Weight Function Method for computations of crack face displacements and stress intensity factors of center cracks

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ABSTRACT. The weight function method provides a powerful and reliable tool for the determination of the stress intensity factor around the crack tip in a linearly elastic cracked solid subjected to arbitrary loading conditions. However, it is difficult to exactly compute the crack face displacement whose partial derivative is responsible for the weight function calculation. In the present paper, only one reference stress intensity factor is used for the purpose of establishing a general expression of the crack face displacement. Then, the generalized and simple expression is applied to calculate the weight function and the stress intensity factor of the center crack configuration. The calculation of the weight function is reduced to the simple integration of the correction function and of the partial derivative of the crack face displacement. It is shown that the present expressions for the computations of the crack face displacement and its partial derivative are in good agreement with their exact solutions.

KEYWORDS. Weight function; Crack face displacement; Stress intensity factor; Center crack configuration.

INTRODUCTION

As is known, the stress intensity factors (SIFs) dominate the singular stress states around the crack tip. Thus, the calculation of the SIFs is of quite importance for assessing the load capacity, fatigue crack growth rate and fracture failure control of a cracked component. Although the SIFs are possibly available in some of the SIF handbooks [1], the documented solutions are sometimes inapplicable in practical problems due to complicated non-linear stress fields. The weight function method provides a reliable method to calculate the SIF around a crack tip in a 2D linearly elastic body subjected to any arbitrarily chosen load systems [2]. Wu and Carlsson [3] presented the generalized weight function method based on the physical judgment on the shape of the opened crack, and subsequently derived the SIFs for center cracks in finite width plates with mixed boundary conditions. Shen and Glinka [4] used two linearly independent reference stress intensity factors together with the characteristic properties of the weight function to determine the three unknown parameters in the universal weight function. Overall, the weight function method makes it possible to determine exact and reliable stress intensity factors which are useful for the evaluation of fatigue crack growth and residual strength of aircraft structures in service.

Nevertheless, the weight function is strongly dependent on the solution of the crack face displacement whose functional dependence on the crack is undetermined. Therefore, it is necessary to make appropriate assumption on the opened crack shape. The present paper is aimed at developing an approximate and simple expression of the crack face displacement for collinear cracks and center crack subjected to mode I loading conditions.

WEIGHT FUNCTION METHOD

The weight function method has been widely applied to determine the SIFs of cracked structures since it is able to take complex loading conditions into consideration. It has been shown that, if the SIF $K(a)^{(1)}$ and the crack face displacement $u^{(1)}(x, a)$ of any linearly elastic cracked solid are known as functions of the crack length a for a symmetrical load system (1), then for the same cracked solid subjected to any other symmetrical load system (2), the SIF $K(a)^{(2)}$ can be obtained by the simple integration of the weight function $h(x, a)$ and the stress function $\sigma^{(2)}(x)$:

$$K(a)^{(2)} = \int_0^a \sigma^{(2)}(x) \cdot h(x,a) dx \tag{1}$$

where the weight function, independent of $\sigma^{(2)}(x)$, is defined as:

$$h(x,a) = \frac{H}{K(a)^{(1)}} \frac{\partial u^{(1)}(x,a)}{\partial a} \tag{2}$$

In Eq. (1) and (2), a is the half or full crack length for edge cracks and center cracks, respectively; H is a material constant, $H=E$ for plane stress condition and $H=E/(1-\nu^2)$ for plane strain condition with E the Young's modulus and ν the Poisson's ratio; $K(a)^{(1)}$ and $u^{(1)}(x, a)$ are, respectively, the known reference stress intensity factor and the crack face displacement in mode I loading conditions for the known load system (1); and $\sigma^{(2)}(x)$ is the stress distribution function across the plane of the crack in the crack free solid subjected to the load system (2).

From Eq. (1), it is known that if the weight function is set to be a known function for a particular crack geometry, the SIF for any stress distribution is able to be calculated by integrating Eq. (1). Therefore, the main problem is to exactly determine the weight function $h(x,a)$ and the corresponding partial derivative $\partial u^{(1)}(x,a)/\partial a$ of the crack face displacement. For a reference SIF $K(a)^{(1)}$, it is possibly available in the SIF handbooks for a wide range of crack configurations and loading conditions. However, there are often no detailed databases about the crack face displacement $u^{(1)}(x, a)$, and it is difficult to rigorously calculate $u^{(1)}(x, a)$ since the functional dependence of $u^{(1)}(x, a)$ on both x and a is undetermined. Thus, approximate $u^{(1)}(x, a)$-solutions across the crack line of a cracked solid are of primary importance for determining the weight function $h(x, a)$ and stress intensity factor $K(a)^{(2)}$ [5].

CRACK OPENING DISPLACEMENTS ACROSS THE CENTER CRACK

In Eq. (1) and (2), if $K(a)^{(1)}=K(a)^{(2)}=K(a)$, and $\sigma^{(1)}(x)= \sigma^{(2)}(x)= \sigma(x)$. Then, substituting (2) into (1), it leads to:

$$K(a)^2 = H\int_0^a \sigma(x) \cdot \frac{\partial u(x,a)}{\partial a} dx \tag{3}$$

So, the crack face displacement $u(x, a)$ in eqn. (3) is the only unknown function dependent on the crack line x and half of the crack length a. As is well known that $u(x=a, a)=0$ for any crack tip. As a result, Eq. (3) becomes:

$$K(a)^2 = H\frac{\partial}{\partial a}\int_0^a \sigma(x) \cdot u(x,a) dx \tag{4}$$

Integrating Eq. (4) over the half crack length a, we have:

$$\int_0^a K(a)^2 da = H\int_0^a \sigma(x) \cdot u(x,a) dx \tag{5}$$

Based on the previous works [3], the crack face displacement $u(x, a)$ is assumed to be dependent on x and a:

$$u(x,a) = u_0(a)u_0^*\left(\frac{x}{a}\right) + u_1(a)u_1^*\left(\frac{x}{a}\right) \tag{6}$$

where $u_0(a)$ and $u_1(a)$ are the unknown functions relative to the half crack length a; $u_0^*(x/a)$ describes the deformation of the center crack; $u_1^*(x/a)$ is assumed to be the higher order of $u_0^*(x/a)$.

To approximately determine the crack face displacements $u(x, a)$ across the whole crack line of a central through crack, $u(x, a)$ should also follow the given criterions below [3, 5]:

(I) exhibiting proper limiting behavior near the crack tip;

(II) deforming as a shape of the ellipse when the cracked solid is subjected to a remotely uniform stress field;

(III) demonstrating the consistent behavior for the small crack;

(IV) $\frac{\partial u(x,a)}{\partial a}\Big|_{x=0} = 0$.

If an infinite solid with a central through crack of the length $2a$ is subjected to a remotely uniform tensile stress filed σ_0, the crack face displacements and the SIF are, respectively, presented as:

$$u_y(x,a) = \pm\frac{2\sigma_0}{H}\sqrt{\xi(2a-\xi)} \quad \text{and} \quad K(a) = \sigma_0\sqrt{\pi a} \tag{7}$$

where $\xi=0$ is the coordinate with its origin at one of the crack tips, and $\xi=2a$ is the other crack tip.

Based on eqn. (7), if a finite solid, with a center crack of the length $2a$, is also undergoing a remotely uniform tensile stress σ_0, the crack face displacements and the SIF are able to be written as:

$$u_y(x,a) = \pm\frac{2f(a)\sigma_0}{H}\sqrt{(a+x)(a-x)} \quad \text{and} \quad K(a) = f(a)\sigma_0\sqrt{\pi a} \tag{8}$$

where x is the coordinate with its origin at the crack center; $x=\pm a$ are set to be the two crack tips of the crack; $2b$ is the width of the elastic cracked solid; $f(a)$ is defined as the correction function which is dependent on the crack geometry and the size of the cracked solid.

In the criterion (II), it is assumed that the crack face displacement of the center crack deforms as a shape of the ellipse when the cracked body is subjected to a uniformly distributed stress field perpendicular to the crack line. So, the shape of the opened crack can be expressed as an elliptic function:

$$u_0^*\left(\frac{x}{a}\right) = \sqrt{1-\left(x/a\right)^2} \tag{9}$$

As a consequence, from Eq. (8) and (9) the first term in the right hand side of Eq. (6) is determined as:

$$u_0(a)u_0^*\left(\frac{x}{a}\right) = \frac{2f(a)\sigma_0 a}{H}\sqrt{1-\left(\frac{x}{a}\right)^2} \tag{10}$$

Here, the first term $u(x, a)$ satisfies the criterions (I) and (II). So, the second term $u(x, a)$ should make the full expression of $u(x, a)$ satisfy all the four criterions. Upon that, $u_1^*\left(\frac{x}{a}\right)$ is taken as a higher order of $u_0^*\left(\frac{x}{a}\right)$:

$$u_1^*\left(\frac{x}{a}\right) = \left[1-\left(x/a\right)^2\right]^{3/2} \tag{11}$$

To make $u(x, a)$ demonstrate the consistent behaviour for the small crack, $u_1(a)$ should have the characteristic properties: $u_1(a)=0(1/a)$ if the half crack length a tends to be zero. Based on the above criterions, an approximate and simple expression of the crack face displacement is derived as:

$$u(x,a)=u_0(a)u_0^*\left(\frac{x}{a}\right)+u_1(a)u_1^*\left(\frac{x}{a}\right)=\frac{2f(a)\sigma_0 a}{H}\sqrt{1-\left(\frac{x}{a}\right)^2}+\frac{g(a)}{a}\left[1-\left(\frac{x}{a}\right)^2\right]^{3/2} \tag{12}$$

where $g(a)$ is the only unknown function of the half crack length a.
Substituting eqn. (12) into (5), and assuming the stress function $\sigma(x)$ is equal to a constant value σ_0, it leads to:

$$\pi\sigma_0^2\int_0^a f(a)^2\, ada=H\left[\int_0^a\frac{2f(a)\sigma_0^2 a}{H}\sqrt{1-\left(\frac{x}{a}\right)^2}dx+\int_0^a\frac{\sigma_0 g(a)}{a}\left[1-\left(\frac{x}{a}\right)^2\right]^{3/2}dx\right] \tag{13}$$

and solving Eq. (13), the unknown function $g(a)$ is determined as:

$$g(a)=16\sigma_0\int_0^a f(a)^2\, ada-8\sigma_0 f(a)a^2\Big/(3H) \tag{14}$$

Finally, substituting eqn. (14) into (12), the fully expression of the crack face displacements is derived as:

$$u(x,a)=\frac{2f(a)\sigma_0 a}{H}\sqrt{1-\left(\frac{x}{a}\right)^2}+\frac{16\sigma_0\int_0^a f(a)^2\, ada-8\sigma_0 f(a)a^2}{3Ha}\left[1-\left(\frac{x}{a}\right)^2\right]^{3/2} \tag{15}$$

RESULTS AND DISCUSSIONS

Calculations of $u(x,a)$ and $\partial u(x,a)/\partial a$ for collinear cracks

To check the accuracy of the expression for $u(x, a)$ of the central through crack, an array of collinear cracks in an infinite plate, subjected to a uniformly tensile stress field σ_0, is taken into account. So, the correction function $f(a)$ is:

$$f(a)=\sqrt{\frac{2b}{\pi a}\tan\left(\frac{\pi a}{2b}\right)} \tag{16}$$

where $2a$ is the full crack length; and $2b$ is set as the distance between the two adjacent crack center lines.
Substituting eqn. (16) into (15) and simplifying the expression, the dimensionless displacement is determined as:

$$\begin{aligned}\frac{Hu(x,a)}{\sigma_0 b}&=2\sqrt{\frac{2a}{\pi b}\tan\left(\frac{\pi a}{2b}\right)}\sqrt{1-\left(x/a\right)^2}\\&+\left[\frac{32}{3a\pi}\int_0^a\tan\left(\frac{\pi a}{2b}\right)da-\frac{8}{3}\sqrt{\frac{2a}{\pi b}\tan\left(\frac{\pi a}{2b}\right)}\right]\left[1-\left(x/a\right)^2\right]^{3/2}\end{aligned} \tag{17a}$$

Also, a generalized formula for the crack face displacements has been given by Wu and Carlsson [3]:

$$\frac{Hu(x,a)}{\sigma_0 b}=\frac{4b}{a}\left(\frac{2}{\pi}\right)^2\left[-\ln\left(\cos\left(\frac{\pi a}{2b}\right)\right)\right]\sqrt{1-\left(\frac{x}{a}\right)^2} \tag{18a}$$

For the same situation, the exact $u(x, a)$-solution was presented as [3]:

$$\frac{Hu(x,a)}{\sigma_0 b}=\frac{4}{\pi}\left\{\ln\left[\cos\left(\frac{\pi x}{2b}\right)+\sqrt{\cos^2\left(\frac{\pi x}{2b}\right)-\cos^2\left(\frac{\pi a}{2b}\right)}\right]-\ln\cos\left(\frac{\pi a}{2b}\right)\right\} \tag{19a}$$

Fig.1(a) presents the variations of the dimensionless crack face displacements $Hu(x, a)/(\sigma_0 b)$ for collinear cracks in an infinite plate, respectively, determined by Eq. (17a), (18a) and (19a). It is clearly shown that if the values of $a/b \le 0.5$, values of $Hu(x, a)/(\sigma_0 b)$ calculated through Eq. (17a) and (18a) are in good agreement with the exact $u(x, a)$-solutions given by eqn. (19a). For this situation, the maximum difference between the results by Eq. (18a) and (19a) is about 0.359% occurred at a/b=0.5 and x/a=0.95; but the maximum difference between Eq. (17a) and (19a) is only 0.0071% when a/b=0.5 and x/a=0. Once the values of $a/b \ge 0.7$, the differences between the crack face displacements by Eq. (18a) and (19a) increase sharply from 1.9% (a/b=0.7 and x/a=0.95) to 17.06% (a/b=0.95 and x/a=0.95). But, the differences between Eq. (17a) and (19a) are still in the small range of 0.092% to 5.1%. Therefore, it is concluded that the present expression of the crack face displacement for collinear cracks can give much better $u(x, a)$-solutions than that calculated by eqn. (18a) since the higher order term $\sqrt{[1-(x/a)^2]}$ is taken into account. Fig.1(b) gives the relationship between $Hu(x, a)/(\sigma_0 b)$ and a/b-values for collinear cracks according to eqn.(17a). It is then seen that the crack face displacements increase with the increasing values of a/b.

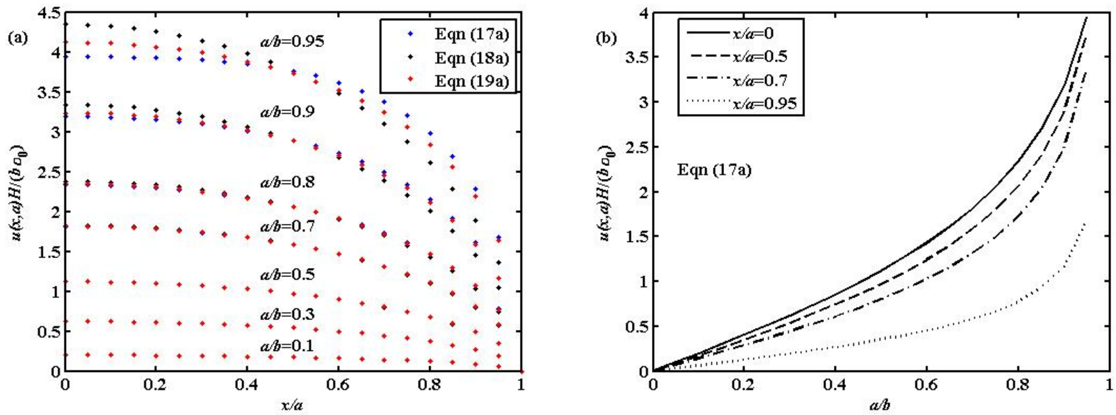

Figure1: Comparisons of the dimensionless crack face displacements for collinear cracks (a) $Hu(x, a)/(\sigma_0 b)$ *versus* x/a; (b) $Hu(x, a)/(\sigma_0 b)$ *versus* a/b.

According to Eq. (17a), (18a) and (19a), the dimensionless partial derivatives of u(x, a) for an array of collinear cracks are, respectively, derived as (17b), (18b) and (19b) below:

$$\begin{aligned}\frac{H\partial u(x,a)}{\sigma_0\partial a}=&\,2\left(\frac{x}{a}\right)^2\frac{\sqrt{\frac{2b}{\pi a}\tan\left(\frac{\pi a}{2b}\right)}}{\sqrt{1-\left(\frac{x}{a}\right)^2}}+\sqrt{1-\left(\frac{x}{a}\right)^2}\left\{\frac{8}{a^2}\left(\frac{2bx}{\pi a}\right)^2\ln\left[\tan^2\left(\frac{\pi a}{2b}\right)+1\right]\right\}\\&+\sqrt{1-\left(\frac{x}{a}\right)^2}\,\frac{\frac{\pi a}{2b}\left[\tan^2\left(\frac{\pi a}{2b}\right)+1\right]+\left(1-\frac{8x^2}{a^2}\right)\tan\left(\frac{\pi a}{2b}\right)}{\sqrt{\frac{\pi a}{2b}\tan\left(\frac{\pi a}{2b}\right)}}+\left[1-\left(\frac{x}{a}\right)^2\right]^{3/2}\left[\frac{32b}{3\pi a}\tan\left(\frac{\pi a}{2b}\right)\right]\\&-\left[1-\left(\frac{x}{a}\right)^2\right]^{3/2}\left\{\left(\frac{4\sqrt{2b}}{\sqrt{3\pi a}}\right)^2\ln\left[\tan^2\left(\frac{\pi a}{2b}\right)+1\right]+\frac{\frac{\pi a}{2b}\left[\tan^2\left(\frac{\pi a}{2b}\right)+1\right]+\tan\left(\frac{\pi a}{2b}\right)}{\frac{3}{4}\sqrt{\frac{\pi a}{2b}\tan\left(\frac{\pi a}{2b}\right)}}\right\}\end{aligned} \tag{17b}$$

$$\frac{H\partial u(x,a)}{\sigma_0\partial a}=\left(\frac{4b}{\pi a}\right)^2\ln\left[\cos\left(\frac{\pi a}{2b}\right)\right]\left[\frac{1-2\left(x/a\right)^2}{\sqrt{1-\left(x/a\right)^2}}\right]+\frac{8b}{\pi a}\tan\left(\frac{\pi a}{2b}\right)\sqrt{1-\left(x/a\right)^2} \tag{18b}$$

$$\frac{H\partial u(x,a)}{\sigma_0\partial a}=2\tan\left(\frac{\pi a}{2b}\right)+\frac{\sin\left(\frac{\pi a}{b}\right)}{\left[\cos\left(\frac{\pi x}{2b}\right)+\sqrt{\cos^2\left(\frac{\pi x}{2b}\right)-\cos^2\left(\frac{\pi a}{2b}\right)}\right]\sqrt{\cos^2\left(\frac{\pi x}{2b}\right)-\cos^2\left(\frac{\pi a}{2b}\right)}} \tag{19b}$$

Fig.2 shows the variations of the dimensionless partial derivatives $\partial u(x,a)/\partial a$ which is used for calculating the weight function $h(x, a)$. Through Fig.2 (a), it is clearly found that for the values of $a/b\leq 0.5$, the dimensionless derivatives, separately, obtained by Eq. (17b) and (18b) agree quite well with those from the exact $u(x, a)$-solutions presented in Eq. (19b). For $0.7\leq a/b\leq 0.9$, the predicted values of $\partial u(x,a)/\partial a$ by Eq. (17b) can give lower errors than Eq. (18b) due to the consideration of the higher order of $\sqrt{1-(x/a)^2}$. From Fig.2 (b), it is obviously observed that once the values of $a/b>0.9$, great differences are presented between the partial derivatives determined by Eq. (17b) and (18b) to (19b).

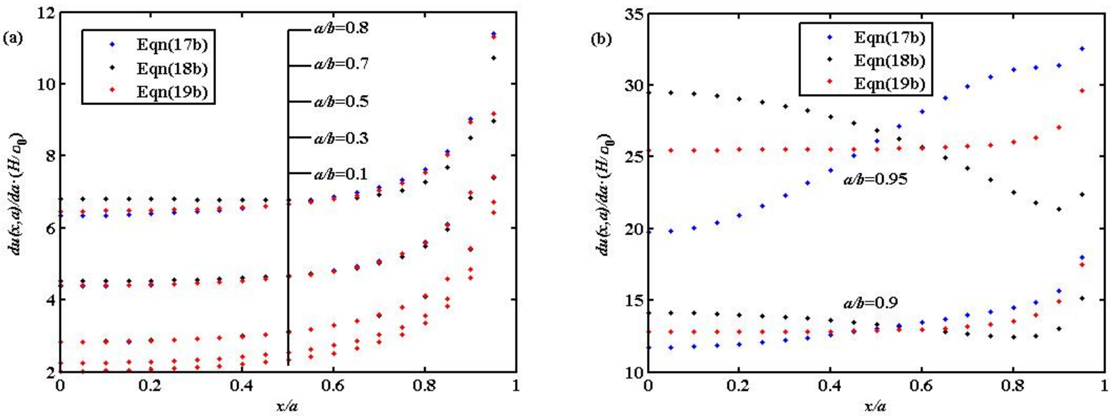

Figure 2: Dimensionless partial derivatives of collinear cracks (a) $0\leq a/b\leq 0.8$; (b) $0.95\geq a/b\geq 0.9$.

Calculations of weight functions and stress intensity factors

To calculate the weight function through Eq. (2), the only unknown information is $K(a)^{(1)}$. To make the problem easier, it is better to choose the uniformly distributed stress field as the reference load system (1) since the basic assumption relevant to the crack shape is most suited for this type of loading condition. Therefore, the formula for computing the weight function of a finite width plate is in the form given below:

$$h(x,a)=\frac{H}{f(a)\sigma_0\sqrt{\pi a}}\frac{\partial u(x,a)}{\partial a} \tag{20}$$

For a finite width plate with a central through crack, the correction function, dependent on the crack length $2a$ and the width of the finite plate $2b$, is given as [1]:

$$f(a)=1+0.128p-0.288p^2+1.525p^3 \tag{21}$$

where $p=a/b$ is defined as the ratio between the crack length $2a$ to the width of the plate $2b$.
Substituting (21) into (15), the partial derivative $u(x, a)$ for the center crack is solved as the function of a and x:

$$
\begin{aligned}
\frac{\partial u(x,a)}{\partial a} = \frac{2\sigma_0}{H}&\left\{\left[f(a)+f'(a)a\right]\sqrt{1-\left(\frac{x}{a}\right)^2}+f(a)\left(\frac{x}{a}\right)^2\Big/\sqrt{1-\left(\frac{x}{a}\right)^2}\right\} \\
&+\frac{16\sigma_0}{3H}\left\{\frac{3G(a)}{a}\left[1-\left(\frac{x}{a}\right)^2\right]+\left[\frac{G'(a)}{a}-\frac{G(a)}{a^2}\right]\left[1-\left(\frac{x}{a}\right)^2\right]^{3/2}\right\} \\
&-\frac{8\sigma_0}{3H}\left\{3f(a)\left(\frac{x}{a}\right)^2\left[1-\left(\frac{x}{a}\right)^2\right]+\left[f'(a)a+f(a)\right]\left[1-\left(\frac{x}{a}\right)^2\right]^{3/2}\right\}
\end{aligned} \tag{22}
$$

where $f'(a)$, $G(a)$ and $G(a)$ are, respectively, determined as:

$$
\begin{cases}
f'(a)=\dfrac{1}{b}\left(0.128-0.576p+4.575p^2\right) \\
G(a)=a^2\left(0.5b^6+0.0853p-0.1399p^2+0.5953p^3+0.0789p^4-0.1255p^5+0.2907p^6\right) \\
G'(a)=2a\left(0.5b^6+0.0853p-0.1399p^2+0.5953p^3+0.0789p^4-0.1255p^5+0.2907p^6\right) \\
\qquad +a\left(0.0853p-0.2798p^2+1.7858p^3+0.3156p^4-0.6274p^5+1.7442p^6\right)
\end{cases} \tag{23}
$$

Subsequently, substituting (22) into (20), the weight function $h(x, a)$ for the center crack in a finite plate by the present method is obtained and written as:

$$
\begin{aligned}
h(x,a)\sqrt{\pi a}=\frac{H}{f(a)\sigma_0}\frac{\partial u(x,a)}{\partial a}=2&\left\{\left[1+\frac{f'(a)a}{f(a)}\right]\sqrt{1-\left(\frac{x}{a}\right)^2}+\left(\frac{x}{a}\right)^2\Big/\sqrt{1-\left(\frac{x}{a}\right)^2}\right\} \\
&+\frac{16}{3}\left\{\frac{3G(a)}{f(a)a}\left[1-\left(\frac{x}{a}\right)^2\right]+\left[\frac{G'(a)}{f(a)a}-\frac{G(a)}{f(a)a^2}\right]\left[1-\left(\frac{x}{a}\right)^2\right]^{3/2}\right\} \\
&-\frac{8}{3}\left\{3\left(\frac{x}{a}\right)^2\left[1-\left(\frac{x}{a}\right)^2\right]+\left[\frac{f'(a)a}{f(a)}+1\right]\left[1-\left(\frac{x}{a}\right)^2\right]^{3/2}\right\}
\end{aligned} \tag{24}
$$

The weight function for a central through crack in an infinitely wide plate has been derived by Paris and Sih [4]:

$$
h(x,a)\sqrt{\pi a}=\sqrt{\frac{a+x}{a-x}}+\sqrt{\frac{a-x}{a+x}} \tag{25}
$$

From Eq. (25), if x/a-value tends to be zero, the dimensionless weight function $h(x, a)\sqrt{(\pi a)}$ tends to be a constant value of 2. This is the limiting condition for collinear cracks in an infinite plate and for center cracks in a finite plate. Based on Eq. (17b), (24) and (25), Fig.3(a) shows the variations of the weight function $h(x, a)\sqrt{(\pi a)}$ for collinear cracks in an infinite plate and for center cracks in a finite plate at the center of the crack $x/a=0$. According to Eq. (1), (24) and (25), the SIFs for center cracks in mode I conditions are computed and presented in Fig.3(b). It is obviously found that the Eq. (25) used for crack problems will result in significant error for the calculations of weight functions and SIFs in finite plates since the correction function has not been considered.

CONCLUSIONS

A general approach has been presented to calculate the crack face displacement of collinear cracks in an infinite plate and of center cracks in a finite plate. The approximate and simple expression is able to be determined based on only one reference stress intensity factor and some criterions around the crack tip. The crack face displacement and its partial derivative, respectively, computed by the present expression and the exact solution are confirmed to be in good agreement if the values of $a/b \leq 0.8$. The calculation of the weight function for center cracks in mode I loading conditions is reduced to the simple quadrature of the correction function and of the partial derivative of the crack face displacement. It is concluded that the limiting value of the stress intensity factor in an infinite plate might introduce significant error into the weight function and stress intensity factor evaluation.

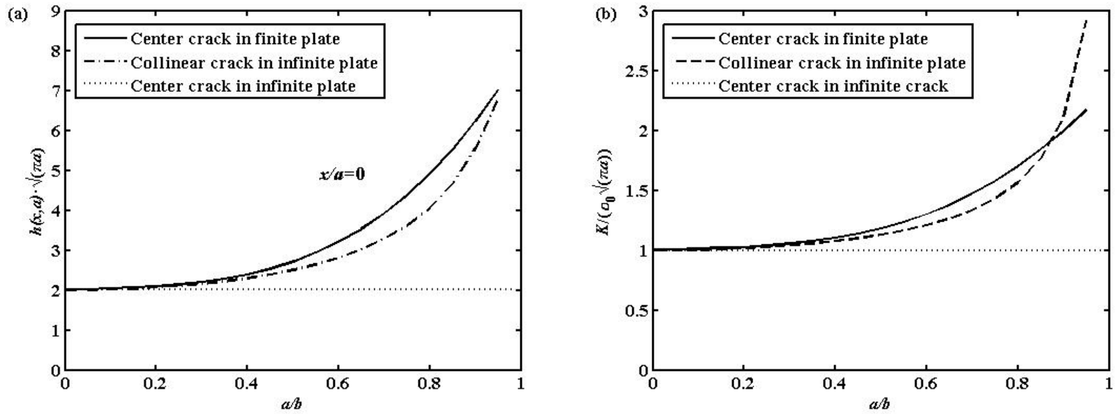

Figure 3: Comparisons of collinear cracks and center crack (a) weight function variation; (b) SIF variation.

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