is evaluated with respect to these requirements. Also application with the stress-based hypotheses is discussed and illustrated using the experimental data for aluminum and magnesium welded joints under constant and variable amplitude loadings.

**KEYWORDS.** Multiaxial fatigue; Non-proportional loadings; Non-proportionality measures; Welded joints; Stress-based criteria.

### INTRODUCTION

**M**aterials and structural components subjected to multiaxial fatigue loadings can react differently depending on the non-proportionality of the loading. Some materials (brittle material state) exhibit a longer fatigue lifetime under non-proportional loadings compared to the proportional ones. Other materials (ductile material state)



react with a shorter fatigue life to non-proportional loadings [1]. This behavior is also exhibited by welded joints of steel, aluminum and magnesium [2-6]. If fatigue life evaluation is performed using stress-based hypotheses, taking non-proportionality into account in a correct manner is a rather difficult task. Most of the stress-based hypotheses compute a longer fatigue life under non-proportional loadings unless there is some explicit non-proportionality measure ‘built in’ into the hypothesis. Examples of such hypotheses are ESSH [7], SSCH [3, 5], MWCM [8].

For other stress-based hypotheses an external non-proportionality factor can be introduced and so improve fatigue assessment under non-proportional loadings.

Such non-proportionality factors similarly to the stress-based hypotheses can be subdivided into two large families: the integral and the critical plane based factors. To the first family belong the non-proportionality factors introduced by Bishop [9], Gaier [10], Sonsino [7] and the new non-proportionality factor based on the statistical correlation of stress components [11], which is presented in the current paper. These factors make use of integral stress values. The non-proportionality factors by Susmel (MWCM) [8] and Kanazawa [12] are critical plane based values. For factors of this type a critical plane (in both cases plane with the highest shear stress amplitude) is determined and then certain stress relations, which belong to the critical plane are computed and used as a non-proportionality measure. The non-proportionality factors introduced by Gaier and Bishop are an improvement over the factor introduced by Chu et al. [13], which does depend on the choice of coordinate system.

Most of the non-proportionality factors under discussion attend values between 0 and 1 also evaluation of variable amplitude loadings can be performed using all the factors. Other important features are independence on material properties (only the time-dependent stress tensor path is used for evaluation of the non-proportionality) as well as independence on the choice of the coordinate system.

As an illustration for the application of the non-proportionality factors the Findley criterion is used to evaluate fatigue life of aluminum and magnesium welded joints.

## NON-PROPORTIONALITY FACTORS OF INTEGRAL TYPE

This class of non-proportionality factors involves some form of integral evaluation of stresses in different planes. Depending on the situation those can be planes orthogonal to the specimen surface (plane stress state) or arbitrary oriented planes (general stress state). Computation of these factors to variable amplitude loadings can require a sufficiently fine sampling of the time-dependent loading path and/or a sophisticated integration procedure. With the only exception all factors discussed in this section yield values between 0 and 1, with 0 being the value for the proportional loading and 1 being ascribed to the ‘most non-proportional case’. Exact definition of the ‘most non-proportional case’ is different for each factor.

### *Non-proportionality factors due to Bishop and Gaier*

Bishop proposed two measures of the non-proportionality or, more precisely, out-of-phase measures [9]. Non-proportionality obtained due to the presence of mean stresses in the case of in-phase time-dependent parts is neglected. The measures are based on the notion of the principal axes of a tensor path.

A stress tensor

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{pmatrix} \quad (1)$$

is mapped to the vector

$$\boldsymbol{\sigma} \mapsto \mathbf{x} = (\sigma_x, \sqrt{2}\tau_{xy}, \sigma_y, \sqrt{2}\tau_{xz}, \sqrt{2}\tau_{yz}, \sigma_z) \quad (2)$$

The scaling factor  $\sqrt{2}$  for the shear stresses results in the identity:

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = \boldsymbol{\sigma}_1 : \boldsymbol{\sigma}_2 \quad (3)$$

for any two vectors  $\mathbf{x}_1$  and  $\mathbf{x}_2$ , on which the tensors  $\boldsymbol{\sigma}_1$  and  $\boldsymbol{\sigma}_2$  respectively are mapped. The identity (3) in turn results in the independence of the choice of the coordinate system for the values introduced below. A tensor path is defined if



the tensor  $\sigma(t)$  is time-dependent with  $t \in [a, b]$  for finite real values  $a$  and  $b$ . The respective time-dependent vector  $\mathbf{x}(t)$  will be also called tensor path.

First of all the length of the tensor path  $\mathbf{x}(t)$  computes to

$$L = \int_a^b \|\dot{\mathbf{x}}(t)\| dt$$

and the mean value (or centroid) of the time-dependent vector  $\mathbf{x}(t)$  is given by the integral

$$\bar{\mathbf{x}} = \frac{1}{L} \int_a^b \mathbf{x}(t) \|\dot{\mathbf{x}}(t)\| dt$$

With  $\mathbf{y}(t) = \mathbf{x}(t) - \bar{\mathbf{x}}$  the rectangular moment-of-inertia tensor can be defined:

$$\mathbf{I} = \int_a^b \mathbf{y}(t) \otimes \mathbf{y}(t) \|\dot{\mathbf{y}}(t)\| dt$$

or componentwise

$$I_{ij} = \int_a^b y_i y_j \|\dot{\mathbf{y}}(t)\| dt; \quad i, j = 1, 2, \dots, 6$$

Spherical moment of inertia tensor is given by

$$\mathbf{I}^S = -(\mathbf{I} - tr(\mathbf{I})\mathbf{I}^{id})$$

where  $tr(\mathbf{I})$  is the trace of the tensor  $\mathbf{I}$  and  $\mathbf{I}^{id}$  is the identity tensor. If the tensor path is given in the form of discrete sample points  $\sigma^1 = \sigma(t_1)$ ,  $\sigma^2 = \sigma(t_2)$ , ...,  $\sigma^{N+1} = \sigma(t_{N+1})$  or the respective vector values  $\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^{N+1}$ , the path is assumed to be linear between the samples and the following approximation formulas hold:

$$L = \sum_{k=1}^N L^k \tag{4}$$

$$\bar{\mathbf{x}} = \frac{1}{2L} \sum_{k=1}^N L^k (\mathbf{x}^{k+1} + \mathbf{x}^k) \tag{5}$$

$$I_{ij} = \frac{1}{6} \sum_{k=1}^N L^k (y_i^k y_j^k + y_i^{k+1} y_j^{k+1} + (y_i^k + y_i^{k+1})(y_j^k + y_j^{k+1})) \tag{6}$$

with  $L^k = \mathbf{x}^{k+1} - \mathbf{x}^k$  and  $\mathbf{y}^k = \mathbf{x}^k - \bar{\mathbf{x}}$ . The formulas (4)-(6) can be used in order to compute the tensor  $\mathbf{I}$  for a variable amplitude loading.

The rectangular moment-of-inertia tensor  $\mathbf{I}$  is symmetric and hence has 6 real eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_6$  and its respective eigendirections  $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_6$  comprise an orthonormal coordinate system. These eigendirections are the principle directions of the tensor path  $\mathbf{x}(t)$ . Two non-proportionality measures can be defined now:

$$m_1 = \frac{\max_{t \in [a, b]} \mathbf{p}_1 \cdot \mathbf{x}(t)}{\max_{t \in [a, b]} \{\mathbf{p}_i \cdot \mathbf{x}(t), i = 2, \dots, 6\}} \tag{7}$$

$$m_2 = \sqrt{\frac{\lambda_2}{\lambda_1}} \tag{8}$$

The value  $m_1$  lies between 0 and  $+\infty$ , if it is close to zero, the tensor path  $\mathbf{x}(t)$  is nearly in-phase (Fig. 1), if the tensor path contains narrow long spikes (Fig. 2), it results in  $m_1 > 1$ . The value  $m_2$  lies between 0 and 1, however values near 0 can be attained if the tensor path  $\mathbf{x}(t)$  have some short significantly non-proportional parts (narrow spikes).

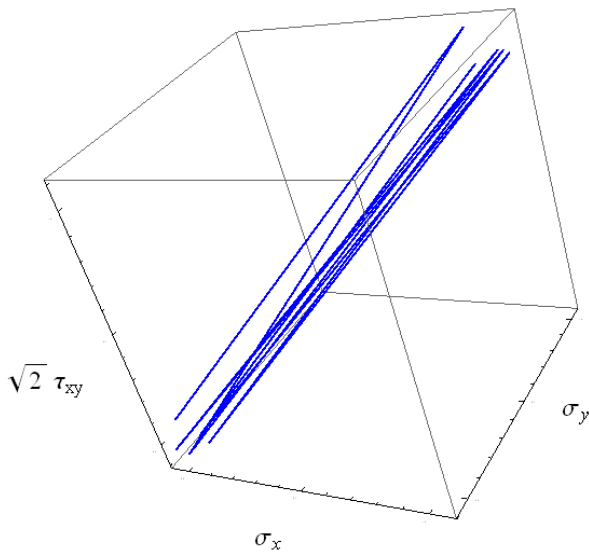


Figure 1: A nearly proportional time-dependent tensor path (plane stress):  $m_1, m_2, d \approx 0$ .

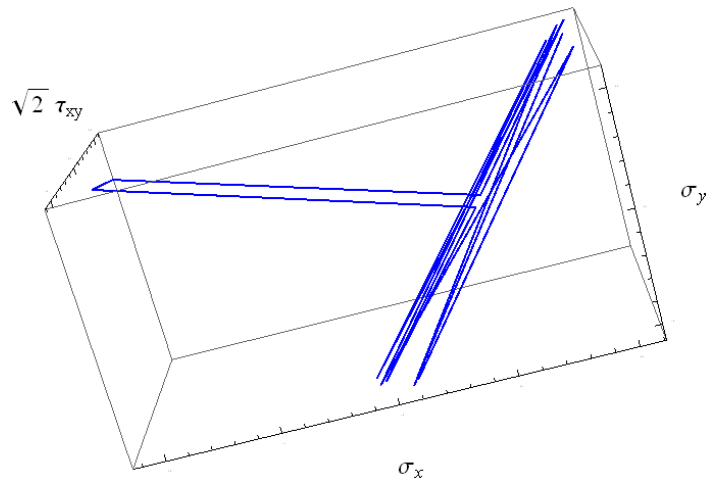


Figure 2: A non-proportional time-dependent tensor path (plane stress). Non-proportionality due to a narrow spike:  $m_1 > 1, m_2, d \approx 0$ .

Gaier et al. [10] employ an idea, which is very similar to Bishop's. For a tensor path given by discrete samples  $\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^{N+1}$  they construct a tensor-of-inertia, which is similar to the tensor  $\mathbf{I}^S$  defined above. The individual components  $I_{ij}^S$  for  $i, j = 1, \dots, 6$  are given by the formulas:

$$I_{ii}^S = \sum_{k=1}^{N+1} \sum_{\substack{j=1, \\ j \neq i}}^6 (x_j^k)^2 \quad (9)$$

$$I_{ij}^S = - \sum_{k=1}^{N+1} x_i^k \cdot x_j^k, i \neq j \quad (10)$$

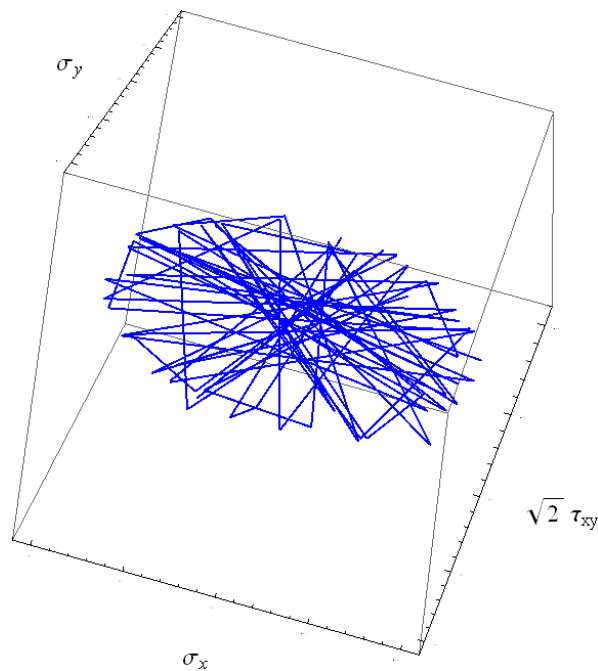


Figure 3: A non-proportional time-dependent tensor path (plane stress):  $m_1, m_2, d \approx 1$ .



If  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_6$  are the eigenvalues of the tensor  $\mathbf{I}^s$ , components of which are given by the Eq. (9) and (10), then the non-proportionality measure of stresses is given by:

$$d_{NP} = \sqrt{\frac{\lambda_6}{\lambda_5}} \tag{11}$$

The formula (11) is the ratio of the second largest axis to the largest axis of the moment-of-inertia ellipsoid of the points  $\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^{N+1}$ , if a unit mass is assigned to each of them. The measure  $d_{NP}$  attends values between 0 and 1 and similar to the non-proportionality measure  $m_2$  (Eq. (8)) does not account for short but highly non-proportional pieces in a mostly proportional tensor path. A major difference between the values  $m_1, m_2$  and the value  $d_{NP}$  is, that no subtraction of the mean stresses  $\bar{\mathbf{x}}$  occurs as  $d_{NP}$  is computed. All the non-proportionality measures  $m_1, m_2$  and  $d_{NP}$  yield the value 1 e.g. for the loading of the type:

$$\sigma_x = \sqrt{2}A \cos \omega t, \tau_{xy} = A \sin \omega t$$

with all other stress components being equal to 0 (cf. Fig. 3). Also all three factors can be computed for a plane stress state with vector path  $\mathbf{x}(t)$  restricted to three components only. In this case the moment-of-inertia tensors will have the  $3 \times 3$  instead of the  $6 \times 6$  shape.

#### *Correlation-based non-proportionality factor*

This factor was briefly introduced in [11], here it is presented and discussed in a greater detail.

The non-proportionality of the loading implies, that time-dependent normal and shear stress components in different planes are essentially of different shapes. Statistical correlation  $Cor(f, g)$  of two functions  $f, g$  provides a numerical measure for how ‘similar’ the shapes of the functions are. In particular  $Cor(f, g) = \pm 1$  means that the functions differ by a scaling factor and  $Cor(f, g) = 0$  means that  $f$  and  $g$  are ‘fully uncorrelated’. In order to give the precise definition for  $Cor(f, g)$  some notions from statistics are required.

In what follows the functions  $f$  and  $g$  are considered to be dependent on a time variable  $t$ , which belongs to a finite interval  $[a, b]$ . The mean value (or expected value)  $E(f)$  of  $f$  and its variance  $Var(f)$  on the interval  $[a, b]$  are given by [14]:

$$E(f) = \frac{1}{b-a} \int_a^b f(t) dt \tag{12}$$

$$Var(f) = E((f - E(f))^2) = \frac{1}{b-a} \int_a^b (f(t) - E(f))^2 dt = E(f^2) - E^2(f) \tag{13}$$

The covariance  $Cov(f, g)$  of two functions is defined by:

$$\begin{aligned} Cov(f, g) &= E((f - E(f)) \cdot (g - E(g))) = E(f \cdot g) - E(f) \cdot E(g) \\ &= \frac{1}{b-a} \int_a^b f(t) \cdot g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt \end{aligned} \tag{14}$$

Using the notions introduced above, the correlation coefficient of  $f$  and  $g$  can be defined:

$$Cor(f, g) = \frac{Cov(f, g)}{\sqrt{Var(f) \cdot Var(g)}} \tag{15}$$

This value is bounded  $-1 \leq Cor(f, g) \leq 1$  also if  $f$  and  $g$  are trigonometric functions of the same frequency with zero mean and a phase-shift  $\delta$  between them, i.e.:



$$f(t) = A_1 \sin t, \quad g(t) = A_2 \sin(t + \delta)$$

with  $A_1$  and  $A_2$  being the respective amplitudes, the correlation coefficient  $Cor(f, g)$  over the full period  $[0, 2\pi]$  computes to:

$$Cor(f, g) = \cos \delta$$

That is the correlation coefficient can be used to generalize the notion of phase-shift for arbitrary functions. Now let the stress tensor be defined by the matrix

$$S = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

with respect to a coordinate system  $xy\zeta$  and

$$Q = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix}$$

a coordinate transformation matrix into the coordinate system  $x'y'\zeta'$ , such that the stress tensor admits with respect to this coordinate system the form:

$$S' = Q^T S Q$$

Consider two components  $\sigma'_{ij}$  and  $\sigma'_{kl}$  of  $S'$ , they can be expressed in terms of the components of  $S$  and  $Q$  as follows:

$$\sigma'_{ij} = \sum_{m=1}^3 \sum_{n=1}^3 q_{im} q_{jn} \sigma_{mn}$$

$$\sigma'_{kl} = \sum_{m=1}^3 \sum_{n=1}^3 q_{km} q_{ln} \sigma_{mn}$$

If the components of  $S$  are functions and their covariances can be calculated, then the covariance of  $\sigma'_{ij}$  and  $\sigma'_{kl}$  computes to

$$Cov(\sigma'_{ij}, \sigma'_{kl}) = Cov\left(\sum_{m=1}^3 \sum_{n=1}^3 q_{im} q_{jn} \sigma_{mn}, \sum_{m=1}^3 \sum_{n=1}^3 q_{km} q_{ln} \sigma_{mn}\right) = \sum_{m=1}^3 \sum_{n=1}^3 \sum_{s=1}^3 \sum_{t=1}^3 q_{im} q_{jn} q_{ks} q_{lt} Cov(\sigma_{mn}, \sigma_{st}).$$

That is covariances of the components of  $S'$  can be computed, if the covariances of the components of  $S$  and the matrix  $Q$  are known. The same applies to the variances, since  $Var(\sigma'_{mn}) = Cov(\sigma'_{mn}, \sigma'_{mn})$ . Therefore the correlation coefficients of any two components of the stress tensor with respect to the coordinate system  $x'y'\zeta'$  can be expressed using the covariances of the components with respect to the coordinate system  $xy\zeta$  and the components of the coordinate transformation matrix. That is only six variances and 15 covariances of the components must be computed explicitly for a stress tensor representing a general stress state in order to be able to compute variances and covariances of the components with respect to any coordinate system. Similar considerations hold true of course for a plane stress state. Now consider a plane stress state given by the stress tensor

$$S = \begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{pmatrix} \tag{16}$$

with the time-dependent components  $\sigma_x, \sigma_y, \tau_{xy}$ . The tensor  $S$  defined above can be transformed (rotated) into another coordinate system by means of the transformation matrix

$$T = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

The transformed tensor computes to

$$S' = T^T S T = \begin{pmatrix} \sigma'_x & \tau'_{xy} \\ \tau'_{xy} & \sigma'_y \end{pmatrix}$$

with the components equal to

$$\sigma'_x = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\sigma'_y = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_y - \sigma_x}{2} \cos 2\theta - \tau_{xy} \sin 2\theta$$

$$\tau'_{xy} = \frac{\sigma_y - \sigma_x}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

Now a measure for non-proportionality of the time-dependent stress tensor  $S$ , which does not depend on a coordinate system, can be introduced. The mean of the square of the correlation  $Cor^2(\sigma'_x, \tau'_{xy})$  over all coordinate systems, i.e. over all angles  $\theta \in [0, \pi]$

$$M = \frac{1}{\pi} \int_0^\pi Cor^2(\sigma'_x, \tau'_{xy}) d\theta$$

provides such a measure. It is equal to 1 for a proportional time-dependent stress tensor and it is equal to 0 for a stress tensor of the form

$$S = \begin{pmatrix} \frac{\sigma_I + \sigma_{II}}{2} + \frac{\sigma_I - \sigma_{II}}{2} \sin \omega t & \frac{\sigma_I - \sigma_{II}}{2} \cos \omega t \\ \frac{\sigma_I - \sigma_{II}}{2} \cos \omega t & \frac{\sigma_I + \sigma_{II}}{2} - \frac{\sigma_I - \sigma_{II}}{2} \sin \omega t \end{pmatrix}$$

A tensor of this shape results in unchanging principal values  $\sigma_I, \sigma_{II}$ , but constantly rotating principle directions (Fig. 4). That means especially, that each plane experiences the same shear stress amplitude.

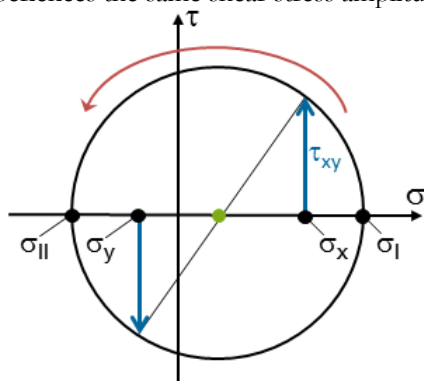


Figure 4: Mohr-circle, which does not change its shape, but rotates constantly:  $f_{NP} = 1$ .

Finally the non-proportionality factor, which can be used in further fatigue life evaluations, is defined using the value  $M$  :

$$f_{NP} = 1 - M = 1 - \frac{1}{\pi} \int_0^\pi Cor^2(\sigma'_x, \tau'_{xy}) d\theta. \tag{17}$$



In the Eq. (17) the value  $Cor^2(\sigma'_x, \tau'_{xy})$  is used instead of  $Cor(\sigma'_x, \tau'_{xy})$  since this way the sign (direction) of the phase-shift is ignored. For a general stress state the value  $M$  is computed as follows:

$$M = \min_{\xi \in [0, \pi)} \frac{1}{2\pi^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} Cor^2(\sigma'_x, \tau'_{xy}) d\theta d\psi$$

and the non-proportionality factor analogous to Eq. (16):

$$f_{NP} = 1 - M = \min_{\xi \in [0, \pi)} \frac{1}{2\pi^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} Cor^2(\sigma'_x, \tau'_{xy}) d\theta d\psi \tag{18}$$

The minimization over the angle  $\xi$  leads to a compatibility with the factor defined by the Eq. (17). For a plane stress state the Eq. (18) will always yield the same or higher value  $f_{NP}$  as the Eq. (17). All the integration and minimization operations employed in the Eq. (17), (18) can be computed numerically in an efficient manner.

*Non-proportionality factor employed by the EESH*

The effective equivalent stress hypothesis (EESH) [7] is designed specifically for evaluation of a time-dependent plane stress state. The non-proportionality factor is based on the integral of the shear stress amplitude taken over all planes, which are orthogonal to the component surface. If the  $\sigma_x$  and  $\tau_{xy}$  components of the stress tensor defined by the Eq. (16) have a phase-shift  $\delta$  between them, the value  $\tau_{arith}(\delta)$  can be defined:

$$\tau_{arith}(\delta) = \frac{1}{\pi} \int_0^{\pi} \tau_a(\theta) d\theta. \tag{19}$$

Using the Eq. (19) the non-proportionality factor can be defined:

$$f_{EESH} = \frac{\tau_{arith}(\delta)}{\tau_{arith}(\delta = 0^\circ)} \tag{20}$$

which is known to admit values between 1 ( $\delta = 0^\circ$ ) and approx. 1.1 ( $\delta = 90^\circ$ ). In case of variable amplitude loading the Rainflow-counting is used in each plane given by the angle  $\theta$  in order to obtain the value  $\tau_a(\theta)$ . Subsequently the arithmetic mean of the amplitudes of all the Rainflow-cycles in the given plane is computed.

**CRITICAL-PLANE-BASED NON-PROPORTIONALITY MEASURES**

The non-proportionality measures of this type make use of the stress values related to a certain plane, known as the critical plane, in order to evaluate the non-proportionality of a time-dependent stress tensor. Two values of this type are considered: the non-proportionality factor according to Kanazawa [12] and the multiaxiality factor used in the MWCM [8]. For both factors critical plane is defined to be the one with the highest shear stress amplitude. So the first step is to identify the critical plane, which can be done by a direct computation for a constant amplitude loading and methods like the Maximum Variance Method [15] can be used for a constant amplitude loading.

*The non-proportionality factor according to Kanazawa*

In Kanazawas original paper [12] the factor is defined for a plane stress state as follows:

$$f_{Kanazawa} = \frac{\tau_{a,45^\circ}}{\tau_{a,crit}} \tag{21}$$

In the Eq. (21)  $\tau_{a,crit}$  denotes the shear stress amplitude in the critical plane and  $\tau_{a,45^\circ}$  the shear stress amplitude in the plane, which has the angle of  $45^\circ$  to the critical plane. This definition is straight-forward in the case of a planar loading





and there is no obvious extension to the general 3D stress case, since there is an infinite number of planes, which enclose the angle of  $45^\circ$  with the critical plane. It can be proposed to minimize the value (21) over all such planes:

$$f_{Kanazawa} = \min_{\theta} \frac{\tau_{a,45^\circ}(\theta)}{\tau_{a,crit}} \quad (22)$$

with  $\theta \in [0, 2\pi)$  being the angle, which defines the orientation of the plane. The value according to the Eq. (22) can be computed numerically.

*The multiaxiality factor according to Susmel*

In [8] the multiaxiality factor  $\rho$  is used to take the effects of multiaxiality including non-proportionality of the load into account.

$$\rho = \frac{\sigma_{n,max}}{\tau_{a,crit}} \quad (23)$$

with  $\tau_{a,crit}$  being as before and  $\sigma_{n,max}$  being the maximal normal stress occurring in the critical plane. It can be split into the amplitude and the mean stress component:  $\sigma_{n,max} = \sigma_{n,a} + \sigma_{n,m}$ , this in turn allows to decompose the multiaxiality factor  $\rho$ :

$$\rho = \frac{\sigma_{n,a}}{\tau_{a,crit}} + \frac{\sigma_{n,m}}{\tau_{a,crit}}$$

and to introduce a mean stress sensitivity parameter  $m$  as follows:

$$\rho = \frac{\sigma_{n,a}}{\tau_{a,crit}} + m \frac{\sigma_{n,m}}{\tau_{a,crit}} \quad (24)$$

The factor defined by the Eq. (23) or (24) was introduced in order to be used with the Modified Wöhler Curve Method (MWCM) [8], but can be seen as a method-independent measure for non-proportionality. However it can attend values from  $-\infty$  to  $+\infty$ , which make require additional work in order to use it with some other criterion. In this paper it is used for evaluation with MWCM only.

**EVALUATION OF FATIGUE LIFE RESULTS FOR ALUMINUM AND MAGNESIUM WELDED JOINTS**

**F**atigue life test results under constant and variable amplitude loading for aluminum thin-walled laserbeam welded joints are presented in the thesis [5]. FE-modelling was carried out in order to obtain linear elastic local stresses according to the notch stress concept with fictitious radius  $r_{ref} = 0.05$  mm [16]. For magnesium laserbeam welded joints the results for constant amplitudes are provided in the thesis [6] and variable amplitude loadings are the subject of an ongoing research project funded by the DFG (German Research Foundation). Fatigue life evaluations were performed using two well-known methods: the Findley-criterion [17] and SIH (Shear Stress Intensity Hypothesis) [18]. The non-proportionality factor proposed within EESH is used also within EESH and the multiaxiality factor proposed within the MWCM is also applied with the MWCM only. Some proposals to adapt the two latter non-proportionality factors are provided in the previous section, however a further investigation is required if these factors are going to be applied outside of their respective hypotheses.

In general the following way to apply the non-proportionality factors together with the stress-based hypotheses is proposed:

$$\tau_a^* = (1 + (w - 1)f)\tau_a \quad (25)$$

with  $\tau_a$  a shear stress amplitude value involved in the respective hypothesis,  $w$  a hypothesis-dependent material parameter,  $f$  is the non-proportionality factor and  $\tau_a^*$  the shear stress amplitude corrected with respect to the non-proportionality of the loading. In order to apply the Eq. (25) it is an important assumption, that  $0 \leq f \leq 1$ . For the non-



proportionality measure  $m_1$  given by Eq. (7) an exception is made, since for the evaluated tensor paths it is always  $0 \leq m_1 \leq 1$ .

*Experimental results*

The experimental results were obtained in a tension-torsion testing rig using the specimen geometry as shown in Fig. 5. For both magnesium and aluminum welded joints the same specimen geometry was used. The tested aluminum alloys are EN AW 5042 and EN AW 6082 T6, the magnesium alloys are AZ31 and AZ61. The type of alloy does not show a significant influence of the fatigue strength of the welded joints.

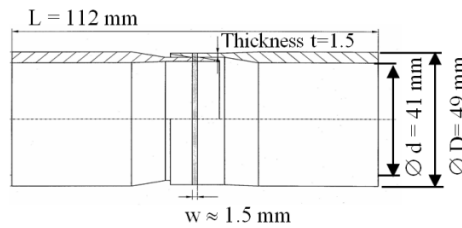


Figure 5: Specimen geometry.

The Wöhlerlines for the aluminum alloy EN AW 5042 (cf. [5]) and for magnesium alloy AZ31 (cf. [6]) are shown in Fig. 6 and Fig. 7 respectively.

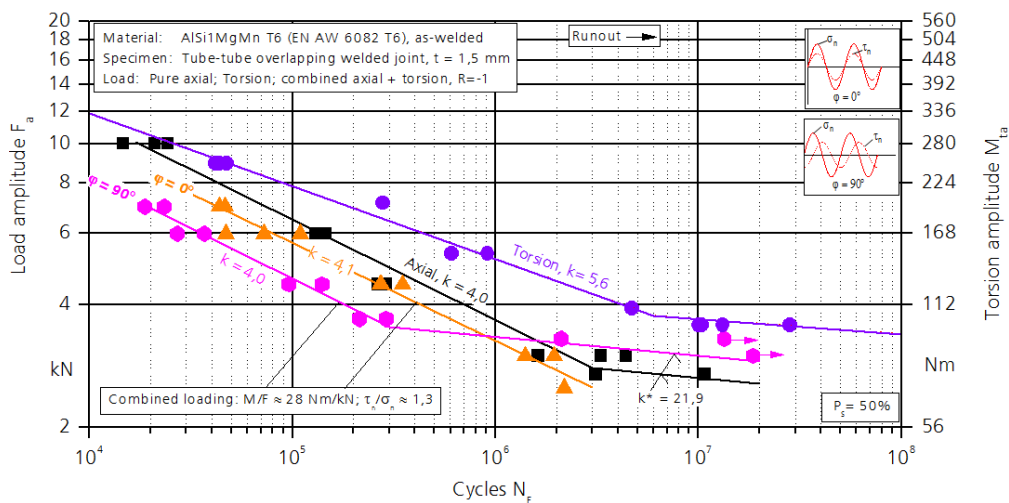


Figure 6: Experimental results for aluminum alloy EN AW 6082 T6 under constant amplitudes

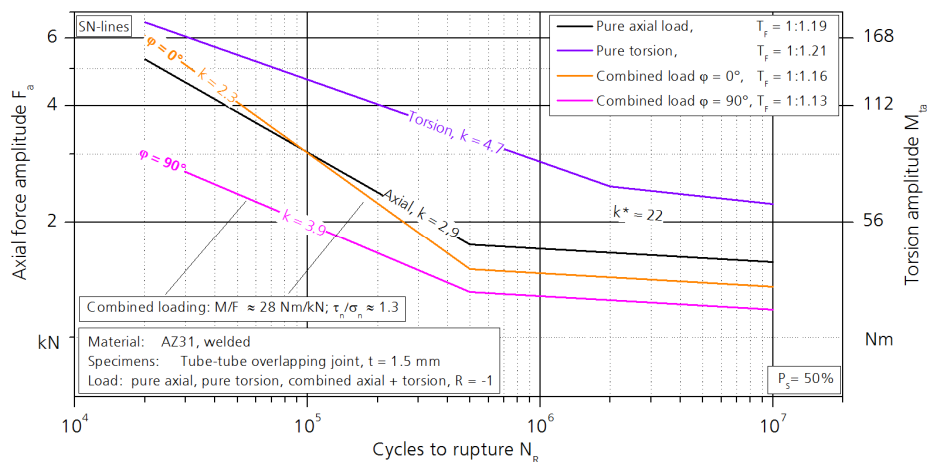


Figure 7: Experimental results for magnesium alloy AZ31 under constant amplitudes.

Both materials show a fatigue life decrease under non-proportional loadings in the finite fatigue life region. The magnesium alloy AZ31 shows this behavior below the knee point of the Wöhlerlines as well. As can be seen in the Tab. 1-6 the non-proportionality factors computed using different methods lie in a narrow interval between 0.6 and 0.75. If in the Eq. (25) it is assumed  $w = \sqrt{2}$ . Results as shown in Fig. 8 and Fig. 9 can be obtained using the correlation-based factor and the hypotheses Findley [17] and SIH [18]. Other factors will lead to similar (slightly more conservative) results.

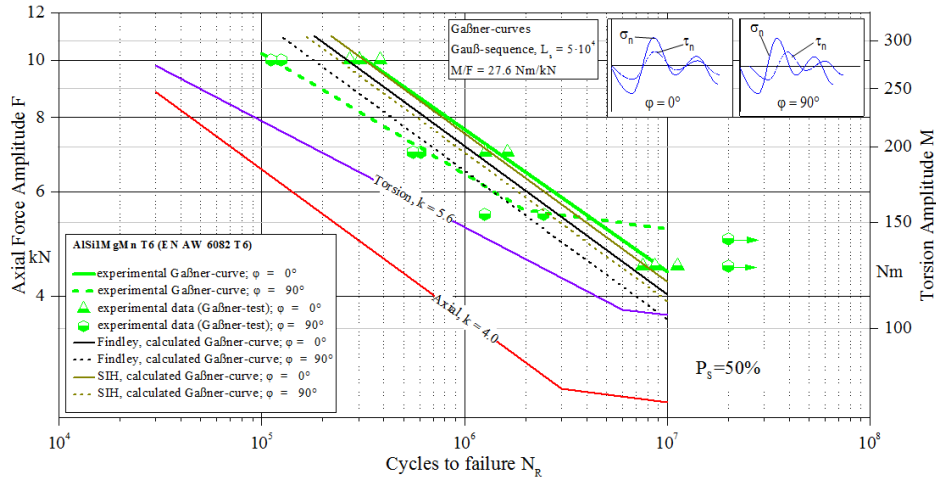


Figure 8: Experimental and computational Gassner-lines for aluminum alloy EN AW 6082 T6 with the application of non-proportionality factors.

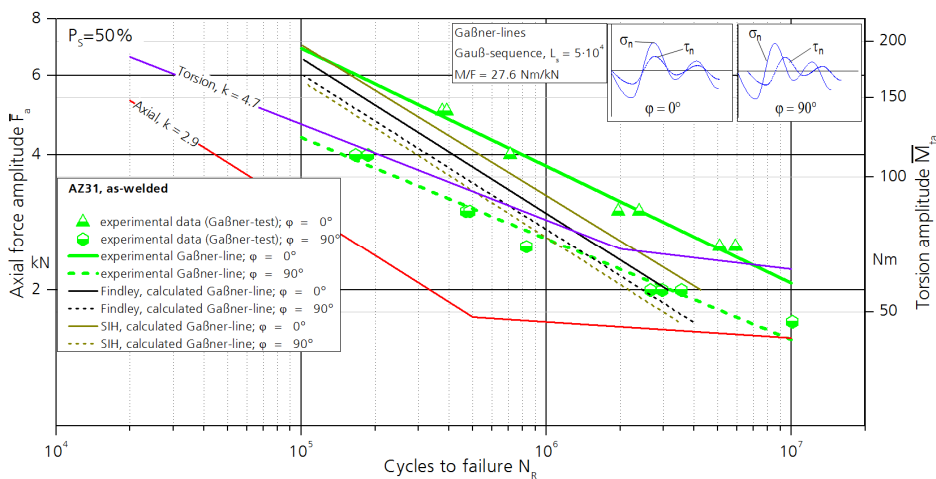


Figure 9: Experimental and computational Gassner-lines magnesium alloy AZ31 T6 with the application of non-proportionality factors.

Both figures show conservative estimates for high number of cycles. The inverse slope of the Gassner-lines is not correctly computed using the hypotheses (especially the Palmgren-Miner damage accumulation rule, which is used for the fatigue life computations in this case).

Aluminium		Magnesium	
CAL, $\delta = 90^\circ$	VAL, $\delta = 90^\circ$	CAL, $\delta = 90^\circ$	VAL, $\delta = 90^\circ$
0.659	0.658	0.677	0.675

Table 1: Non-proportionality factor according to Kanazawa, plane stress state.



Aluminium		Magnesium	
CAL, $\delta = 90^\circ$	VAL, $\delta = 90^\circ$	CAL, $\delta = 90^\circ$	VAL, $\delta = 90^\circ$
0.610	0.609	0.623	0.622

Table 2: Non-proportionality factor according to Kanazawa, generalized for spatial stress state

Aluminium		Magnesium	
CAL, $\delta = 90^\circ$	VAL, $\delta = 90^\circ$	CAL, $\delta = 90^\circ$	VAL, $\delta = 90^\circ$
0.689	0.685	0.679	0.675

Table 3: Non-proportionality factor according to Bishop,  $m_1$

Aluminium		Magnesium	
CAL, $\delta = 90^\circ$	VAL, $\delta = 90^\circ$	CAL, $\delta = 90^\circ$	VAL, $\delta = 90^\circ$
0.755	0.750	0.746	0.742

Table 4: Non-proportionality factor according to Bishop,  $m_2$

Aluminium		Magnesium	
CAL, $\delta = 90^\circ$	VAL, $\delta = 90^\circ$	CAL, $\delta = 90^\circ$	VAL, $\delta = 90^\circ$
0.689	0.687	0.679	0.677

Table 5: Non-proportionality factor according to Gaier,  $d$

Aluminium		Magnesium	
CAL, $\delta = 90^\circ$	VAL, $\delta = 90^\circ$	CAL, $\delta = 90^\circ$	VAL, $\delta = 90^\circ$
0.590	0.590	0.590	0.590

Table 6: Correlation-based non-proportionality factor,  $f_{NP}$

*Results obtained using the MWCM according to Susmel*

For this hypothesis only the results for aluminum alloy under constant amplitudes are shown. Evaluation of fatigue life of welded joint using the notch stress concept for thick-walled structures is discussed in [19]. Here MWCM is applied along with the notch stress concept for thin-walled structures.

Aluminium
CAL, $\delta = 90^\circ$
1.97

Table 7: Multiaxiality factor according to Susmel,  $\rho$

The value of the non-proportionality factor  $\rho$  is shown in Tab. 7. If the value  $\rho_{lim}$  as it is described in [9] is taken into account, the results as they are shown in Fig. 10 can be obtained. In the region of finite fatigue life the results lie in a narrow scatter band around the line  $N_{exp} = N_{calc}$  and towards the knee point of the Wöhlercurves change on the non-

conservative side. It can also be noted, that results under non-proportional loading are evaluated in a slightly more non-conservative way than the results for the proportional one.

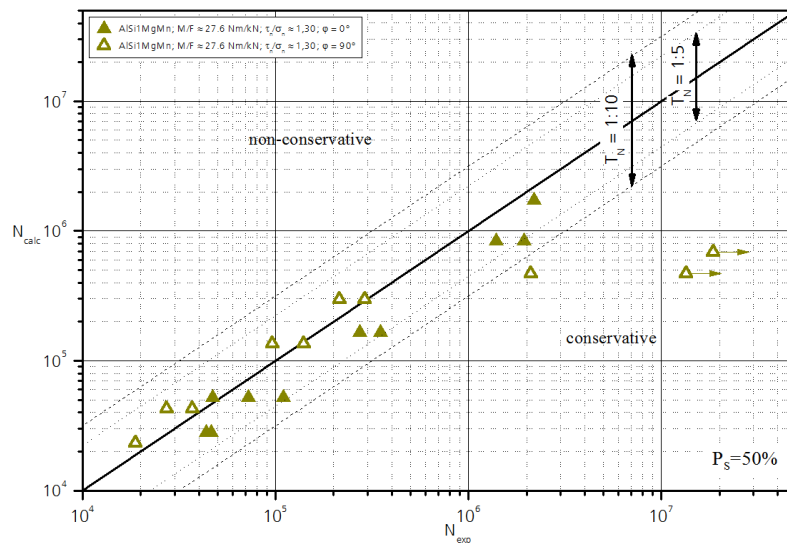


Figure 10: Experimental results for the EN AW 6082 T6 alloy. Experimental vs. calculated (MWCM) fatigue life.

*Results obtained using EESH according to Sonsino*

Evaluations using EESH for aluminum welded joints were performed in the doctoral thesis [5]. There good agreement of experimental and computational results could be observed. Computations for magnesium welded joints under constant amplitude loadings were performed in the thesis [6]. EESH shows there non-conservative computational results. The same behavior is observed under variable amplitude loadings (Fig. 11).

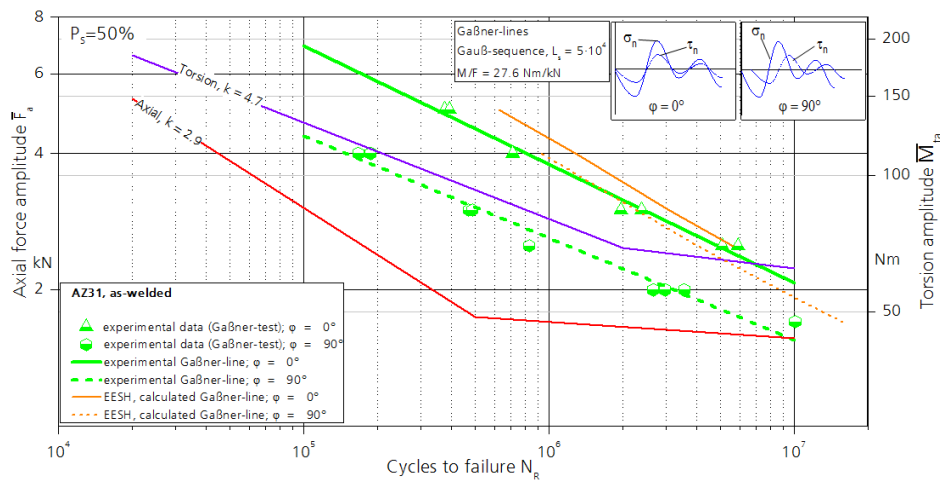


Figure 11: Experimental Gassner-curves and Gassner-curves computed using EESH.

According to the computed value of the non-proportionality measure (cf. Eq. (20), Tab. 8) EESH computes a 10% fatigue strength decrease for the non-proportional loading.

Magnesium
VAL, $\delta = 90^\circ$
1.104

Table 8: Non-proportionality factor used in EESH (Sonsino),  $f_{EESH}$



## CONCLUSION

A number of measures for the non-proportionality of the loadings were presented and used for evaluation of fatigue life of aluminum and magnesium welded joints. In general if a non-proportionality measure yields values between 0 and 1 it can be applied with any stress-based fatigue life hypothesis. The Eq. (25) can be used to modify a suitable stress value and hence take the non-proportionality into account. So it is a plausible requirement for a non-proportional measure to yield values between 0 and 1.

There are also measures such as the factor  $\rho$  used in MWCM (Eq. (23)), which is required to interpolate between axial and torsion Wöhler-lines and cannot be easily interchanged with any other non-proportionality measure discussed in this paper.

Also the non-proportionality measures are subdivided into integral and critical-plane-based ones. The latter group contains two measures discussed in this paper: the measure according to Kanazawa (Eq. (21), (22)) and the measure  $\rho$  used in MWCM according to Susmel (Eq. (23)). The value of this measures can rapidly change due to a small change in the loading, since such a change can lead to a change in the position of the critical plane. As a result two very similar loads can get very different non-proportionality evaluation. This problem can be avoided if an integral measure of non-proportionality is used.

In order to use non-proportionality measures safely for arbitrary loadings, experimental results under loadings with different degrees of non-proportionality are required. In particular this can mean different phase-shifts between axial and torsion loadings and different amplitude ratios.

All non-proportional measures presented in this paper are “global” i.e. they deliver information on the non-proportionality of a loading path as a whole. The question arises, whether these measures still work in an adequate manner, if a loading path consists of a number of pieces with different degrees of non-proportionality. Is it possible to provide some “local” measure of non-proportionality?

As can be observed for the magnesium welded joints taking into account non-proportionality does improve the evaluation, however it can be further improved for variable amplitude loadings if the slopes of the Gassner-lines are taken into account correctly by the hypotheses.

## REFERENCES

- [1] Sonsino, C.M., Influence of material's ductility and local deformation mode on multiaxial fatigue response, *Int. J. Fatigue*, 33(8) (2011) 930-947.
- [2] Sonsino, C. M., Küppers, M., Eibl, M., Zhang, G., Fatigue strength of laser welded thin steel structures under multiaxial loading, *Int. J. Fatigue*, 28 (5-6) (2006) 657-62.
- [3] Störzel, K., Wiebesiek, J., Bruder, T., Hanselka, H., Betriebsfeste Bemessung von mehrachsigen belasteten Laserstrahlschweißverbindungen aus Stahlblechen des Karosseriebaus. Fraunhofer-Institut für Betriebsfestigkeit und Systemzuverlässigkeit LBF, LBF-Bericht Nr. FB-235, Darmstadt (2008).
- [4] Küppers, M., Betriebsfestigkeit von Aluminiumschweißverbindungen unter mehrachsigen Spannungszuständen mit konstanten und veränderlichen Hauptspannungsrichtungen. Dissertation: Fachbereich Maschinenbau, Fachgebiet SzM, TU Darmstadt (2006), LBF-Bericht Nr. FB-233
- [5] Wiebesiek, J., Festigkeitshypothesen zum Schwingfestigkeitsverhalten von dünnwandigen Laserstrahlschweißverbindungen aus Aluminium unter mehrachsigen Beanspruchungen mit konstanten und veränderlichen Hauptspannungsrichtungen, Dissertation, Shaker Verlag, Aachen (2012).
- [6] Exel, N., Schwingfestigkeit laserstrahlgewetzter Magnesiumlegierungen unter mehrachsigen proportionalen und nichtproportionalen Beanspruchungen, Dissertation., Shaker Verlag, Aachen (2013).
- [7] Sonsino, C. M., Multiaxial fatigue of welded joints under in-phase and out-of-phase local strains and stresses, *Int. J. Fatigue*, 17(1) (1995) 55-70.
- [8] Susmel, L., *Multiaxial notch fatigue*, Woodhead, Cambridge (2009).
- [9] Bishop, J. E., Characterizing the non-proportional and out-of-phase extent of tensor paths, *Fatigue Fract. Engng. Mater. Struct.* 23 (2000) 1019-1032.
- [10] Gaier, C., Lukacs, A., Hofwimmer, K., Investigations on a statistical measure of non-proportionality of stresses, *Int. J. Fatigue*, 26 (2004) 331-337.



- [11] Bolchoun, A., Sonsino, C.M., Kaufmann, H., Melz, T., Multiaxial random fatigue of magnesium laserbeam-welded joints – Experimental results and numerical fatigue life evaluation, *Proc. Engng.*, 101 (2015) 61-68.
- [12] Kanazawa, K., Miller, K.J., Brown, M. W., Cyclic deformation of 1% Cr-Mo-V steel under out-of-phase loads, *Fatigue Engng. Mater. Struct.*, 2 (1979) 217-228.
- [13] Chu, C.C., Conle, F.A., Hübner, A., An integrated uniaxial and multiaxial fatigue life prediction method, VDI-Verlag, Düsseldorf, VDI-Bericht 1283, (1996) 337-348.
- [14] Capasso, V., Backstein, D., *An Introduction to Continuous-Time Stochastic Processes*, Birkhäuser, Boston (2005).
- [15] Susmel, L., A simple and efficient numerical algorithm to determine the orientation of the critical plane in multiaxial fatigue problems, *Int. J. Fatigue*, 32 (2010) 1875-1883.
- [16] Radaj, D., Sonsino, C.M., Fricke, W., *Fatigue assessment of welded joints by local approaches*, 2nd ed., Woodhead, Cambridge, (2006).
- [17] Findley, W.N., A theory of the effect of mean stress on fatigue of metals under combined torsion and axial loading or bending, *J. Eng. Ind. (ASME)*, 81 (1959) 301-306.
- [18] Zenner, H., Richter, I., Eine Festigkeitshypothese für die Dauerfestigkeit bei beliebigen Beanspruchungskombinationen, *Konstruktion*, 29 (1) (1977) 11-18.
- [19] Susmel, L., Sonsino, C.M., Tovo, R., Accuracy of the Modified Wöhler Curve Method applied along with the  $r_{ref} = 1$  mm concept in estimating lifetime of welded joints subjected to multiaxial fatigue loading, *Int. J. Fatigue*, 33 (2011) 1075-1091.