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ELEMENTARY PROOF OF SCHWEITZER'S THEOREM ON HILBERT C^*-MODULES IN WHICH ALL CLOSED SUBMODULES ARE ORTHOGONALLY CLOSED

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Abstract

Let A and B be C^*-algebras and let X be an A-B-imprimitivity bimodule. Schweitzer showed the theorem that if every closed right B-submodule of X is orthogonally closed, then there are families $\{\mathcal{H}_i\}_{i\in I}$, $\{\mathcal{K}_i\}_{i\in I}$ of Hilbert spaces such that A (resp. B) is isomorphic to the c_0-direct sum $\Sigma^{\oplus}_{i\in I}C(\mathcal{H}_i)$ of all compact operators $C(\mathcal{H}_i)$ on $\mathcal{H}_i$ (resp. $\Sigma^{\oplus}_{i\in I}C(\mathcal{K}_i)$ of all compact operators $C(\mathcal{K}_i)$ on $\mathcal{K}_i$) as a C^*-algebra, and X is isomorphic to the c_0-direct sum $\Sigma^{\oplus}_{i\in I}C(\mathcal{K}_i, \mathcal{H}_i)$ as a Hilbert C^*-module, where $C(\mathcal{K}_i, \mathcal{H}_i)$ denotes the Hilbert C^*-module consisting of all compact operators from $\mathcal{K}_i$ into $\mathcal{H}_i$. In this paper, we give an alternative proof, of this theorem, which is shorter and more elementary than the original one.

1. Introduction

Let A be a C^*-algebra and let X be a Hilbert A-module with an A-valued inner product $\langle \cdot, \cdot \rangle$. For any closed subspace Y of X, we denote by $Y^{\perp}$ the orthogonally complemented subspace of Y in X, i.e.,

$$Y^{\perp} = \{ x \in X \mid \langle x, y \rangle = 0 \text{ for all } y \in Y \}.$$

We say that a closed A-submodule Y of a Hilbert A-module X is *orthogonally complemented* in X if X coincides with $Y \oplus Y^{\perp}$, and that a closed A-submodule Y of a Hilbert A-module X is *orthogonally closed* in X if $(Y^{\perp})^{\perp} = Y$. If Y is orthogonally complemented in X, then it is orthogonally closed in X. But the converse is not necessarily true.

Suppose that X is a full (right) Hilbert A-module. Several years ago, Magajna [5] proved that A is a C^*-algebra which admits a full Hilbert A-module X such that every closed right A-submodule of X is orthogonally closed if and only if A is isomorphic to a C^*-subalgebra of the C^*-algebra $C(\mathcal{H})$ of all compact operators on some Hilbert space $\mathcal{H}$. In the sequel, Schweitzer [7, Theorem 1] elaborated on Magajna's theorem, that is, he showed the following theorem:

Theorem 1. *Let A and B be C^*-algebras and let X be an A-B-imprimitivity bimodule. If*

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every closed right B-submodule of X is orthogonally closed, then there are families $\{\mathcal{H}_i\}_{i\in I}$, $\{\mathcal{K}_i\}_{i\in I}$ *of Hilbert spaces such that* $A \cong \sum^{\oplus}_{i\in I} C(\mathcal{H}_i)$, $B \cong \sum^{\oplus}_{i\in I} C(\mathcal{K}_i)$, *and* $X \cong \sum^{\oplus}_{i\in I} C(\mathcal{K}_i, \mathcal{H}_i)$, *where the symbol " $\cong$ " means isomorphic.*

Remark that it is trivial that the converse holds in Theorem 1. As a corollary, furthermore immediately we have the following:

Corollary 2. *Let A and B be C^*-algebras and let X be an A-B-imprimitivity bimodule. Then every closed right B-submodule of X is orthogonally closed if and only if every closed right B-submodule of X is orthogonally complemented in X.*

In this paper, we give an alternative proof of Theorem 1 above based on the representation theory of Hilbert C^*-modules. Our proof presented in this paper is more elementary than the original one in the sense that we essentially use nothing particular except the basic fact that any Hilbert C^*-module admits a faithful representation.

2. Alternative proof of Theorem 1

Recall the definition of a Hilbert C^*-module. Let A be a C^*-algebra. By a *left Hilbert A-module*, we mean a left A-module X equipped with an A-valued pairing $\langle \cdot , \cdot \rangle$, called an A-valued inner product, satisfying the following conditions:

(a) $\langle \cdot , \cdot \rangle$, is sesquilinear. (We make the convention that $\langle \cdot , \cdot \rangle$ is linear in the first variable and is conjugate-linear in the second variable.)

(b) $\langle x , y \rangle = \langle y , x \rangle^*$ for all $x , y \in X$.

(c) $\langle ax , y \rangle = a\langle x , y \rangle$ for all $a \in A$ and $x, y \in X$.

(d) $\langle x , x \rangle \geqq 0$ for all $x \in X$, and $\langle x , x \rangle = 0$ implies that $x = 0$.

(e) X is complete with respect to the norm $\| x \| = \| \langle x , x \rangle \|^{\frac{1}{2}}$.

We remark that the Hilbert A-module is always assumed to be a vector space over the field of complex numbers. Hence every Hilbert A-module is a Banach space with respect to the norm $\| \cdot \|$. Furthermore, X is said to be *full* if X satisfies an additional condition:

(f) the closed linear span of $\{\langle x , y \rangle \mid x , y \in X\}$ coincides with A.

Let B be a C^*-algebra. Right Hilbert B-modules are defined similarly, except that we require that B should act on the right of X, that the B-valued inner product $\langle \cdot , \cdot \rangle$ should be conjugate-linear in the first variable, and that $\langle x , yb \rangle = \langle x , y \rangle b$ for all $b \in B$ and $x, y \in X$.

Let A and B be C^*-algebras. We denote by ${}_A\langle \cdot , \cdot \rangle$ the A-valued inner product on the left Hilbert A-module and by $\langle \cdot , \cdot \rangle_B$ the B-valued inner product on the right Hilbert B-module, respectively. By an *A-B- imprimitivity bimodule X*, we mean a full left Hilbert A-module and full right Hilbert B-module X satisfying

(g) ${}_A\langle xb , y \rangle = {}_A\langle x , yb^* \rangle$ and $\langle ax , y \rangle_B = \langle x , a^*y \rangle_B$ for all $a \in A$, $b \in B$ and x, y X;

(h) ${}_A\langle x , y \rangle \cdot z = x \cdot \langle y , z \rangle_B$ for all $x , y , z \in X$.

Two C^*-algebras A and B are said to be *Morita equivalent* if there exists an A-B-imprimitivity bimodule. We remark that in this paper, Morita equivalence means strong Morita equivalence in the sense of Rieffel (cf. [6, Remark 3.15]). The reader is referred to [4], [6] for Hilbert C^*-modules and Morita equivalence.

Let A and B be C^*-algebras, and suppose, for simplicity, that X is an A-B-imprimitivity bimodule. Recall that a ***representation*** of X is a triple (π_A, π_X, π_B) consisting of nondegenerate representations π_A and π_B of A and B on Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, together with a linear map $\pi_X : X \to \mathcal{B}(\mathcal{H}_B, \mathcal{H}_A)$ such that

(R1) $\pi_X(ax) = \pi_A(a)\pi_X(x)$,

(R2) $\pi_X(xb) = \pi_X(x)\pi_B(b)$,

(R3) $\pi_A({}_A\langle x, y\rangle) = \pi_X(x)\pi_X(y)^*$, and

(R4) $\pi_B(\langle x, y\rangle_B) = \pi_X(x)^*\pi_X(y)$

for all $a \in A$, $x, y \in X$, and $b \in B$, where $\mathcal{B}(\mathcal{H}_B, \mathcal{H}_A)$ denotes the set of all bounded linear operators from $\mathcal{H}_B$ into $\mathcal{H}_A$.

It is known that any A-B-imprimitivity bimodule X admits representations. For example, given a representation π_B of B on a Hilbert space $\mathcal{H}$, a representation $(\pi_A, \mathcal{K})$ of A can be given by the induced representation acting on the B-balanced tensor product $X \otimes_B \mathcal{H} (= \mathcal{K})$ according to $\pi_A(a)(x \otimes \xi) = (ax) \otimes \xi$ and then π_X is defined by $\pi_X(x)(\xi) = x \otimes \xi$ for $x \in X$ and $\xi \in \mathcal{H}$. Note that if π_B is irreducible, so is the induced representation π_A.

Now we are in a position to give an alternative proof of Theorem 1. Note that our proof is constructive in the sense of giving a concrete isomorphism between X and $\Sigma^{\oplus}_{i \in I} C(\mathcal{K}_i, \mathcal{H}_i)$.

Proof of Theorem 1. We first remark that by assumption, A and B are Morita equivalent. By [5, Theorem 1], there is a family $\{\mathcal{K}_i\}_{i \in I}$ of Hilbert spaces such that B is isomorphic to the c_0-direct sum $\Sigma^{\oplus}_{i \in I} C(\mathcal{K}_i)$. Taking into account the Rieffel correspondence (cf. [6, Theorem 3.22]), the proof can be reduced to the one for the case where if B is isomorphic to $C(\mathcal{K})$ on some Hilbert space $\mathcal{K}$, then there is a Hilbert space $\mathcal{H}$ such that $A \cong C(\mathcal{H})$ and $X \cong C(\mathcal{K}, \mathcal{H})$.

Suppose that $(\pi_B, \mathcal{K})$ is a faithful representation of B such that $\pi_B(B) = C(\mathcal{K})$. Then B is simple and of type I. Since π_B is irreducible, there exists a representation (π_A, π_X, π_B) of X on the space $\mathcal{B}(\mathcal{K}, \mathcal{H})$ of all bounded linear operators from $\mathcal{K}$ into some Hilbert space $\mathcal{H}$ such that $(\pi_A, \mathcal{H})$ is an irreducible representation of A. It then follows from (R4) and (R1) that π_X and π_A are faithful, since π_B is faithful. Since simplicity and type I-ness of C^*-algebras are preserved by Morita equivalence, A is also simple and of type I. Hence A is also an elementary C^*-algebra. Thus we see that $\pi_A(A) = C(\mathcal{H})$.

Take any $x \in X$. Then $\pi_X(x)^*\pi_X(x) = \pi_B(\langle x, x\rangle_B) \in C(\mathcal{K})$, and so $|\pi_X(x)| = (\pi_X(x)^*\pi_X(x))^{\frac{1}{2}}$ is a compact operator on $\mathcal{K}$. Consider the polar decomposition $\pi_X(x) = u|\pi_X(x)|$, where u is a partial isometry in $\mathcal{B}(\mathcal{K}, \mathcal{H})$. Then $u|\pi_X(x)|$ is a compact operator in $\mathcal{B}(\mathcal{K}, \mathcal{H})$. Thus $\pi_X(X) \subset C(\mathcal{K}, \mathcal{H})$. Since $\pi_X(X)$ is a closed $C(\mathcal{H})$-$C(\mathcal{K})$-submodule of the $C(\mathcal{H})$-$C(\mathcal{K})$-imprimitivity bimodule $C(\mathcal{K}, \mathcal{H})$ and since $C(\mathcal{K}, \mathcal{H})$ has no non-trivial closed $C(\mathcal{H})$-$C(\mathcal{K})$-submodule, we conclude that $\pi_X(X) = C(\mathcal{K}, \mathcal{H})$. Q.E.D.

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