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Focussed on multiaxial fatigue

On the applicability of multi-surface, two-surface and non-linear kinematic hardening models in multiaxial fatigue

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ABSTRACT. In this work, a comparison between NLK and Mróz-Garud's multi-surface formulations is presented. A unified common notation is introduced to describe the involved equations, showing that the Mróz-Garud model can be regarded as a particular case of the NLK formulation. It is also shown that the classic two-surface model, which is an unconventional simplified plasticity model based on the translation of only two surfaces, can also be represented using this formulation. Such common notation allows a direct quantitative comparison among multi-surface, two-surface, and NLK hardening models.

KEYWORDS. Multiaxial fatigue; Incremental plasticity; Kinematic hardening; Non-proportional variable amplitude loads.

INTRODUCTION

The Bauschinger effect, commonly called *kinematic hardening*, can be modeled in stress spaces by allowing the yield surface to translate with no change in its size or shape. So, in the deviatoric stress space, kinematic hardening maintains the radius S of the yield surface fixed while its center is translated, changing the associated generalized plastic modulus P that defines the slope between stress and plastic strain increments in the Prandtl-Reuss plastic flow rule, also known as the normality rule.

There are several models to calculate the current value of P as the yield surface translates, as well as the direction of such translation, to obtain the associated plastic strain increments. Most of these hardening models can be divided into three classes: Mróz multi-surface (or multi-linear), non-linear, and two-surface kinematic models.

Mróz [1] defined in 1967 the first multi-surface kinematic hardening model to approximately describe the behavior of elastoplastic solids through a family of nested surfaces in the stress space, the innermost being the yield surface associated with the material yield strength. It assumes that P is piecewise constant, resulting in a multi-linear description of the stress-strain curve, i.e. the non-linear shape of the elastoplastic stress-strain relation is approximated by several linear segments. The Mróz model can induce a few numerical problems, which can result in hardening surfaces improperly intersecting in more than one point under finite strain increments. Garud [2] proposed a geometrical correction that avoids intersection problems even for coarse integration increments.

In spite of its limitations, several multiaxial fatigue works use the Mróz-Garud model to predict the stress-strain behavior under combined loading, especially due to its ability to store plastic memory effects under variable amplitude loading. This popularity is understandable, since the multi-linear stress-strain curves generated by the Mróz-Garud model provide good results for balanced loadings. However, such multi-linear models cannot predict any uniaxial ratcheting or mean stress relaxation caused by unbalanced loadings, since their idealized hysteresis loops always close because they are unrealistically assumed as perfectly symmetric. In addition, under several non-proportional loading conditions, these models predict multiaxial ratcheting with a constant rate that never decays, severely overestimating the ratcheting effect measured in practice [3]. As a result, multi-surface kinematic hardening models should only be confidently applied to balanced proportional loading histories.

To correctly predict the stress-strain history associated with unbalanced loadings, it is necessary to introduce non-linearity in the hardening surface translation equations, the main characteristic of the non-linear kinematic (NLK) models. NLK models are more general than Mróz-Garud because they use non-linear equations to describe the surface translation direction and the value of P, leading thus to a more precise description of the non-linear stress-strain curves. Armstrong and Frederick's original formulation [4] was improved by Chaboche [5], which indirectly introduced the Mróz nested-surface idea to NLK models, however in a non-linear instead of multi-linear formulation. Therefore, both NLK and Mróz-Garud formulations have several common features, as discussed further in this work.

A third class of kinematic hardening models involves the so-called two-surface models, which use a rather simplified formulation that combines elements of both non-linear and Mróz multi-surface kinematic models.

In this work, instead of defining the nested hardening surfaces in the 6D stress or 6D deviatoric stress spaces, a 5D reduced order deviatoric stress space E_{5s} is adopted, using the Mises yield function to describe each surface. This 5D space has two advantages over the usual 6D formulations: it is a non-redundant representation of the deviatoric stresses, which decreases the computational cost of stress-strain calculations; and the radius S of the yield surface is equal to the yield strength without the need to include the scaling factor $\sqrt{2/3}$ required in 6D formulations. Besides, even though all kinematic hardening equations are presented here in the 5D space, their conversion to 6D versions is trivial.

MRÓZ MULTI-SURFACE MODELS

In Fig. 1, the first (and innermost) circle is the monotonic yield surface, with radius $r_1 \equiv S_Y$. In addition, $M - 1$ *hardening surfaces* with radii $r_1 < r_2 < \ldots < r_{M+1}$ are defined, along with an outermost *failure surface* whose radius r_{M+1} is equal to the true rupture stress σ_U of the material. Their centers are located at points $\vec{s}'_{\alpha i}$ with $i = 2, 3, \ldots, M + 1$, respectively. These $M + 1$ nested circles cannot cross one another, must have increasing radii, and for a virgin material they all are initially concentric at the origin of the E_{5s} space, i.e. initially their $\vec{s}'_{\alpha i} = 0$. Moreover, the failure surface never translates, i.e. its center always remains at the origin of the E_{5s} space, $\vec{s}'_{\varepsilon_{M+1}} \equiv 0$. In fact, any stress point that reaches its boundary causes the material to fracture due to ductility exhaustion, which is equivalent to the criterion $|\vec{s}'| = r_{M+1} = \sigma_U$.

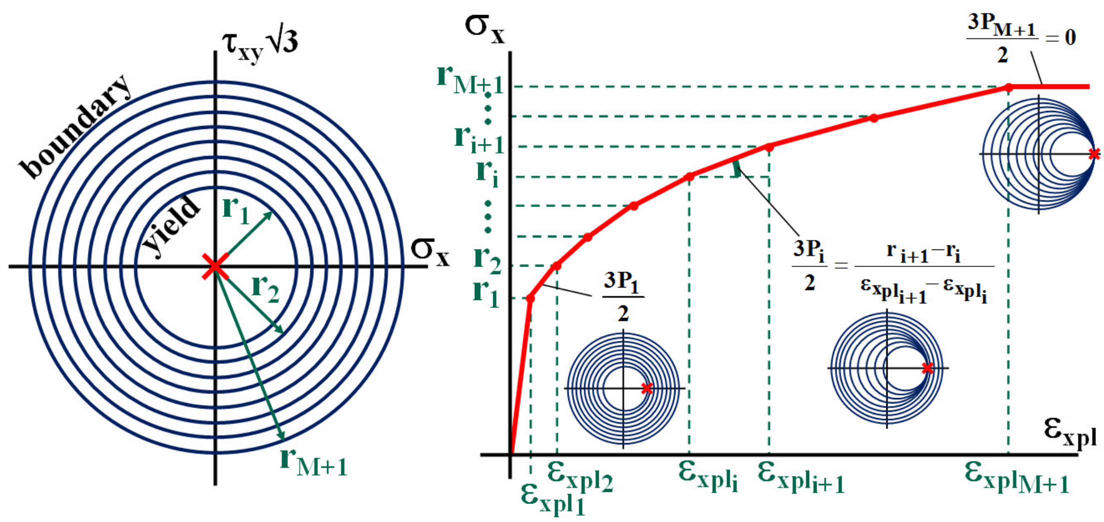

Figure 1: Yield, hardening and failure surfaces in the $\sigma_x \times \tau_{xy}\sqrt{3}$ 2D sub-space of E_{5s} (left) and corresponding radii r_i and generalized plastic moduli P_i obtained from the piecewise linearization of the uniaxial stress versus plastic strain curve (right).

Except for the failure surface, all other hardening surfaces can translate as the material strain-hardens, as shown in Fig. 1(right). The centers of the hardening surfaces (which are circles in the 2D example from Fig. 1) move as the material plastically deforms and hardens, because they are successively pushed by the inner surfaces. The radii r_i of the various hardening surfaces are equal to the stress levels associated with the plastic strains ε_{xpli} that delineate the multi-linear representation of the stress-strain curve, fitted to properly describe the stress-strain $\sigma \times \varepsilon$ behavior of the material. The difference between the radii of each pair of consecutive surfaces is defined as $\Delta r_i \equiv r_{i+1} - r_i$. In principle, all hardening surfaces radii r_i may change during plastic deformation as a result of isotropic and NP hardening effects.

The backstress vector $\vec{\beta}'$, which locates the current yield surface center $\vec{\beta}' \equiv \vec{s}'_{c_1}$, can be decomposed as the sum of up to M surface backstresses $\vec{\beta}'_1, \ldots, \vec{\beta}'_M$ that describe the relative positions $\vec{\beta}'_i \equiv \vec{s}'_{c_i} - \vec{s}'_{c_{i+1}}$ between centers of the consecutive hardening surfaces, see Fig. 2. Note that the length (norm) $|\vec{\beta}'_i|$ of each surface backstress is always between $|\vec{\beta}'_i| = 0$, if the surface centers $\vec{s}'_{c_i}$ and $\vec{s}'_{c_{i+1}}$ coincide (as in an unhardened condition), and $|\vec{\beta}'_i| = \Delta r_i$, if the surfaces are mutually tangent (a saturation condition with maximum hardening).

The main rules in the evolution of these yield and hardening surfaces are: (i) they must translate as rigid bodies when the point $\vec{s}'$ that defines the current deviatoric stress state in the E_{5s} space reaches their boundaries, to guarantee that such stress point is never outside any surface. In other words, neither the yield surface nor any hardening surface translate unless the stress state is at their boundary ($|\vec{s}' - \vec{s}'_{c_i}| = r_i$) trying to move outwards; and (ii) the hardening surfaces cannot cross through one another, therefore they gradually become mutually tangent to one another at the current stress point $\vec{s}'$ as the material plastically deforms.

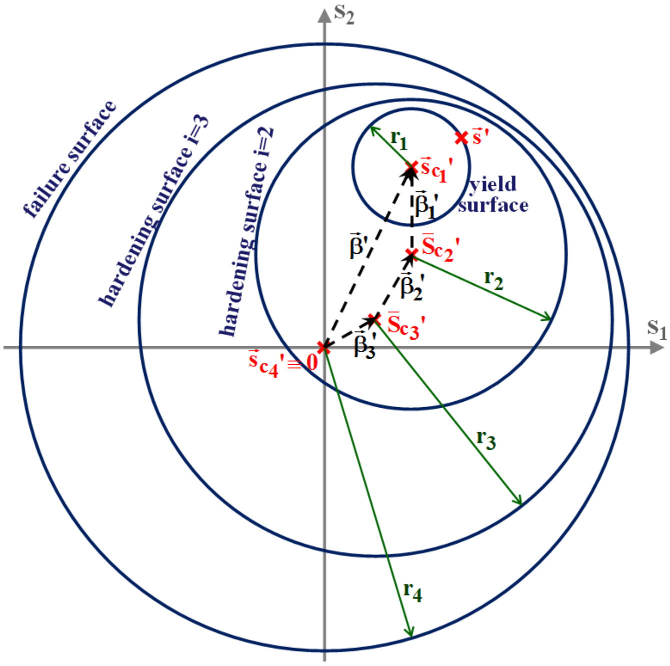

Figure 2: Yield, hardening, and failure surfaces in the deviatoric space for $M = 3$, showing the backstress vector $\vec{\beta}'$ that defines location of the yield surface center $\vec{\beta}' \equiv \vec{s}'_{c_1}$ and its components $\vec{\beta}'_1$, $\vec{\beta}'_2$, and $\vec{\beta}'_3$ that describe the relative positions between the centers of consecutive surfaces.

In the Mróz multi-surface formulation, the outermost surface that is moving at any instant is called the *active surface*, denoted here as the surface with index i_A. Any changes in the stress state that happen inside the yield surface are assumed elastic, not resulting in any surface translation as long as $|\vec{s}' - \vec{s}'_{c_1}| = r_1$, therefore no surface would be active in this case and thus $i_A = 0$. Each surface is associated with a generalized plastic modulus P_i ($i = 1, 2, \ldots, M + 1$), which altogether define a *field of hardening moduli*. The value of P is then chosen as the P_i from the active surface $i = i_A$. Fig. 3 illustrates two consecutive surfaces i and $i + 1$ ($i \geq 1$), with radii r_i and r_{i+1} in the E_{5s} 5D deviatoric stress space. Assuming that the current active surface is $i = i_A$, then according to the Mróz model all outer hardening surfaces do not translate, therefore the increments of the respective backstresses are $d\vec{\beta}'_{i+1} = d\vec{\beta}'_{i+2} = ... = d\vec{\beta}'_M = 0$, resulting in $d\vec{s}'_{c_i} = d\vec{\beta}'_i + d\vec{s}'_{c_{i+1}} = d\vec{\beta}'_i$.

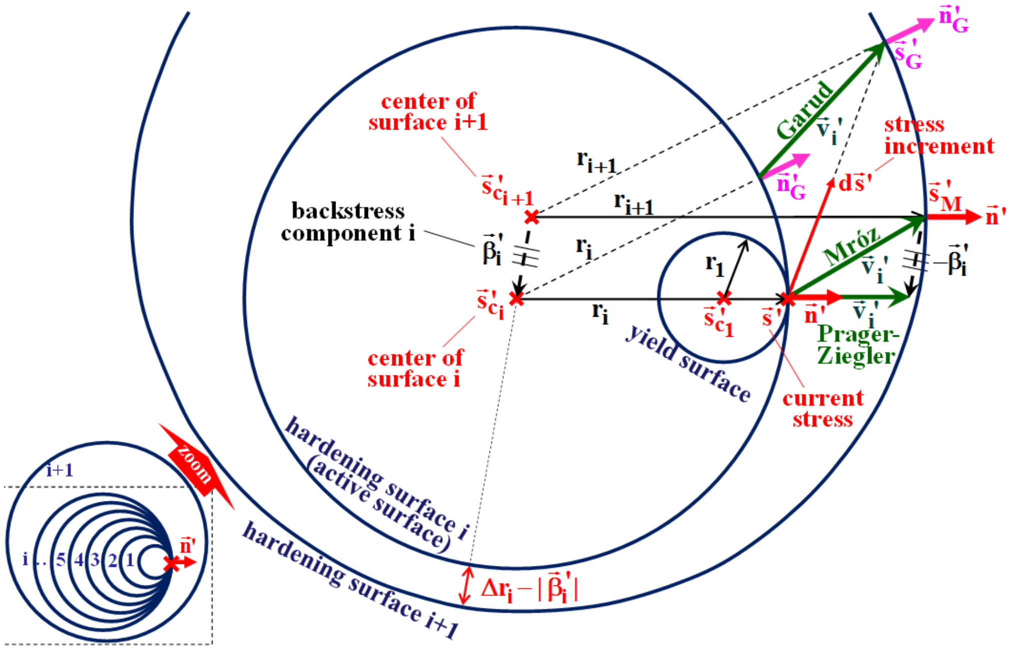

Figure 3: Illustration of Mróz, Garud, and Prager-Ziegler surface translation rules used to model kinematic hardening in the Mróz multi-surface formulation in the E_{5s} space.

The Mróz multi-surface formulation assumes that, during plastic straining, all inner surfaces $1, 2, \ldots, i_A - 1$ must translate altogether with the active surface $i = i_A$, therefore their centers do not move relatively to each other, resulting in $d\vec{\beta}'_1 = d\vec{\beta}'_2 = ... = d\vec{\beta}'_{i-1} = 0$. Thus, translation rules in the Mróz multi-surface formulation only need to be applied to the evolution of the backstress $d\vec{\beta}'_i$ of the active surface $i = i_A$, giving

$$d\vec{\beta}'_i = \begin{cases} d\vec{s}'_{c_i} = d\vec{s}'_{c_1} = d\vec{\beta}', & \text{if } i = i_A \\ 0, & \text{if } i \neq i_A \end{cases} \tag{1}$$

Moreover, since these inner hardening surfaces $1, 2, \ldots, i_A - 1$ are all mutually tangent at the current stress state $\vec{s}'$ perpendicular to the normal vector $\vec{n}'$, their backstresses are all parallel to $\vec{n}'$ and have reached their saturation (maximum) values.

The kinematic rule for the translation $d\vec{\beta}'_i$ of the active yield surface can be defined from an assumed translation direction $\vec{v}'_i$. Prager [6] assumed that $\vec{v}'_i$ is parallel to the direction of the normal unit vector $\vec{n}'$, i.e. $d\vec{\beta}'_i$ happens at the current stress state $\vec{s}'$ in such normal direction $\vec{n}'$. Ziegler, on the other hand, assumed that $d\vec{\beta}'_i$ happens in the radial direction $\vec{s}' - \vec{s}'_{c_i}$ from the surface center [7]. For the Mises yield surface, both Prager's and Ziegler's rules result in the same Prager-Ziegler direction $\vec{v}'_i = \vec{n}'$, see Fig. 3, which can be calculated from the normalized difference between the current stress state $\vec{s}'$ and the yield surface center $\vec{s}'_{c_i}$:

$$\vec{n}' = \frac{\vec{s}' - \vec{s}'_{c_i}}{|\vec{s}' - \vec{s}'_{c_i}|} = \frac{\vec{s}' - \vec{s}'_{c_i}}{r_i} \tag{2}$$

For Mises materials, the translation direction of Prager-Ziegler's kinematic rule is then

$$\vec{v}'_i \equiv \vec{n}' \cdot (r_{i+1} - r_i) = \vec{n}' \cdot \Delta r_i \quad \textit{(Prager-Ziegler)} \tag{3}$$

But Prager-Ziegler's translation rule is too simplistic to model kinematic hardening. An improved rule was adopted by Mróz [1], who assumed that the translation $d\vec{\beta}'_i$ of the active surface occurs in a direction $\vec{v}'_i$ defined by the segment that joins the current stress state $\vec{s}' = \vec{s}'_{c_i} + \vec{n}' \cdot r_i$ with the corresponding "image stress point" $\vec{s}'_M = \vec{s}'_{c_{i+1}} + \vec{n}' \cdot r_{i+1}$ that has the same normal unit vector $\vec{n}'$ at the next hardening surface *i + 1*, see Fig. 3. The Mróz translation direction is then

$$\vec{v}'_i \equiv \underbrace{(\vec{s}'_{c_{i+1}} + \vec{n}' \cdot r_{i+1})}_{\vec{s}'_M \text{ image point}} - \underbrace{(\vec{s}'_{c_i} + \vec{n}' \cdot r_i)}_{\text{current point}} = \vec{n}' \cdot \underbrace{(r_{i+1} - r_i)}_{\Delta r_i} + \underbrace{\vec{s}'_{c_{i+1}} - \vec{s}'_{c_i}}_{-\vec{\beta}'} = \vec{n}' \cdot \Delta r_i - \vec{\beta}'_i \quad (Mróz) \tag{4}$$

Note that the Mróz translation direction is the combination of Prager-Ziegler's hardening term with a "dynamic recovery" direction $-\vec{\beta}'_i$. This last term induces the center of the active surface to translate back towards the center of the next hardening surface, attempting to dynamically recover from the hardening state described by the surface backstress $\vec{\beta}'_i$ that separates their centers. This dynamic recovery term is able to consider a fading memory of the plastic strain path, which is necessary to model mean stress relaxation and ratcheting effects.

However, the Mróz rule can induce a few numerical problems, which can result in surfaces intersecting in more than one point under finite load increments. Garud proposed an empirical correction that avoids intersection problems even for coarse integration increments, as detailed in [2].

A major concern of the Mróz multi-surface formulation is that the directions of the calculated stress or strain paths may significantly vary depending on the number of surfaces used. Even worse, better predictions are not necessarily obtained from using a larger number of surfaces. As a result, the number of hardening surfaces that results in the best calculation accuracy is a finite number that would also need to be calibrated, an undesirable feature.

Moreover, the multi-linear formulation adopted by Mróz and Garud models is not able to correctly predict ratcheting and mean stress relaxation effects. A better approach is to replace such multi-linear models with a non-linear kinematic hardening formulation, described next.

NON-LINEAR KINEMATIC (NLK) HARDENING MODELS

The multi-linear stress-strain curves generated by the Mróz and Garud multi-surface models provide good results for balanced proportional loadings, which by definition do not induce ratcheting or mean stress relaxation. However, such piecewise linear models cannot predict any uniaxial ratcheting or mean stress relaxation effects caused by unbalanced proportional loadings. This shortcoming is due to the linearity of the Mróz and Garud surface translation rules and the resulting multi-linearity of the approximated stress-strain representation, which describes all elastoplastic hysteresis loops using multiple straight segments, instead of predicting the experimentally observed curved paths caused by those non-linear effects. Such straight segments generate unrealistic perfectly symmetric hysteresis loops that always close under constant amplitude proportional loadings.

In addition, for NP loadings, the Mróz and Garud multi-surface models may predict multiaxial ratcheting with a constant rate that never decays, severely overestimating the ratcheting effect measured in practice. As a result, Mróz or Garud kinematic strain-hardening models should only be applied to balanced loading histories, severely limiting their applicability.

Another serious flaw of the Mróz and Garud models becomes evident for a NP path example where the stress state $\vec{s}'$ follows the contour of the active yield surface. In this example, the stress increment $d\vec{s}'$ would always be tangent to such active surface, therefore it would induce $d\vec{s}'^T \cdot \vec{n}' = 0$ and no plastic strain would be predicted from the normality rule. This conclusion is physically inadmissible, since it would assume zero plastic straining along a load path with radius r_A larger than the yield surface radius r_1. The Mróz and Garud models would predict that the yield surface can translate tangentially to the active surface without generating plastic strains.

These major drawbacks are a consequence of Mróz and Garud multi-surface kinematic hardening models being of an "uncoupled formulation" type, as qualified by Bari and Hassan in [8]. Such uncoupling means that the generalized plastic modulus $P = P_i$ in this formulation is not a function of the load translation direction $\vec{v}'_i$, since it is a constant for each surface. Such "uncoupled procedure" (where P and $\vec{v}'_i$ are independent) provides undesirable additional degrees of freedom to the Mróz and Garud models that allow, for instance, *90°* out-of-phase tension-torsion predictions of resulting plastic strain amplitudes that are not a monotonic function of the applied stress amplitudes, as they should be [9]. These

wrong Mróz and Garud predictions are both qualitatively and quantitatively dependent on the number of surfaces adopted in the model, without any clear convergence.
To correctly predict the stress-strain history in unbalanced loadings, P and $\vec{v}'_i$ must be coupled, in addition to introducing non-linearity in the surface translation equations, generating the non-linear kinematic (NLK) models discussed next. The first non-linear kinematic (NLK) hardening model was proposed by Armstrong and Frederick in 1966 [4]. Their original single-surface model did not include any hardening surface, but their yield surface already translated according to a non-linear rule.
Chaboche et al. [5] significantly improved Armstrong-Frederick's model capabilities by indirectly introducing the concept of multiple hardening surfaces. As demonstrated by Ohno and Wang in [10], this improvement allowed the NLK formulation to use the same representation of the hardening state as the one in the Mróz multi-surface formulation, which includes one yield surface, $M - 1$ hardening surfaces, and one failure surface, all of them nested within each other without crossing. Therefore, the improved NLK models also adopt a multi-surface formulation, using a "coupled procedure" based on non-linear translation rules. However, unless otherwise noted, the denomination "multi-surface model" is traditionally associated in the literature with the Mróz "uncoupled procedure" based on multi-linear translation rules and piecewise-constant generalized plastic moduli P_i [8].
In summary, despite their significant differences, both Mróz and NLK approaches can be represented using the same multi-surface formulation. As in the Mróz models, instead of defining these surfaces in the 6D stress or deviatoric stress spaces, in this work a 5D reduced order deviatoric stress space E_{5s} is adopted, using the Mises yield function to describe each surface.
In the multi-surface version of NLK models, the backstress $\vec{\beta}'$ that locates the center of the hardening surface is also decomposed as the sum of M surface backstresses $\vec{\beta}'_1, \vec{\beta}'_2, ..., \vec{\beta}'_M$, that describe the relative positions $\vec{\beta}'_i \equiv \vec{s}'_{c_i} - \vec{s}'_{c_{i+1}}$ between the centers of consecutive surfaces, exactly as in the Mróz multi-surface formulation. From these relations, the center of the yield surface ($i = 1$) or of any hardening surface ($2 \leq i \leq M$) can be written as the 5D vector

$$\vec{s}'_{ci} = \vec{\beta}'_i + \vec{\beta}'_{i+1} + \vec{\beta}'_{i+2} + ... + \vec{\beta}'_{M-2} + \vec{\beta}'_{M-1} + \vec{\beta}'_M \quad (5)$$

Once again, the length $|\vec{\beta}'_i|$ of each hardening surface backstress is always between 0 (in the unhardened condition) and its saturation value Δr_i (a maximum hardening condition when these surfaces become tangent, see Fig. 3), while the failure surface is always centered at the origin, i.e. $\vec{s}'_{c_{M+1}} \equiv 0$.

As usual, the hardening surfaces cannot pass through one another, remaining nested at all times, since $|\vec{\beta}'_i| \leq \Delta r_i$. But during plastic straining in the NLK multi-surface formulation, the yield and *all* hardening surfaces do translate, as opposed to the Mróz formulation, where all surfaces outside the active one would not move. The NLK models do not use an "active surface" concept, since *all* hardening surfaces become active during plastic straining. The yield and hardening surfaces from the NLK models behave as if they were all attached to one another with non-linear spring-slider elements, causing coupled translations even before they enter in contact, i.e. even for $|\vec{\beta}'_i| \leq \Delta r_i$. Therefore, any hardening surface translation causes all surfaces to translate, usually with different magnitudes and directions, even before they become tangent to each other.
Pairs of consecutive hardening surfaces i and $i + 1$ may eventually become mutually tangent at the saturation condition $|\vec{\beta}'_i| = \Delta r_i$, when their respective translations have the same magnitude and direction $d\vec{\beta}'_i = d\vec{s}'_{c_i} - d\vec{s}'_{c_{i+1}} = 0$, so when there is no surface backstress variation for this pair. So, plastic straining causes increments $d\vec{\beta}'_i \neq 0$ in all surface backstresses, except for the ones from the saturated surfaces. Quantitatively, these surface translation rules state that, during plastic straining,

$$d\vec{\beta}'_i = \begin{cases} p_i \cdot \vec{v}'_i \cdot dp, & \text{if } |\vec{\beta}'_i| < \Delta r_i \\ 0, & \text{if } |\vec{\beta}'_i| = \Delta r_i \end{cases} \quad (6)$$

for the yield ($i = 1$) and all hardening surfaces $i = 2, \ldots, M$, where $dp = (2/3) \cdot |d\vec{e}'_{pl}|$ is the equivalent plastic strain increment, $\vec{v}'_i$ is the surface translation direction vector (in the 5D space E_{5s}), and p_i is a generalized plastic modulus coefficient that must be calibrated for every surface, used in the calculation of P.

The main difference among the several NLK hardening models proposed in the literature rests in the equation of the hardening surface translation direction $\vec{v}_i'$, which for most models can be condensed into the general equation [11]

$$\vec{v}_i' = \underbrace{\vec{n}' \cdot \Delta r_i}_{\text{Prager-Ziegler}} - \chi_i{*} \cdot m_i{*} \cdot \gamma_i \cdot [\underbrace{\delta_i \cdot \vec{\beta}_i'}_{\text{dynamic recovery}} + \underbrace{(1-\delta_i)\cdot(\vec{\beta}_i'^T \cdot \vec{n}')\cdot \vec{n}'}_{\text{radial return}}] \tag{7}$$

where the scalar functions χ_i^* and m_i^* are defined as

$$\chi_i{*} \equiv \left(\frac{|\vec{\beta}_i'|}{\Delta r_i}\right)^{\chi_i} \quad \text{and} \quad m_i{*} \equiv \begin{cases} \left[\vec{\beta}_i'^T \cdot \vec{n}' / |\vec{\beta}_i'|\right]^{m_i}, & \text{if } \vec{\beta}_i'^T \cdot \vec{n}' \geq 0 \\ 0, & \text{if } \vec{\beta}_i'^T \cdot \vec{n}' < 0 \end{cases} \tag{8}$$

The calibration parameters for each surface i are the ratcheting exponent χ_i, the multiaxial ratcheting exponent m_i, the ratcheting coefficient γ_i, and the multiaxial ratcheting coefficient δ_i, scalar values that are listed in Tab. 1 for several popular NLK hardening models. Note that several literature references represent the NLK hardening parameters Δr_i, p_i, and χ_i as $r^{(i)}$, $c^{(i)}$, and $\chi^{(i)}$, but this notation is not used in this work to avoid mistaking the *(i)* superscripts for exponents.

Year	*Kinematic hardening model*	χ_i	m_i	γ_i	δ_i
1949	Prager [6]	0	0	0	1
1966	Armstrong-Frederick [4]	0	0	$0 \leq \gamma_i \leq 1$	1
1967	Mróz [1]	0	0	1	1
1983	Chaboche [12]	1	0	1	1
1986	Burlet-Cailletaud [13]	0	0	$0 \leq \gamma_i \leq 1$	0
1993	Ohno-Wang I [14-15]	∞	1	1	1
1993	Ohno-Wang II [14-15]	$0 \leq \chi_i < \infty$	1	1	1
1995	Delobelle [16]	0	0	$0 \leq \gamma_i \leq 1$	$0 \leq \delta_i \leq 1$
1996	Jiang-Sehitoglu [17-18]	$0 \leq \chi_i < \infty$	0	1	1
2004	Chen-Jiao [19]	$0 \leq \chi_i < \infty$	1	1	$0 \leq \delta_i \leq 1$
2005	Chen-Jiao-Kim [20]	$0 \leq \chi_i < \infty$	$-\infty < m_i < \infty$	1	1

Table 1: Calibration parameters for the general translation direction from Eq. (8).

The translation direction $\vec{v}_i'$ of each hardening surface i, shown in Eq. (7), can be separated into three components: (i) the Prager-Ziegler term, in the normal direction $\vec{n}'$ perpendicular to the yield surface at the current stress point $\vec{s}'$; (ii) the dynamic recovery term, in the opposite direction of the backstress component of the considered surface, which acts as a recall term that gradually erases plastic memory with an intensity proportional to the product of the ratcheting terms $\chi_i^* \cdot m_i^* \cdot \gamma_i \cdot \delta_i$; and (iii) the radial return term, in the opposite direction of the normal vector $\vec{n}'$, which mostly affects multiaxial ratcheting predictions. Fig. 4 shows the geometric interpretation of these three components.

TWO-SURFACE KINEMATIC HARDENING MODEL

The two-surface model proposed by Dafalias and Popov [21] and independently by Krieg [22] is an unconventional plasticity model based on the translation of only two moving surfaces: a single hardening surface ($i = 2$), usually called bounding or limit surface, and an inner yield surface ($i = 1$), shown in Fig. 5. The outer failure surface ($i = 3$) is also present, however it does not translate.

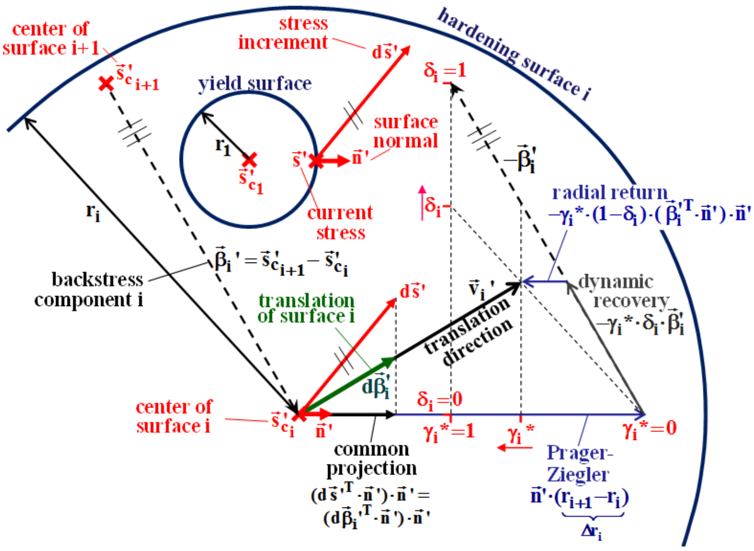

Figure 4: Geometric interpretation of the three components of the translation direction $\vec{v}_i'$ of hardening surface *i*: Prager-Ziegler's, dynamic recovery, and radial return terms, where the equivalent parameter $\gamma_i^* \equiv \chi_i^* \cdot m_i^* \cdot \gamma_i$.

In this model, every time an elastic stress state $\vec{s}'$ reaches the yield surface, the length of the current Mróz translation vector between the yield and bounding surfaces is stored as the initial reference length $v_{in} = |\vec{v}_1'| = |\vec{s}_M' - \vec{s}'|$, where $\vec{s}_M'$ is the Mróz (stress) image point, see Fig. 5. The generalized plastic modulus P is then calculated at each stress increment from the current length $|\vec{v}_1'|$ of the Mróz translation vector and its initial value v_{in} through:

$$P = P_2 + f(v_{in}) \cdot |\vec{v}_1'| / (v_{in} - |\vec{v}_1'|) \qquad (9)$$

where P_2 is the value of P calibrated for the bounding surface, and $f(v_{in})$ is a continuous material function that needs to be calibrated. The generalized plastic modulus P continuously varies in a non-linear way between the elastic value $P = \infty$ and the saturated $P = P_2$, instead of assuming a piecewise-constant value $P = P_i$ (from the active surface $i = i_A$) as in the original Mróz multi-surface formulation. Some implementations [22] of this model adopt a slightly different metric to define generalized plastic modulus P and the initial reference length v_{in}, replacing $|\vec{v}_1'|$ with the projection $\vec{v}_1'^T \cdot \vec{n}'$ onto the yield surface normal vector.

The translation direction of the surface backstress $\vec{\beta}_1'$ is defined by $\vec{v}_1' = \vec{s}_M' - \vec{s}' = \vec{n}' \cdot \Delta r_1 - \vec{\beta}_1'$, the Mróz surface translation rule, while the translation direction of the yield surface can adopt any linear or non-linear hardening rule such as the ones listed in Tab. 1. A non-linear rule is suggested, to avoid the same drawbacks from the Mróz and Garud multi-linear models for unbalanced uniaxial or NP loadings, thus allowing the prediction of ratcheting.

Contrary to the Mróz or NLK multi-surface formulations, the two-surface kinematic hardening model does not adopt a rule to explicitly calculate the surface translation direction between surfaces $i = 2$ and $i = 3$ (respectively the bounding and failure surfaces in the two-surface formulation). The continuous definition of the modulus P from Eq. (9), as opposed to the piecewise-constant $P = P_i$ from the Mróz multi-surface formulation, introduces a non-linear component in the two-surface model that allows it to reasonably predict uniaxial and multiaxial ratcheting under constant amplitude loading. However, the quality of such ratcheting predictions strongly depends on the non-linear kinematic hardening rule employed to define the translation direction of the yield surface [8]. In practice, non-linear kinematic models should be preferred over two-surface models for cyclic unbalanced loadings that cause ratcheting or mean stress relaxation. Due to its computational simplicity, the two-surface model has been widely adopted for the prediction of the deformation behavior of metals under monotonic and constant amplitude loadings. The need for only $M = 2$ moving surfaces makes it attractive to model pressure-sensitive or even anisotropic materials, whose elaborate yield function would result in a high computational cost in Mróz or NLK multi-surface formulations with $M >> 2$. Even though two-surface models are not the best option for unbalanced or variable amplitude loadings, they can provide excellent results for monotonic loading applications, such as in sheet metal forming [23].

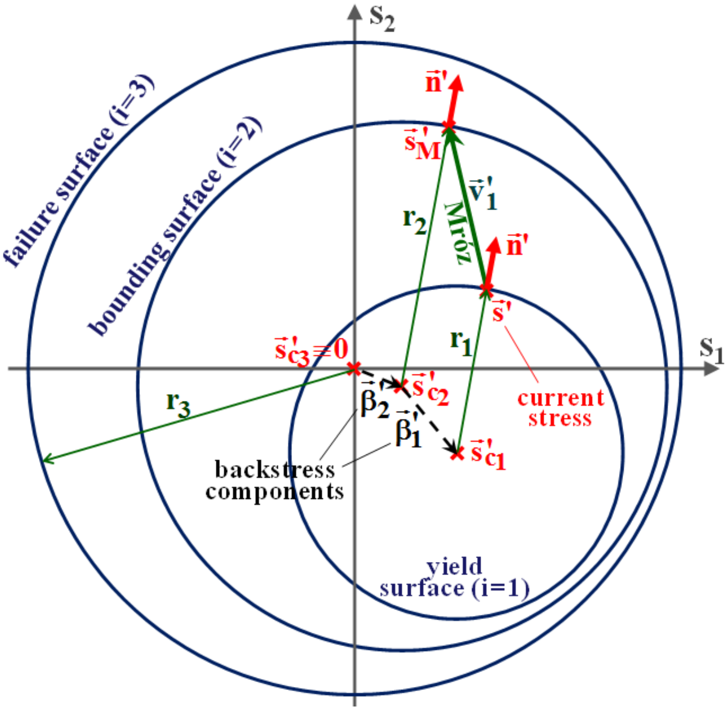

Figure 5: Yield, bounding, and failure surfaces in the s_1-s_2 deviatoric space for the two-surface model.

In summary, the two-surface model and its variations combine elements of both Mróz and NLK multi-surface models, as compared in Tab. 2. Similarly to Mróz multi-surface models, it is an "uncoupled formulation" since the generalized plastic modulus P is not a function of the translation directions, only of the distance $|\vec{v}_i'|$. On the other hand, similarly to NLK multi-surface models, the yield and all hardening surfaces translate, the surface translation direction can (and should) use a non-linear equation, and the generalized plastic modulus P continuously varies instead of being assigned piecewise-constant values.

	Mróz multi-surface	*two-surface*	*NLK multi-surface*
Number M of yield and hardening surfaces:	$M \geq 2$	$M = 2$	$M \geq 1$
Surface translation during plastic straining:	no translation outside active surface	the yield and all hardening surfaces (including the bounding surface) translate	
Surface translation direction $\vec{v}_i'$:	defined by linear rules such as Mróz	Mróz for $d\vec{\beta}_1'$; linear or non-linear rule for $d\vec{\beta}_1' + d\vec{\beta}_2'$	defined by non-linear rules
Surface backstress variation $d\vec{\beta}_1'$ during plastic straining:	$d\vec{\beta}_1'$ from the active surface $i = i_A$ is the only $d\vec{\beta}_i' \neq 0$	all surface backstress increments $d\vec{\beta}_1'$ (for $i = 1, 2, \ldots, M$) are different than zero	
Generalized plastic modulus P:	piecewise-constant $P = P_i$ from the active surface $i = i_A$	non-linear and continuously varying, calculated from relative positions among the yield and all hardening surfaces	
Consistency condition that prevents $\vec{s}'$ from moving outside any surface:	used to calculate the backstress increments $d\vec{\beta}_1'$ and associated surface translations $d\vec{s}_{c_i}'$		used to calculate the non-linear P

Table 2: Comparison among the Mróz multi-surface, two-surface, and NLK multi-surface model formulations, to predict multiaxial kinematic hardening effects.

CONCLUSIONS

In this work, it is concluded that the Mróz-Garud multi-surface formulation can lead to very poor multiaxial stress-strain predictions in the presence of significant mean stresses, severely limiting their application in multiaxial fatigue analysis. The two-surface model better accounts for unbalanced loadings, however its application is mostly recommended for monotonic plastic processes, since it does not appropriately deal with complex variable amplitude loading histories, with decreasing amplitudes and several layers of hysteresis loops within loops, which are very common under spectrum loading. On the other hand, NLK models can accurately deal with non-proportional loadings, either balanced or unbalanced. Moreover, contrary to two-surface models, they deal with complex variable amplitude spectrum loading, as long as a sufficiently high number of backstress components (i.e. number of hardening surfaces) is adopted in a refined NLK formulation, to efficiently store plastic memory effects. Therefore, this analysis shows that NLK models should be preferred over multi-surface and two-surface models in multiaxial fatigue calculations under variable amplitude loading.

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