An Alternative Proof of the Duality Theorem for Crossed Products of Hilbert C*-Modules by Abelian Group Actions

AN ALTERNATIVE PROOF OF THE DUALITY THEOREM FOR CROSSED PRODUCTS OF HILBERT C^*-MODULES BY ABELIAN GROUP ACTIONS

Masaharu KUSUDA*

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Abstract

We give an alternative proof of the duality theorem for crossed products of Hilbert C^*-modules by abelian group actions by using the duality theorem for crossed products of Hilbert C^*-modules by coactions.

1. Introduction

Let (A, G, α) be a C^*-dynamical system, that is, a triple (A, G, α) consisting of a C^*-algebra A, a locally compact group G with left invariant Haar measure ds and a group homomorphism α from G into the automorphism group of A such that $G \ni t \to \alpha_t(x)$ is continuous for each x in A in the norm topology. Denote by $L^1(A, G)$ the Banach*-algebra of all Bochner integrable A-valued functions on G (see [4, 7.6] for the Banach*-algebra structure). Then the C^*-crossed product $A \times_\alpha G$ of A by G is the enveloping C^*-algebra of $L^1(A, G)$, and we denote by $A \times_{\alpha,r} G$ the reduced crossed product which is a certain quotient of $A \times_\alpha G$. Suppose that X is an A-Hilbert module with an α-compatible action η of G. Let $\mathcal{C}(L^2(G))$ be the set of all compact operators on $L^2(G)$ and let $X \otimes \mathcal{C}(L^2(G))$ be the external tensor product of X and $\mathcal{C}(L^2(G))$, which is an $A \otimes \mathcal{C}(L^2(G))$-Hilbert module, where we always take the minimum C^*-tensor product for C^*-algebras.

In [2, Theorem 3.6], the author has proved the first duality theorem:

Theorem (Duality I). *If G is abelian, then there exists the dual action $\widehat{\eta}$ of the dual group $\widehat{G}$ of G on the crossed product $X \times_\eta G$ such that the $\left((A \times_\alpha G) \times_{\widehat{\alpha}} \widehat{G}\right)$-Hilbert module $(X \times_\eta G) \times_{\widehat{\eta}} \widehat{G}$ is isomorphic to the $(A \otimes \mathcal{C}(L^2(G)))$-Hilbert module $X \otimes \mathcal{C}(L^2(G))$.*

Further in the same paper, he also has proved the second duality theorem:

Theorem (Duality II). *If G is a locally compact group, then there exist a coaction δ_A of G on the reduced crossed product $A \times_{\alpha,r} G$ and a coaction δ_X of G on the reduced crossed product $X \times_{\eta,r} G$ such that the $((A \times_{\alpha,r} G) \times_{\delta_A} G)$-Hilbert module $(X \times_{\eta,r} G) \times_{\delta_X} G$ is isomorphic to the $(A \otimes \mathcal{C}(L^2(G)))$-Hilbert module $X \otimes \mathcal{C}(L^2(G))$.*

These theorems were proved by the author to be mutually independent. The purpose of this paper is to give an alternative proof of the first duality theorem by using the second

* Department of Mathematics

duality theorem. One merit of the alternative proof to be presented is that the proof is much shorter and much simpler than the original one. However the relation between the dual action $\widehat{\widehat{\eta}}$ of $\widehat{\eta}$ on $(X \times_\eta G) \times_{\widehat{\eta}} \widehat{G}$ and the action $\eta \otimes \mathrm{Ad}\rho$ on $X \otimes \mathcal{C}(L^2(G))$ does not follow immediately from the proof, where ρ is the right regular representation of G on $L^2(G)$. On the other hand, one merit of the original proof is that the relation between $\widehat{\widehat{\eta}}$ and $\eta \otimes \mathrm{Ad}\rho$ follows easily from the proof. Nevertheless, in almost all applications, it would be sufficient only to use that $(X \times_\eta G) \times_{\widehat{\eta}} \widehat{G}$ is isomorphic to $X \otimes \mathcal{C}(L^2(G))$.

2. Notation and Preliminaries

First recall the definition of a Hilbert C^*-module. Let A be a C^*-algebra. By a *left Hilbert A-module* (or a *left A-Hilbert module*), we mean a left A-module X equipped with an A-valued pairing $\langle \cdot \, , \, \cdot \rangle$ (called an A-valued inner product), which satisfies the following conditions:

(H1) $\langle \cdot \, , \, \cdot \rangle$ is sesquilinear. (We make the convention that $\langle \cdot \, , \, \cdot \rangle$ is linear in the first variable and is conjugate-linear in the second variable.)

(H2) $\langle x \, , \, y \rangle = \langle y \, , \, x \rangle^*$ for all $x, y \in X$.

(H3) $\langle ax \, , \, y \rangle = a\langle x \, , \, y \rangle$ for all $x, y \in X$ and all $a \in A$.

(H4) $\langle x \, , \, x \rangle \geqq 0$ for all $x \in X$, and $\langle x \, , \, x \rangle = 0$ implies that $x = 0$.

(H5) X is a Banach space with respect to the norm $\|x\| = \|\langle x \, , \, x \rangle\|^{\frac{1}{2}}$.

Let B be a C^*-algebra. Right Hilbert B-modules are defined similarly except that we require that B should act on the right of X, that the B-valued inner product $\langle \cdot \, , \, \cdot \rangle$ should be conjugate-linear in the first variable, and that $\langle x \, , \, yb \rangle = \langle x \, , \, y \rangle b$ for all $x, y \in X$ and all $b \in B$.

A *representation* of a left A- and right B-Hilbert module X is a triple (π_A, π_X, π_B) consisting of nondegenerate representations π_A and π_B of A and B on Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, together with a linear map $\pi_X : X \to \mathcal{B}(\mathcal{H}_B, \mathcal{H}_A)$ such that

(R1) $\pi_X(ax) = \pi_A(a)\pi_X(x)$ and $\pi_X(xb) = \pi_X(x)\pi_B(b)$,

(R2) $\pi_A({}_A\langle x, y \rangle) = \pi_X(x)\pi_X(y)^*$ and $\pi_B(\langle x, y \rangle_B) = \pi_X(x)^*\pi_X(y)$

for all $a \in A, x, y \in X$, and $b \in B$, where $\mathcal{B}(\mathcal{H}_B, \mathcal{H}_A)$ denotes the set of all bounded linear operators from $\mathcal{H}_B$ into $\mathcal{H}_A$.

Let (A, G, α) and (B, G, β) be C^*-dynamical systems. Suppose that η is an α-compatible and β-compatible action of G on a left A- and right B-Hilbert module X, that is, η is a group homomorphism from G into the group of invertible linear transformations on X such that

(E1) $\eta_t(a \cdot x) = \alpha_t(a)\eta_t(x)$ and $\eta_t(x \cdot b) = \eta_t(x)\beta_t(b)$;

(E2) ${}_A\langle \eta_t(x), \eta_t(y) \rangle = \alpha_t({}_A\langle x, y \rangle)$ and $\langle \eta_t(x), \eta_t(y) \rangle_B = \beta_t(\langle x, y \rangle_B)$

for each $t \in G, a \in A, b \in B, x, y \in X$; and such that $t \to \eta_t(x)$ is continuous from G into X for each $x \in X$ in norm.

Then there exists a left $(A \times_\alpha G)$- and right $(B \times_\beta G)$- Hilbert module $X \times_\eta G$

containing a dense subspace $K(X,G)$ such that

$$(f\cdot x)(s) = \int_G f(t)\eta_t(x(t^{-1}s))dt,$$

$$(x\cdot g)(s) = \int_G x(t)\beta_t(g(t^{-1}s))dt,$$

$${}_{A\times_\alpha G}\langle x\ ,\ y\rangle(s) = \int_G {}_A\langle x(st^{-1})\ ,\ \eta_s(y(t^{-1}))\rangle dt,$$

$$\langle x,y\rangle_{B\times_\beta G}(s) = \int_G \beta_{t^{-1}}(\langle x(t)\ ,\ y(ts)\rangle_B)dt$$

for $f\in K(A,G), x,y\in K(X,G)$, and $g\in K(B,G)$. We call $X\times_\eta G$ the *(full) crossed product* of X by G. Here $K(X,G)$ (resp. $K(A,G)$ and $K(B,G)$) denotes the set of continuous functions from G into X (resp. A and B) with compact support.

From now on, without loss of generality we may suppose that X is a right Hilbert A-module with the A-inner product $\langle\cdot\ ,\ \cdot\rangle$. We define a linear operator $\Theta_{x,y}$ on X by

$$\Theta_{x,y}(z) = x\cdot\langle y\ ,\ z\rangle$$

for all $x,y,z\in X$. We denote by $\mathcal{K}(X)$ the C^*-algebra generated by the set $\{\Theta_{x,y}\mid x\, y\in X\ \}$. Then X is a left $\mathcal{K}(X)$-Hilbert module with respect to the natural left action defined by $t\cdot x = t(x)$ for $t\in\mathcal{K}(X)$ and $x\in X$, with the inner product ${}_{\mathcal{K}(X)}\langle x\ ,\ y\rangle\equiv\Theta_{x,y}$.

Throughout this paper, for a given representation $(\pi,\mathcal{H})$ of A, we always denote by $\widetilde{\pi}$ the representation of A on the Hilbert space $L^2(\mathcal{H},G)$ defined by

$$(\widetilde{\pi}(a)\xi)(t) = \pi(\alpha_{t^{-1}}(a))\xi(t)$$

for $a\in A, \xi\in L^2(\mathcal{H},G)$, where $L^2(\mathcal{H},G)$ is the Hilbert space of all square integrable functions from G into $\mathcal{H}$. Define a unitary representation λ^A on $L^2(\mathcal{H},G)$ by

$$(\lambda^A{}_s\xi)(t) = \xi(s^{-1}t).$$

Then $(\widetilde{\pi},\lambda^A,L^2(\mathcal{H},G))$ is a covariant representation of A, and the corresponding representation $\widetilde{\pi}\times\lambda^A$ of $A\times_\alpha G$ is defined by

$$(\widetilde{\pi}\times\lambda^A)(x) = \int_G \widetilde{\pi}(x(s))\lambda^A{}_s\, ds$$

for $x\in K(A,G)$. If π is faithful, then $(\widetilde{\pi}\times\lambda^A)(A\times_\alpha G)$ is called the *reduced C^*-crossed product* of A by G and we denote it by $A\times_{\alpha,r}G$.

For $\mathcal{K}(X)$, we consider the C^*-dynamical system $(\mathcal{K}(X),G,\mathrm{Ad}\eta)$. Then η on the left $\mathcal{K}(X)$-Hilbert module X becomes an $\mathrm{Ad}\eta$-compatible action of G. Let $(\pi_\mathcal{K},\pi_X,\pi_A)$ be a representation of X into $\mathcal{B}(\mathcal{H}_A,\mathcal{H}_\mathcal{K})$. Define a representation $\widetilde{\pi}_X$ of X into $\mathcal{B}(L^2(\mathcal{H}_A,G),$ $L^2(\mathcal{H}_\mathcal{K},G))$ by

$$(\widetilde{\pi}_X(x)\xi)(t) = \pi_X(\eta_{t^{-1}}(x))\xi(t)$$

for $x\in X$, $t\in G$ and $\xi\in L^2(\mathcal{H}_A,G)$. Then the representation $(\widetilde{\pi}_\mathcal{K},\widetilde{\pi}_X,\widetilde{\pi}_A,\lambda^\mathcal{K},\lambda^A)$ of X into $\mathcal{B}(L^2(\mathcal{H}_A,G),L^2(\mathcal{H}_\mathcal{K},G))$ satisfies the covariant condition, that is,

$$(\widetilde{\pi}_X(\eta_s(x))\xi)(t) = ((\lambda^\mathcal{K}{}_s\widetilde{\pi}_X(x)\lambda^{A*}_s)\xi)(t)$$

for $s, t \in G$ and $\xi \in L^2(\mathcal{H}_A, G)$. Hence we can define the representation $\widetilde{\pi}_X \times \lambda^A$ of $X \times_\eta G$ into $\mathcal{B}(L^2(\mathcal{H}_A, G), L^2(\mathcal{H}_\mathcal{K}, G))$ by

$$(\widetilde{\pi}_X \times \lambda^A)(x) = \int_G \widetilde{\pi}_X(x(s)){\lambda^A}_s \, ds$$

for $x \in K(X, G)$. Suppose that $\pi_\mathcal{K}$ and π_A are faithful representations of $\mathcal{K}(X)$ and A, respectively. Then π_X is automatically faithful and we consider the representation $\widetilde{\pi}_X \times \lambda^A$ of $X \times_\eta G$ into $\mathcal{B}(L^2(\mathcal{H}_A, G), L^2(\mathcal{H}_\mathcal{K}, G))$. We say that $(\widetilde{\pi}_X \times \lambda^A)(X \times_\eta G)$ is the *reduced crossed product* of X by G, and denote it by $X \times_{\eta,r} G$. It is easy to verify that $X \times_{\eta,r} G$ is a right $(A \times_{\alpha,r} G)$-Hilbert module. We remark that $X \times_{\eta,r} G$ does not depend on the choice of a pair of faithful representations $\pi_\mathcal{K}$ and π_A of $\mathcal{K}(X)$ and A.

We denote by W_G the unitary operator on $L^2(G \times G)$ defined by

$$(W_G\xi)(s, t) = \xi(s, s^{-1}t) \text{ for } \xi \in L^2(G \times G) \text{ and } s, t \in G.$$

Let λ be the left regular representation of G on $L^2(G)$, and define the representation $\widetilde{\lambda}$ of $L^1(G)$ on $L^2(G)$ by

$$\widetilde{\lambda}(f) = \int_G f(s)\lambda_s ds$$

for $f \in L^1(G)$. Then the reduced group C^*-algebra $C_r^*(G)$ of G is defined as the norm closure of $\widetilde{\lambda}(L^1(G))$ in the set of all bounded linear operators on $L^2(G)$. If there is no confusion, we write $\lambda(f)$ for $\widetilde{\lambda}(f)$ above.

Let A be a C^*-algebra and denote by $M(A \otimes_{\min} C_r^*(G))$ the multiplier algebra for the minimum C^*-tensor product $A \otimes_{\min} C_r^*(G)$. We then define the C^*-subalgebra $\widetilde{M}(A \otimes_{\min} C_r^*(G))$ of $M(A \otimes_{\min} C_r^*(G))$ by

$$\begin{aligned}&\widetilde{M}(A \otimes_{\min} C_r^*(G)) = \\ &\{m \in M(A \otimes_{\min} C_r^*(G)) \,|m(1 \otimes z), (1 \otimes z)m \in A \otimes_{\min} C_r^*(G) \text{ for all } z \in C_r^*(G)\}.\end{aligned}$$

Consider the homomorphism δ_G from $C_r^*(G)$ into $\widetilde{M}(C_r^*(G) \otimes_{\min} C_r^*(G))$ defined by

$$\delta_G(\lambda(f)) = W_G(\lambda(f) \otimes 1){W_G}^* \quad \text{for} \quad f \in L^1(G).$$

Let δ_A be a coaction of a locally compact group G on A, that is, δ_A is an injective homomorphism from A into $\widetilde{M}(A \otimes_{\min} C_r^*(G))$ satisfying:
(C1) there is an approximate identity $\{e_i\}$ for A such that $\delta_A(e_i) \to 1$ strictly in $\widetilde{M}(A \otimes_{\min} C_r^*(G))$;
(C2) $(\delta_A \otimes \mathrm{id})(\delta_A(a)) = (\mathrm{id} \otimes \delta_G)(\delta_A(a))$ for all $a \in A$, where we always denote by *id* the identity map on each considered set.

Let $C_0(G)$ be the set of all continuous functions on G vanishing at infinity. We denote by M_f the multiplication operator on $L^2(G)$ given by $f \in C_0(G)$ which is defined by

$$(M_f\xi)(t) = f(t)\xi(t)$$

for all $\xi \in L^2(G)$. Then the *crossed product* $A \times_{\delta_A} G$ of A by δ_A is the C^*-subalgebra of $M(A \otimes \mathcal{C}(L^2(G)))$ generated by the set $\{\delta_A(a)(1 \otimes M_f) | a \in A, f \in C_0(G)\ \}$. We

say that a linear map δ_X from a right A-Hilbert module X into the multiplier module $M(X \otimes C_r^*(G))$ is a δ_A-*compatible coaction* of the locally compact group G on X if δ_X satisfies the following conditions:
(D1) $\delta_X(x)(1_A \otimes z)$ lies in $X \otimes C_r^*(G)$ for all $x \in X$ and $z \in C_r^*(G)$;
(D2) $\delta_X(x \cdot a) = \delta_X(x) \cdot \delta_A(a)$ for all $x \in X$ and $a \in A$;
(D3) $\delta_A({}_A\langle x\ ,\ y\rangle) = \langle \delta_X(x)\ ,\ \delta_X(y)\rangle_{M(A\otimes_{\min} C_r^*(G))}$;
(D4) $(\delta_X \otimes \mathrm{id}) \circ \delta_X = (\mathrm{id} \otimes \delta_G) \circ \delta_X$.

For simplicity, suppose that C^*-algebras A and $\mathcal{K}(X)$ are concretely represented on Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_\mathcal{K}$, respectively. Given a δ_A-compatible coaction δ_X of G on X, the *crossed product* $X \times_{\delta_X} G$ of X by δ_X is the right $(A \times_{\delta_A} G)$-Hilbert closed submodule of $M(X \otimes C_r^*(G)) \subset \mathcal{B}(L^2(\mathcal{H}_A, G), L^2(\mathcal{H}_\mathcal{K}, G))$ generated by the set $\{\delta_X(x)(1_A \otimes M_f) | x \in X, f \in C_0(G)\ \}$. Then the inner product on $X \times_{\delta_X} G$ is given in terms of the usual operator adjoint $^* : \mathcal{B}(L^2(\mathcal{H}_A, G), L^2(\mathcal{H}_\mathcal{K}, G)) \to \mathcal{B}(L^2(\mathcal{H}_\mathcal{K}, G), L^2(\mathcal{H}_A, G))$ by

$$\langle x\ ,\ y\rangle_{A\times_{\delta_A} G} = x^* y \quad \text{for } x, y \in X \times_{\delta_X} G.$$

Let (A, G, α) be a C^*-dynamical system and let $A \times_{\alpha,r} G$ be the reduced C^*-crossed product of A by G. If π_A is a faithful representation of A on a Hilbert space $\mathcal{H}$, $(\widetilde{\pi}_A \times \lambda^A, L^2(\mathcal{H}, G))$ is a faithful representation of $A \times_{\alpha,r} G$. Then

$$((\widetilde{\pi}_A \times \lambda^A) \otimes \mathrm{id})(\delta(x)) = (1_A \otimes W_G)((\widetilde{\pi}_A \times \lambda^A) \otimes \mathrm{id})(x \otimes 1)(1_A \otimes W_G{}^*)$$

for $x \in A \times_{\alpha,r} G$ defines a nondegenerate coaction δ of G on $A \times_{\alpha,r} G$, which is called a *dual coaction*. Let X be a right A-Hilbert module with an α-compatible action η of G. We then regard X as a $\mathcal{K}(X)$-A-Hilbert module and let $(\pi_\mathcal{K}, \pi_X, \pi_A)$ be a representation of X, where $(\pi_\mathcal{K}, \mathcal{H}_\mathcal{K})$ and $(\pi_A, \mathcal{H}_A)$ are representations of $\mathcal{K}(X)$ and A, respectively. If π_X is a faithful representation of X into $\mathcal{B}(\mathcal{H}_A, \mathcal{H}_\mathcal{K})$, $\widetilde{\pi}_X \times \lambda^A$ is a faithful representation of $X \times_{\eta,r} G$. Denote by $1_\mathcal{K}$ the identity of the multiplier algebra $M(\mathcal{K}(X))$ for $\mathcal{K}(X)$. Then a *dual coaction* δ_X of G on $X \times_{\eta,r} G$ is defined by

$$((\widetilde{\pi}_X \times \lambda^A) \otimes \mathrm{id})(\delta_X(x)) = (1_\mathcal{K} \otimes W_G)((\widetilde{\pi}_X \times \lambda^A) \otimes \mathrm{id})(x \otimes 1)(1_A \otimes W_G{}^*)$$

for $x \in X \times_{\eta,r} G$.

3. An Alternative Proof of the First Duality Theorem

Let (A, G, α) be a C^*-dynamical system and let X be an A-Hilbert module with an α-compatible action η of G. Now we suppose that G is a locally compact abelian group and we denote by $\widehat{G}$ its dual group. Let $(\pi_\mathcal{K}, \pi_X, \pi_A)$ be a representation of X, where $(\pi_A, \mathcal{H})$ is the universal representation of A. We use the notation in §2 without comment. Now we are in a position to give a proof.

Proof of the first duality theorem. Take any $x \in K(X, G)$. Define the unitary representation U of $\widehat{G}$ on $L^2(\mathcal{H}, G)$ by

$$(U_\gamma \xi)(t) = \overline{< t, \gamma >} \xi(t)$$

for $\gamma \in \widehat{G}$ and $\xi \in L^2(\mathcal{H}, G)$, where $< t, \gamma >$ means the value of γ at t. Since $\widehat{\eta}_\gamma(x)(t) = \overline{< t, \gamma >} x(t)$, it follows from a straightforward calculation that

$$\widehat{\eta}_\gamma(x) = U_\gamma x U_\gamma^*.$$

Consider the faithful representation $\widetilde{\pi}_X \times \lambda^A$ of $X \times_{\eta, r} G$ and denote by $\overline{\lambda}$ the regular representation of $\widehat{G}$ on $L^2(\mathcal{H}, G \times \widehat{G})$. Then $(\widetilde{\pi}_X \times \lambda^A)^\sim \times \overline{\lambda}$ is a faithful representation of $(X \times_{\eta, r} G) \times_{\widehat{\eta}, r} \widehat{G}$. Here we remark that $\widetilde{\pi}_X \times \lambda^A$ is $\widehat{G}$-equivariant, that is,

$$(\widetilde{\pi}_X \times \lambda^A)(\widehat{\eta}_\gamma(x)) = U_\gamma (\widetilde{\pi}_X \times \lambda^A)(x) U_\gamma^*$$

for all $\gamma \in \widehat{G}$ and all $x \in K(X, G)$. In fact, we have

$$\begin{aligned}(\widetilde{\pi}_X \times \lambda^A)(\widehat{\eta}_\gamma(x)) &= \int_G \widetilde{\pi}_X(\widehat{\eta}_\gamma(x)(t)) {\lambda^A}_t \, dt = \int_G \widetilde{\pi}_X(\overline{< t, \gamma >} x(t)) {\lambda^A}_t \, dt \\ &= \int_G \overline{< t, \gamma >} \widetilde{\pi}_X(x(t)) {\lambda^A}_t \, dt = \int_G U_\gamma \widetilde{\pi}_X(x(t)) {\lambda^A}_t U_\gamma^* \, dt = U_\gamma (\widetilde{\pi}_X \times \lambda^A)(x) U_\gamma^*.\end{aligned}$$

Let F be the isometry from $L^2(\mathcal{H}, G \times \widehat{G})$ onto $L^2(\mathcal{H}, G \times G)$ defined by

$$(F\xi)(s, t) = \int_{\widehat{G}} \overline{< t, \gamma >} \xi(s, \gamma) \, d\gamma \quad \text{for} \quad \xi \in K(\mathcal{H}, G \times \widehat{G}).$$

Then we have

$$\begin{aligned}
&(F(\widetilde{\pi}_X \times \lambda^A)^\sim(x) F^* \xi)(s, t) \\
&= \int_{\widehat{G}} \overline{< t, \gamma >} ((\widetilde{\pi}_X \times \lambda^A)^\sim(x) F^* \xi)(s, \gamma) \, d\gamma \\
&= \int_{\widehat{G}} \overline{< t, \gamma >} ((\widetilde{\pi}_X \times \lambda^A)(\widehat{\eta}_{-\gamma}(x)) F^* \xi)(s, \gamma) \, d\gamma \\
&= \int_{\widehat{G}} \overline{< t, \gamma >} (U_\gamma^* (\widetilde{\pi}_X \times \lambda^A)(x) U_\gamma F^* \xi)(s, \gamma) \, d\gamma \\
&= \int_{\widehat{G}} \overline{< t, \gamma >} < s, \gamma > ((\widetilde{\pi}_X \times \lambda^A)(x) U_\gamma F^* \xi)(s, \gamma) \, d\gamma \\
&= \int_{\widehat{G}} \overline{< t, \gamma >} < s, \gamma > \int_G \pi_X(\eta_{-s}(x(k))) (U_\gamma F^* \xi)(s - k, \gamma) \, dk d\gamma \\
&= \int_{\widehat{G}} \overline{< t, \gamma >} < s, \gamma > \int_G \pi_X(\eta_{-s}(x(k))) \overline{< s - k, \gamma >} (F^* \xi)(s - k, \gamma) \, dk d\gamma \\
&= \int_{\widehat{G}} \int_G < k - t, \gamma > \pi_X(\eta_{-s}(x(k))) \int_G < l, \gamma > \xi(s - k, l) \, dl dk d\gamma \\
&= \int_{\widehat{G}} \int_G \int_G < l - t, \gamma > \pi_X(\eta_{-s}(x(k))) \xi(s - k, l - k) \, dl dk d\gamma \\
&= \int_G \pi_X(\eta_{-s}(x(k))) \int_{\widehat{G}} \int_G < l - t, \gamma > \xi(s - k, l - k) \, dl d\gamma dk \\
&= \int_G \pi_X(\eta_{-s}(x(k))) \xi(s - k, t - k) \, dk
\end{aligned}$$

$$
\begin{aligned}
&= \int_G \pi_X(\eta_{-s}(x(k)))((1 \otimes W_G^*)\xi)(s-k, t-s)\, dk \\
&= \int_G \pi_X(\eta_{-s}(x(k)))({\lambda^A}_k \otimes 1)((1 \otimes W_G^*)\xi)(s, t-s)\, dk \\
&= ((\widetilde{\pi}_X \times \lambda^A)(x) \otimes \mathrm{id})((1 \otimes W_G^*)\xi)(s, t-s) \\
&= (1 \otimes W_G)((\widetilde{\pi}_X \times \lambda^A)(x) \otimes \mathrm{id})((1 \otimes W_G^*)\xi)(s, t).
\end{aligned}
$$

Thus we obtain that

$$F(\widetilde{\pi}_X \times \lambda^A)^{\sim}(x)F^* = (1 \otimes W_G)((\widetilde{\pi}_X \times \lambda^A)(x) \otimes \mathrm{id})((1 \otimes W_G^*) = ((\widetilde{\pi}_X \times \lambda^A) \otimes \mathrm{id})(\delta(x))$$

for all $x \in K(X, G)$. For any $f \in L^1(\widehat{G})$, we then have

$$
\begin{aligned}
&F((\widetilde{\pi}_X \times \lambda^A)^{\sim} \times \overline{\lambda})\,(x \otimes f)F^* = \int_{\widehat{G}} f(\gamma)(F(\widetilde{\pi}_X \times \lambda^A)^{\sim}(x)F^*)(F(1 \otimes \overline{\lambda}_\gamma)F^*)\, d\gamma \\
&= (F(\widetilde{\pi}_X \times \lambda^A)^{\sim}(x)F^*)(F(1 \otimes \int_{\widehat{G}} f(\gamma)\overline{\lambda}_\gamma\, d\gamma)F^*) = (F(\widetilde{\pi}_X \times \lambda^A)^{\sim}(x)F^*)(1 \otimes M_{\widehat{f}}) \\
&= ((\widetilde{\pi}_X \times \lambda^A) \otimes \mathrm{id})(\delta(x))(1 \otimes M_{\widehat{f}}) = ((\widetilde{\pi}_X \times \lambda^A) \otimes \mathrm{id})(\delta(x)(1 \otimes M_{\widehat{f}})),
\end{aligned}
$$

where $\widehat{f}$ denotes the Fourier transform of f. By the definition of $(X \times_{\eta,r} G) \times_\delta G$, we see that

$$F(\widetilde{\pi}_X \times \lambda^A)^{\sim}\,((X \times_{\eta,r} G) \times_{\widehat{\eta}} \widehat{G}))F^* = (X \times_{\eta,r} G) \times_\delta G.$$

Hence it follows from the second duality theorem that $(X \times_{\eta,r} G) \times_{\widehat{\eta}} \widehat{G} \cong X \otimes \mathcal{C}(L^2(G))$, where "$\cong$" means isomorphic. Since G is abelian (hence amenable), $X \times_{\eta,r} G$ is identified with $X \times_\eta G$ by $\widetilde{\pi}_X \times \lambda^A$ (see [2, Proposition 2.13]). Since $\widetilde{\pi}_X \times \lambda^A$ is $\widehat{G}$-equivariant, it is not hard to show that $(X \times_\eta G) \times_{\widehat{\eta}} \widehat{G} \cong (X \times_{\eta,r} G) \times_{\widehat{\eta}} \widehat{G}$. Thus we see that $(X \times_\eta G) \times_{\widehat{\eta}} \widehat{G} \cong X \otimes \mathcal{C}(L^2(G))$. □

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