# Some Hecke Eigen-Values for Γ(q)



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# Some Hecke Eigen-Values for $\Gamma(q)$

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Abstract

We give some explicit eigen-values of Hecke operators for the cusp forms of weight 2 for the principal congruence subgroups $\Gamma(q)$ for several primes $q$, using the Eichler's formula [1]. For the prime $q \equiv 3 \bmod 4$, the cusp forms correponding to the degree $q-1$ representation of the group $SL_2(\mathbb{Z} / q\mathbb{Z})$ are viewed as the cusp forms for $\Gamma_0(q^2)$. Using these cusp forms, we give an example of the class field over the imaginary quadratic field $\mathbb{Q}(\sqrt{-q})$ for $q = 43$ which is constructed by Shimura's theory of class fields over the quadratic fields.

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## Introduction

In this paper, we give some explicit eigen-values of Hecke operators for the cusp forms of weight 2 for the principal congruence subgroups $\Gamma(q)$ for $q$ = 23,31,43 and 47 corresponding to the representation of degree $q-1$ of the group $SL_2(\mathbb{Z}/q\mathbb{Z})$,using the Eichler's formula [1]. For the prime $q \equiv 3 \bmod 4$, the cusp forms correponding to degree $q-1$ representations are viewed as the cusp forms for $\Gamma_0(q^2)$, and contained in the space $S_{\mathrm{I}}$ or $S_{\mathrm{III}}$ defined in [3]. The cusp forms corresponding to the representations of degree $q$ or $q$+1 are viewed as forms of $\Gamma_0(q)$ with some character of conductor $q$, of which Hecke eigen-values are not treated here, since they are rather easily obtained. We denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. $GL_2(\mathbb{R})$ denotes the group of all non-singular square matrices whose components are contained in $\mathbb{R}$ and $SL_2(\mathbb{Z})$ denotes the group of all square matrices of determinant 1 whose components are contained in $\mathbb{Z}$.

## 1 Cusp forms for $\Gamma(q)$ and $\Gamma_0(q^2)$

Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$ $(\det(\alpha) > 0)$. For a function $f(z)$ on the complex upper half plane $\mathfrak{H}$ and a positive integer $k \geq 2$, we define a function $f|[\alpha]_k$ on $\mathfrak{H}$ by

$$(f|[\alpha]_k)(z) = (\det\alpha)^{k/2}(cz+d)^{-k}f(\alpha(z)),$$

where $\alpha(z) = (az+b)/(cz+d)$. For a positive integer $N$, we define

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \bmod N \right\}.$$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}.$$

We put $\Gamma = \Gamma(1) = SL_2(\mathbb{Z})$. We denote by $S_k(\Gamma_0(N))$ (resp. $S_k(\Gamma(N))$) the space of all holomorphic cusp forms of weight $k$ on $\mathfrak{H}$ satisfying

$$f|[\gamma]_k = f \quad \text{for} \quad \gamma \in \Gamma_0(N) \quad (\text{resp.}\Gamma(N))$$

Further for a character $\psi$ modulo $N$, we denote by $S_k(\Gamma_0(N), \psi)$ the space of all holomorphic cusp forms of weight $k$ satisfying

$$f|[\gamma]_k = \psi(d)f \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

If $\psi$ is the identity character, we simply write $S_k(\Gamma_0(N), \psi)$ as $S_k(\Gamma_0(N))$. Hereafter we take $N = q$, $q$ is an odd prime. We put $\mathfrak{M} = \Gamma/\Gamma(q)$. Throughout this paper we assume $k$ = 2. Let $\mathfrak{S} = S_2(\Gamma(q))$ be the space of cusp forms of weight 2 for $\Gamma(q)$.

Then it is known by Hecke [2], the space $\mathfrak{S}$ is decomposed as

$$(1) \qquad \mathfrak{S} = \sum_{\mu} s^{q-1,\mu}\mathfrak{S}^{q-1,\mu} + \sum_{\nu} s^{q+1,\nu}\mathfrak{S}^{q+1,\nu} + s^q\mathfrak{S}^q + \cdots$$

where $\mathfrak{S}^{q-1,\mu}$ etc. are irreducible components of $\mathfrak{S}$ of the representation $S$ of $\mathfrak{M}$ defined by

$$\begin{pmatrix} \varphi_1(z) \\ \cdot \\ \cdot \\ \varphi_m(z) \end{pmatrix} \circ M = S(M) \begin{pmatrix} \varphi_1(z) \\ \cdot \\ \cdot \\ \varphi_m(z) \end{pmatrix}.$$

Here $\{\varphi_1(z) \cdots \varphi_m(z)\}$ is a basis of $\mathfrak{S}$ and

$$\varphi_i(z) \circ M = (\varphi_i|[M]_2)(z)\,.$$

The positive integer $s^{q-1,\mu}$ is the multiplicity of $\mathfrak{S}^{q-1,\mu}$. The right upper index $q-1$ of $\mathfrak{S}^{q-1,\mu}$, for example, means the degree of the representation and $\mu$ will be explaind later and exact values of $s^{q-1,\mu}$ etc. is given in 3. Tables (Table (A)). We have by (1),

$$(2) \quad \sigma(M) = \sum_{\mu} s^{q-1,\mu}\sigma^{q-1,\mu}(M) + \sum_{\nu} s^{q+1,\nu}\sigma^{q+1,\nu}(M) + \cdots,$$

where $\sigma, \sigma^{q-1,\mu}$, etc. denote the traces on the spaces $\mathfrak{S}, \mathfrak{S}^{q-1,\mu}$, etc.

Suppose $n$ is a quadratic residue mod $q$, and take $t$ so that $n \equiv t^2$ mod $q$. For the Hecke operator $T(n)$ acting on $\mathfrak{S}$, define $\hat{T}(n)$ as

$$\hat{T}(n) = T(n)U_t \text{ where } U_t \in \Gamma \text{ and } U_t \equiv \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \text{ mod } q\,.$$

We put $\mathfrak{T} = \left\{\hat{T}(n) | \quad \chi(n) = 1\right\}$, where $\chi$ denotes the real primitive character modulo $q$ of order 2. Since

$$\hat{T}(n)M = M\hat{T}(n), \text{ for } M \in \Gamma,$$

the trace $\sigma(\hat{T}(n))$ of the representation of $\mathfrak{T}$ becomes

$$(3) \quad \sigma(\hat{T}(n)) = (q-1)\sum_{\mu} \sigma^{q-1,\mu}(\hat{T}(n)) + (q+1)\sum_{\nu} \sigma^{q+1,\nu}(\hat{T}(n)) + \cdots,$$

where $\sigma^{q-1,\mu}$, $\sigma^{q+1,\nu}$, etc. denote the traces on certain subspaces $\mathfrak{U}^{q-1,\mu}$, $\mathfrak{U}^{q+1,\nu}$, etc. of $\mathfrak{S}$, and $\dim \mathfrak{U}^{q-1,\mu} = s^{q-1,\mu}$, $\dim \mathfrak{U}^{q+1,\nu} = s^{q+1,\nu}$, etc (Eichler[1]). In this paper, we exclusively treat the representation of $\mathfrak{M}$ of degree $q-1$. This representation is characterized by the character $\chi(S) = -(\zeta_\mu + \zeta_\mu^{-1})$ where $S$ ia an element $\mathfrak{M}$ of order $\frac{q+1}{2}$ and $\zeta_\mu$ is a $\frac{q+1}{2}$-th root of unity which is not equal to $\pm 1$ and $1 \leq \mu \leq \frac{q-1}{4}$ in the case $q \equiv$ 1,5 mod 12 and $1 \leq \mu \leq \frac{q-3}{4}$ in the case $q \equiv$ 7,11 mod 12 (Hecke[2]). Now take

$$f(z)=\sum_{n=1}^{\infty}\lambda_n e^{\frac{2\pi inz}{q}}\in S_2(\Gamma(q)),$$

which is a common eigen-function of Hecke operators $T(n)$. Then we know by Shimura([4], prop.3.53, prop.3.64)

$$f(qz)=\sum_{n=1}^{\infty}\lambda_n e^{2\pi inz}\in S_2(\Gamma_0(q^2),\varphi)$$

for a suitable character $\varphi$ modulo $q$, which is a common eigen-function of Hecke operators. Further, for another character $\xi$ modulo $q$,

$$f_\xi(qz)=\sum_{n=1}^{\infty}\xi(n)\lambda_n e^{2\pi inz}\in S_2(\Gamma_0(q^2),\varphi\xi^2),$$

$f_\xi(qz)$ is also a common eigen function of all Hecke operators. Further assume $q$ is a prime such that $q\equiv 3 \mod 4$. Hence if we put $\xi=\varphi^{\frac{q-3}{4}}$, then $\varphi\xi^2=1$. Thus any function $f(z)\in S_2(\Gamma(q))$ may be regarded as a function of $\tilde{f}(z)=f_\xi(qz)\in S_2(\Gamma_0(q^2))$. Using the trace formula of Eichler [1], we give several characteristic polynomials $\Phi_{\tilde{f},n}(x)$ of Hecke operators $T(n)$ on the subspace of $S_2(\Gamma_0(q^2))$ corresponding to the primitive functions $\tilde{f}(z)$ for $q$ = 23,31,43,47. We denote by $\chi$ the non- trivial quadratic character of conducter $q$. Then for $\tilde{f}(z)=\Sigma_{n=1}^{\infty}a_n e^{2\pi inz}\in S_2(\Gamma(q^2))$, $\tilde{f}_\chi(z)=\Sigma_{n=1}^{\infty}\chi(n)a_n e^{2\pi inz}$ is also contained in $S_2(\Gamma_0(q^2))$.

Let $S_2^0(\Gamma_0(q^2))$ denote the subspace of all new forms in $S_2(\Gamma_0(q^2))$. We put $S_2(\Gamma_0(q))^\chi=\{f_\chi | f\in S_2(\Gamma_0(q))\}$, then the space $S_2(\Gamma_0(q))^\chi$ is the subspace of $S_2^0(\Gamma_0(q^2))$. We denote by $S_2^n(\Gamma_0(q^2))$ the orthogonal complement of $S_2(\Gamma_0(q))^\chi$ in $S_2^0(\Gamma_0(q^2))$ with respect to the Petersson inner product. Then we know by [3], $S_2^n(\Gamma_0(q^2))$ decomposes into four spaces,

$$S_2^n(\Gamma_0(q^2))=S_{\mathrm{I}}\bigoplus S_{\mathrm{II}}\bigoplus S_{\mathrm{II}_\chi}\bigoplus S_{\mathrm{III}} \tag{4}$$

where

$$\begin{aligned}
S_{\mathrm{I}}&=\{f\in S_2^n(\Gamma_0(q^2)) \mid f|W=f,\ f|\delta_\chi W=f|\delta_\chi\}\\
S_{\mathrm{II}}&=\{f\in S_2^n(\Gamma_0(q^2)) \mid f|W=f,\ f|\delta_\chi W=-f|\delta_\chi\}\\
S_{\mathrm{II}_\chi}&=\{f\in S_2^n(\Gamma_0(q^2)) \mid f|W=-f,\ f|\delta_\chi W=f|\delta_\chi\}\\
S_{\mathrm{III}}&=\{f\in S_2^n(\Gamma_0(q^2)) \mid f|W=-f,\ f|\delta_\chi W=-f|\delta_\chi\}.
\end{aligned}$$

Here $W=\begin{pmatrix}0&-1\\q^2&0\end{pmatrix}$, and for the quadratic character $\chi$, $\delta_\chi$ is the twisting operator

$$f|\delta_\chi=\frac{1}{g(\chi)}\sum_{u=1}^{q-1}\chi(u)f|[\alpha_u]_k \quad \text{for} \quad \alpha_u=\begin{pmatrix}q&u\\0&q\end{pmatrix}$$

where $g(\chi)$ is the Gaussian sum for $\chi$.

We list the dimensions of the space $S_{\mathrm{I}}, S_{\mathrm{II}}$ and $S_{\mathrm{III}}$ in the table (B). (we note $\dim S_{\mathrm{II}} = \dim S_{\mathrm{II}_\chi}$).

## 2 The case $q$ = 43.

In [3, § 5], we have given an example of the form in $S_2(\Gamma_0(q^2))$ which relates to the theory of Shimura's construction of class fields over real quadratic fields for the case $q^2 = 19^2$. In this section we add one example for the case $q^2 = 43^2$. Hereafter we put $q$ = 43. We denote by $\mathfrak{o}_L$ the maximal orders of the number field $L$. Let $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ be a primitive form corresponding to the first factor of the Table (C) (3.a), then we see $a_2 = \sqrt{\beta_2}, a_{11} = <-5, 0, 5, 0, -1>$, $a_{13} = <2, -5, -5, 2, 1>$, $a_{17} = <-2, -8, 3, 2, -1>$ where $\beta_2 = <2, -3, 0, 1, 0>$. We note that the absolute norm of $\beta_2$ is 11. Here $<u_0, u_1, u_2, u_3, u_4>$ denotes $u_0 + u_1\sigma_{22} + u_2\sigma_{22}^2 + u_3\sigma_{22}^3 + u_4\sigma_{22}^4$, where $\sigma_{22} = \zeta_{22} + \zeta_{22}^{-1}$ with a primitive 22-th root of unity $\zeta_{22}$.

Let $K$ be the subfield of $\mathbb{C}$ generated by $a_n$ for all $n$.

Now $f(z)$ and its companions, namely $f^\sigma(z) = \sum_{n=1}^{\infty} a_n^\sigma e^{2\pi i n z}$ for all isomorphism $\sigma$ from $K$ into $\mathbb{C}$ span 10-dimensional subspace of $S_2(\Gamma_0(43^2))$. From this $f(z)$ and its companions, we obtain an abelian variety of dimension [$K$: $\mathbb{Q}$ ] (=10) defined over $\mathbb{Q}$ and an isomorphism $\theta$ of $K$ into $\mathrm{End}(A) \otimes \mathbb{Q}$, $\theta(a)$ is rational for all $a \in K$. Further, we see that there is an automorphism $\rho$ of $K$, other than the identity map, such that $\chi(n)a_n = a_n^\rho$ *for all* $n$. Thus, let $F$ be the invariant subfield of $K$ under $\rho$, then we have [$K$: $F$]=2.

Let $\mathfrak{c} = (\beta_2)\mathfrak{o}_F$ be the ideal in $F$, and $\mathfrak{b} = \sqrt{\beta_2}\mathfrak{o}_K$ be the ideal in $K$. Then $\mathfrak{b}^2 = \mathfrak{c}$ and both the absolute norm of $\mathfrak{b}$ and $\mathfrak{c}$ is 11. Further we see $\beta_2 = <2, -3, 0, 1, 0> = <2, 1, 0, 0, 0> <1, -2, 1, 0, 0>$ and $<1, -2, 1, 0, 0>$ is a unit. So we may put $\mathfrak{c} = (2 + \sigma_{22})$.

We know by [6, prop.8 and prop.9] there exists an endomorphism $\eta$ of $A$ which is defined over the quadratic field $k = \mathbb{Q}(\sqrt{-q})$ and satisfies the following conditions:

(i) $\eta^\varepsilon = -\eta$ *if* $\varepsilon$ *is the generator of* $\mathrm{Gal}(k/\mathbb{Q})$.

(ii) $\eta^2 = \chi(-1)q \cdot \mathrm{id}_A$.

(iii) $\eta \circ \theta(a) = \theta(a^\rho) \circ \eta$ *for every* $a \in K$.

For the ideal $\mathfrak{b}$ given as above, we put

$$\mathfrak{x} = \{t \in A \mid \theta(\mathfrak{b})t = 0\},$$

then $\mathfrak{x}$ is isomorphic to $(\mathfrak{o}_K/\mathfrak{b})^2$ as an $\mathfrak{o}_k$-module. further we put

$$\mathfrak{y} = \{t \in \mathfrak{x} \mid (\eta - \theta(1))t = 0\},$$

$$\mathfrak{z} = \{t \in \mathfrak{x} \mid (\eta + \theta(1))t = 0\}.$$

Now as in [5,Prop.9.2], we see

*The submodules* $\mathfrak{y}$ *and* $\mathfrak{z}$ *are* $\mathfrak{o}_F$*-isomorphic to* $\mathfrak{o}_F/\mathfrak{c}$ *and* $\mathfrak{x} = \mathfrak{y} \oplus \mathfrak{z}$.

Let $k(\mathfrak{x})$ (resp. $k(\mathfrak{y})$) denote the smallest extension of $k$ over which the points of $\mathfrak{x}$ (resp. $\mathfrak{y}$) are rational. Then $k(\mathfrak{x})$ ia an abelian extension of $k$, and making $\mathrm{Gal}(k(\mathfrak{x})/k)$ act on $\mathfrak{y}$ and $\mathfrak{z}$, we obtain an injective homomorphism

$$\mathrm{Gal}(k(\mathfrak{x})/k) \to (\mathfrak{o}_F/\mathfrak{c})^\times \times (\mathfrak{o}_F/\mathfrak{c})^\times .$$

We shall examine the abelian extension $k(\mathfrak{y})/k$ following the method explained in [3, § 5].

**Proposition 2.1** *The field* $k(\mathfrak{y})$ *is a ray class field over k of conductor* $\mathfrak{l}\sqrt{-43}$ *with a prime factor* $\mathfrak{l}$ *of* 11, *one has*

$$r((\alpha)) = \varphi(\alpha)\mu(\alpha \bmod \mathfrak{l})$$

*for every* $\alpha$ *in k prime to* $43\mathfrak{l}$, *where* $\mu$ *is the isomomorphism of* $\mathfrak{o}_k/\mathfrak{l}$ *onto* $\mathfrak{o}_F/\mathfrak{c}$ and $\varphi$ *is a homomorphism of* $(\mathfrak{o}_k/(\sqrt{-43}))^\times$ *into* $(\mathfrak{o}_F/\mathfrak{c})^\times$ *of order 2.*

*proof.* Since every prime factor of the conductor $\mathfrak{f}$ of $k(\mathfrak{y})/k$ divides $43 \cdot N(\mathfrak{l}) = 43 \cdot 11$, [4, § 7.5 and Prop. 7.23] we may put $\mathfrak{f} = \prod_{\mathfrak{p}} \mathfrak{p}^{f_\mathfrak{p}}$ where $\mathfrak{p}$ runs all prime factors of $43 \cdot 11$ By the same argument as in the proof of [5, Th.2.3], we obtain

$$r((m)) = \chi(m)\mu(m \mod \mathfrak{l})$$

for every $m \in \mathbb{Z}$ prime to $43 \cdot 11$. Thus $[k(\mathfrak{y}) : k] = 5$ or 10, and so the $\mathfrak{l}$ exponent $f_\mathfrak{l} = 1$ by [4,Lemma 3.2]. To determine $f_{\mathfrak{l}^\epsilon}$, we use the following fact which is nothing but [5, Th. 2.8] for the present case.

*Let l be a prime which divides* $N(\mathfrak{l})$ *but not* $q = 43$. *Suppose* $\chi(l) = 1$, and $a_l$ is prime to $\mathfrak{c} = (2+\sigma_{22})$. *Then* $\mathfrak{f}$ *is divisible only one prime factor of l. Put* $l = \mathfrak{l} \cdot \mathfrak{l}^\epsilon$ *and suppose* $\mathfrak{l}$ *divides* $\mathfrak{f}$. *Then*

$$r(\mathfrak{l}^\epsilon) \equiv a_l \bmod \mathfrak{c}.$$

Take $l = 11$. Then Since $a_{11} = \langle -5, 0, 5, 0, -1 \rangle$ is prime to $\mathfrak{c} = (2+\sigma_{22})$, we have $f_{\mathfrak{l}^\epsilon} = 0$. So we may put $\mathfrak{f} = \mathfrak{l}^t\sqrt{-43}^{t'}$. By the Hasse's conductor ramification theorem, we find $t = 1$ and $t' = 1$, since $[k(\mathfrak{y}) : k] = 5$ or 10. Thus we have

$$r((\alpha)) = \varphi(\alpha)\mu(\alpha \bmod \mathfrak{l})$$

for every $\alpha$ in $k$ prime to $43\mathfrak{l}$, with a homomorphism $\varphi$ of $(\mathfrak{o}_k/(\sqrt{-43}))^\times$ into $(\mathfrak{o}_F/\mathfrak{c})^\times$ and the isomorphism $\mu$ of $\mathfrak{o}_k/\mathfrak{l}$ onto $\mathfrak{o}_F/\mathfrak{c}$. Now the orders of $(\mathfrak{o}_k/(\sqrt{-43}))^\times$ and $(\mathfrak{o}_F/\mathfrak{c})^\times$ are 42 and 10 respectively, and $\varphi(m) = \chi(m)$ for rational $m$ prime to $43 \cdot 11$, so the order of $\varphi$ is 2. Further, $r(\mathfrak{l}^\epsilon) \equiv a_{11} \equiv -1 \bmod (2+\sigma_{22})$, we have $[k(\mathfrak{y}) : k]$=10. This completes our proof.

**Remark.** If $p$ is a rational prime such that $\chi(p) = 1$ then $p = \alpha\,\alpha'$ for an integer $\alpha$

$\in \mathbb{Q}(\sqrt{-43})$. We take $\alpha$ so that $\chi(\alpha+\alpha') = -1$.

Then we have

$$a_p \equiv \alpha + \alpha' \bmod (2+\sigma_{22}).$$

## 3 Tables

### Table (A)

In the Table(A), the mulplicity $s^{q-1,\mu}$ of $\mathfrak{S}^{q-1,\mu}$ etc. is given in the decomposition of $\mathfrak{S}$. (see the equality (1) in the text)

1. the table of $s^{q-1,\mu}$

| | $\mu \equiv 0 \bmod 6$ | $\mu \equiv 3 \bmod 6$ | $\mu \equiv 1,5 \bmod 6$ | $\mu \equiv 2,4 \bmod 6$ |
|---|---|---|---|---|
| $q \equiv 1 \bmod 12$ | $(q-1)/12$ | $(q-1)/12$ | $(q-1)/12$ | $(q-1)/12$ |
| $q \equiv 5 \bmod 12$ | $(q+7)/12$ | $(q+7)/12$ | $(q-5)/12$ | $(q-5)/12$ |
| $q \equiv 7 \bmod 12$ | $(q+5)/12$ | $(q-7)/12$ | $(q-7)/12$ | $(q+5)/12$ |
| $q \equiv 11 \bmod 12$ | $(q+13)/12$ | $(q+1)/12$ | $(q-11)/12$ | $(q+1)/12$ |

where $1 \leq \mu \leq \frac{q-1}{4}$ in the case $q \equiv 1, 5 \bmod 12$ and $1 \leq \mu \leq \frac{q-3}{4}$ in the case $q \equiv 7, 11 \bmod 12$.

2. the table of $s^{q+1,\nu}$

| | $\nu \equiv 0 \bmod 6$ | $\nu \equiv 3 \bmod 6$ | $\nu \equiv 1,5 \bmod 6$ | $\nu \equiv 2,4 \bmod 6$ |
|---|---|---|---|---|
| $q \equiv 1 \bmod 12$ | $(q-25)/12$ | $(q-13)/12$ | $(q-1)/12$ | $(q-13)/12$ |
| $q \equiv 5 \bmod 12$ | $(q-17)/12$ | $(q-5)/12$ | $(q-5)/12$ | $(q-17)/12$ |
| $q \equiv 7 \bmod 12$ | $(q-19)/12$ | $(q-19)/12$ | $(q-7)/12$ | $(q-7)/12$ |
| $q \equiv 11 \bmod 12$ | $(q-11)/12$ | $(q-11)/12$ | $(q-11)/12$ | $(q-11)/12$ |

where $1 \leq \nu \leq \frac{q-5}{4}$ in the case $q \equiv 1,5 \bmod 12$ and $1 \leq \nu \leq \frac{q-3}{4}$ in the case $q \equiv 7,11 \bmod 12$.

Remark that $s^{q+1,\nu} = \dim S_2(\Gamma_0(q),\psi)$, where $\psi$ ia a Dirichet character of conductor $q$ and of order $t$. $t$ is determined as $\zeta^\nu$ is a primitive $t$-th root of unity,where $\zeta$ is a primitive $\frac{q-1}{2}$-th root of unity.

3. the table $s^{\frac{q+1}{2}}$

$$s^{\frac{q+1}{2}} = \begin{cases} (q-25)/24 & q \equiv 1 \bmod 24 \\ (q-5)/24 & q \equiv 5 \bmod 24 \\ (q-17)/24 & q \equiv 17 \bmod 24 \\ (q-13)/24 & q \equiv 13 \bmod 24 \end{cases}$$

Remark that in this case, $q \equiv 1 \bmod 4$, and $s^{\frac{q+1}{2}} = \frac{1}{2}\dim S_2(\Gamma_0(q,(\frac{-}{q}))$, where $(\frac{-}{q})$ denotes the Legendre symbol of level $q$.

4. the table $s^{\frac{q-1}{2}}$

$$s^{\frac{q-1}{2}} = \frac{1}{2}h(-q) + \begin{cases} (q+5)/24 & q \equiv 7 \bmod 24 \\ (q+1)/24 & q \equiv 11 \bmod 24 \\ (q-7)/24 & q \equiv 19 \bmod 24 \\ (q+13)/24 & q \equiv 23 \bmod 24 \end{cases}$$

In this case, $q \equiv 3 \bmod 4$, and $h(-q)$ denotes the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-q})$.

5. the table $s^q$

$$s^q = \begin{cases} (q-25)/24 & q \equiv 1 \bmod 24 \\ (q-5)/24 & q \equiv 5 \bmod 24 \\ (q+7)/24 & q \equiv 17 \bmod 24 \\ (q-13)/24 & q \equiv 13 \bmod 24 \end{cases}$$

Note that $s^q = \dim S_2(\Gamma_0(q))$.

Table (B)

In this Table(B), $\dim S_{\mathrm{I}}$, $\dim S_{\mathrm{II}}$ $(= \dim S_{\mathrm{II}_\chi})$ and $\dim S_{\mathrm{III}}$ are given in the decomposition of $S_2^n(\Gamma_0(q^2))$. (see the equality (4) in the text)

| | $\dim S_{\mathrm{I}}$ | $\dim S_{\mathrm{II}}$ | $\dim S_{\mathrm{III}}$ |
|---|---|---|---|
| $q \equiv 1 \bmod 12$ | $(q^2-22q+117)/48$ | $(q^2-2q+1)/48$ | $(q^2-6q+5)/48$ |
| $q \equiv 5 \bmod 12$ | $(q^2-22q+85)/48$ | $(q^2-2q-15)/48$ | $(q^2-6q+5)/48$ |
| $q \equiv 7 \bmod 12$ | $(q^2-6q-7)/48$ | $(q^2-14q+49)/48$ | $(q^2+2q-15)/48$ |
| $q \equiv 11 \bmod 12$ | $(q^2-6q-7)/48$ | $(q^2-14q+33)/48$ | $(q^2+2q-47)/48$ |

Table (C)

Table (C) is the table of the product $\Phi_{\tilde{f},n}(x)\cdot \Phi_{\tilde{f}_{\chi},n}(x)$. where $f(z)$ belongs to the space $\mathfrak{U}^{q-1,\mu}$.

(1.a) $q$=23, $\mu$ = 1 or $\mu$ = 5 (contained in $S_{\mathrm{I}}$)

| | |
|---|---|
| $T(2)$ | $(x+1)^2\cdot(x+1)^2$ |
| $T(3)$ | $(x-(-1+\alpha))^2$ |
| $T(5)$ | $(x-(\alpha\beta-\beta)/2)(x+(\alpha\beta-\beta)/2)$ |
| $T(7)$ | $(x+\alpha\beta)(x-\alpha\beta)$ |
| $T(11)$ | $(x-(\alpha\beta+2\beta))(x+\alpha\beta+2\beta)$ |
| $T(13)$ | $(x+1+2\alpha)^2$ |

where $\alpha=\sqrt{3}$, $\beta=\sqrt{2}$.

(1.b) $q=23$, $\mu$ = 2 (contained in $S_{\mathrm{III}}$)

| | |
|---|---|
| $T(2)$ | $(x-\alpha)^2\cdot(x+\alpha)^2$ |
| $T(3)$ | $(x-(1+\alpha))^2\cdot(x-(1-\alpha))^2$ |
| $T(5)$ | $(x-\alpha)(x+\alpha)\cdot(x-\alpha)(x+\alpha)$ |
| $T(7)$ | $(x-(-3+\alpha))(x+(-3+\alpha))\cdot(x-(3+\alpha))(x+3+\alpha)$ |

where $\alpha=\sqrt{3}$.

(1.c) $q=23$, $\mu$ = 4(contained in $S_{\mathrm{III}}$)

| | |
|---|---|
| $T(2)$ | $(x-(1+\beta))^2\cdot(x-(1-\beta))^2$ |
| $T(3)$ | $(x-\beta)^2\cdot(x+\beta)^2$ |
| $T(5)$ | $(x-(1-2\beta))(x+(1-2\beta))\cdot(x-(1+2\beta))(x+1+2\beta)$ |
| $T(7)$ | $(x-(2+\beta))(x+2+\beta)\cdot(x-(2-\beta))(x+(2-\beta))$ |

where $\beta=\sqrt{2}$.

(1.d) $q$=23, $\mu$ = 3 (contained in $S_{\mathrm{I}}$)

| | |
|---|---|
| $T(2)$ | $(x-(-1+\gamma)/2)^2 \cdot (x+(1+\gamma)/2)^2$ |
| $T(3)$ | $(x+1)^2 \cdot (x+1)^2$ |
| $T(5)$ | $(x-(\alpha\gamma-\alpha)/2)(x+(\alpha\gamma-\alpha)/2) \cdot (x-(\alpha\gamma+\alpha)/2)(x+(\alpha\gamma+\alpha)/2)$ |
| $T(7)$ | $(x+(\alpha\gamma+\alpha)/2)(x-(\alpha\gamma+\alpha)/2)) \cdot (x+(\alpha\gamma-\alpha)/2)(x-(\alpha\gamma-\alpha)/2))$ |

where $\alpha=\sqrt{3}$, $\gamma=\sqrt{13}$.

(2.a) $q$=31, $\mu$ = 1 or $\mu$ = 3, 5, 7 (contained in $S_{\mathrm{I}}$)

| | |
|---|---|
| $T(2)$ | $(x^2-a_2x+b_2)^2$ |
| $T(3)$ | $x^4-a_3x^2+b_3$ |
| $T(5)$ | $(x^2-a_5x+b_5)^2$ |

where $a_2$ = $<-1, -1, 0, 0>$, $b_2$ = $<-1, 1, 0, 0>$

$a_3$ = $<6, -1, 0, 1>$, $b_3$ = $<4, 4, 1, 0>$,

$a_5$ = $<0, 3, -1, -1>$, $b_5$ = $<-5, 1, 2, 0>$

and $<u_0, u_1, u_2, u_3>$ denotes $u_0+u_1\sigma_{16}+u_2\sigma_{16}^2+u_3\sigma_{16}^3$, where $\sigma_{16} = \zeta_{16}+\zeta_{16}^{-1}$ with a primitive 16-th root of unity $\zeta_{16}$.

(2.b) $q$ = 31, $\mu$ = 2 or $\mu$ = 6 (contained in $S_{\mathrm{III}}$)

| | |
|---|---|
| $T(2)$ | $(x^3-\beta x^2-4x+(-1+3\beta))^2$ |
| $T(3)$ | $x^6+(3\beta-14)x^4+(62-28\beta)x^2+(-80+56\beta)$ |
| $T(5)$ | $(x^3+(-2+2\beta)x^2+(-3-3\beta)x+(2-7\beta))^2$ |

where $\beta = \sqrt{2}$.

(2.c) $q$=31, $\mu$ =4 (contained in $S_{\mathrm{III}}$)

| | |
|---|---|
| $T(2)$ | $(x+1)^2 \cdot (x^2-3x+1)^2$ |
| $T(3)$ | $(x^2-8) \cdot (x^4-6x^2+4)$ |
| $T(5)$ | $x^2 \cdot (x^2-5)^2$ |

(3.a) $q = 43$, $\mu = 1$ or $\mu = 5,7,9$ (contained in $S_{\mathrm{I}}$)

| | |
|---|---|
| $T(2)$ | $(x^2 - a_2)\cdot(x^4 - b_2x^2 + c_2)$ |
| $T(11)$ | $(x - a_{11})^2\cdot(x^2 - b_{11}x + c_{11})^2$ |
| $T(13)$ | $(x - a_{13})^2\cdot(x^2 - b_{13}x + c_{13})^2$ |
| $T(17)$ | $(x - a_{17})^2\cdot(x^2 - b_{17}x + c_{17})^2$ |

where $a_2 = <2, -3, 0, 1, 0>$, $b_2 = <4, -1, 6, 1, -2>$,
$c_2 = <-1, 7, 17, -1, -5>$,
$a_{11} = <-5, 0, 5, 0, -1>$, $b_{11} = <8, 4, -19, -2, 5>$,
$c_{11} = <9, 18, 3, -6, -2>$,
$a_{13} = <2, -5, -5, 2, 1>$, $b_{13} = <-5, 5, 9, -2, -2>$,
$c_{13} = <-12, -6, 18, 2, -5>$,
$a_{17} = <-2, -8, 3, 2, -1>$, $b_{17} = <1, -2, -4, 1, 1>$,
$c_{17} = <-31, -15, 39, 7, -10>$ and $<u_0, u_1, u_2, u_3, u_4>$ denotes $u_0 + u_1\sigma_{22} + u_2\sigma_{22}^2 + u_3\sigma_{22}^3 + u_4\sigma_{22}^4$, where $\sigma_{22} = \zeta_{22} + \zeta_{22}^{-1}$ with a primitive 22-th root of unity $\zeta_{22}$.

(3.b) $q = 43$, $\mu = 2$ or $\mu = 4,6,8,10$. (contained in $S_{\mathrm{III}}$)

| | |
|---|---|
| $T(2)$ | $x^8 - a_2x^6 + b_2x^4 - c_2x^2 + d_2$ |
| $T(11)$ | $(x^4 - a_{11}x^3 + b_{11}x^2 - c_{11}x + d_{11})^2$ |
| $T(13)$ | $(x^4 - a_{13}x^3 + b_{13}x^2 - c_{13}x + d_{13})^2$ |

where $a_2 = <17, 0, -8, 0, 2>$, $b_2 = <90, -3, -72, 1, 18>$,
$c_2 = <154, 2, -149, -1, 36>$, $d_2 = <40, 17, -25, -6, 4>$,
$a_{11} = <9, 2, -8, 0, 2>$, $b_{11} = <13, 14, -33, -2, 8>$,
$c_{11} = <-80, -32, 147, 1, -40>$, $d_{11} = <-94, 36, 72, -12, -14>$.
$a_{13} = <3, 4, 0, 0, 0>$, $b_{13} = <-15, 9, 13, 0, -3>$,
$c_{13} = <-10, -10, -36, -11, 14>$, $d_{13} = <9, -50, 13, 13, -14>$
and $<u_0, u_1, u_2, u_3, u_4,>$ denotes $u_0 + u_1\sigma_{11} + u_2\sigma_{11}^2 + u_3\sigma_{11}^3 + u_4\sigma_{11}^4$, where $\sigma_{11} = \zeta_{11} + \zeta_{11}^{-1}$ with a primitive 11-th root of unity $\zeta_{11}$.

(4.a) $q = 47$, $\mu = 1$ or $\mu = 5,7,11$ (contained in $S_{\mathrm{I}}$)

| | |
|---|---|
| $T(2)$ | $(x^3 - a_2x^2 + b_2x - c_2)^2$ |
| $T(3)$ | $(x^3 - a_3x^2 + b_3x - c_3)^2$ |
| $T(5)$ | $x^6 - a_5x^4 + b_5x^2 - c_5$ |

where $a_2 = <-1, -1, 0, 0>$, $b_2 = <-2, 1, 0, 0>$, $c_2 = <0, 1, 0, 0>$,
$a_3 = <-1, -4, 0, 1>$, $b_3 = <-5, 4, 1, -1>$, $c_3 = <0, 15, 0, -4>$,
$a_5 = <14, 3, 0, 0>$, $b_5 = <40, 8, 2, 4>$, $c_5 = <4, -12, 14, 11>$.
$<u_0, u_1, u_2, u_3>$ denotes $u_0 + u_1\sigma_{24} + u_2\sigma_{24}^2 + u_3\sigma_{24}^3$, where $\sigma_{24} = \zeta_{24} + \zeta_{24}^{-1}$ with a primitive 24-th root of unity $\zeta_{24}$.

(4.b) $q = 47$, $\mu = 2$ or $\mu = 10$ (contained in $S_{\mathrm{III}}$)

| | |
|---|---|
| $T(2)$ | $(x^4 - \alpha x^3 - 5x^2 + 4\alpha x + 2 - \alpha)^2$ |
| $T(3)$ | $(x^4 + (-1+\alpha)x^3 + (-8-\alpha)x^2 + (4-5\alpha)x + (8-2\alpha))^2$ |
| $T(5)$ | $x^8 + (-24+2\alpha)x^6 + (193-28\alpha)x^4 + (-598+111\alpha)x^2 + (613-140\alpha)$ |

where $\alpha = \sqrt{3}$.

(4.c) $q = 47$, $\mu = 4$ (contained in $S_{\mathrm{III}}$)

| | |
|---|---|
| $T(2)$ | $(x^4 - 3x^3 - x^2 + 6x - 1)^2$ |
| $T(3)$ | $(x^4 - 2x^3 - 5x^2 + 7x - 2)^2$ |
| $T(5)$ | $x^8 - 30x^6 + 279x^4 - 837x^2 + 81$ |

(4.d) $q = 47$, $\mu = 8$ (contained in $S_{\mathrm{III}}$)

| | |
|---|---|
| $T(2)$ | $(x+1)^2 \cdot (x^3 - 2x^2 - 3x + 5)^2$ |
| $T(3)$ | $(x-2)^2 \cdot (x^3 + 2x^2 - 5x - 5)^2$ |
| $T(5)$ | $(x+3)(x-3) \cdot (x^3 + x^2 - 4x + 1)(x^3 - x^2 - 4x - 1)$ |

(4.e) $q = 47$, $\mu = 3$ or $\mu = 9$ (contained in $S_{\mathrm{I}}$)

| | |
|---|---|
| $T(2)$ | $(x^4 + (1+\beta)x^3 + (-5+\beta)x^2 + (-3-4\beta)x + (3-3\beta))^2$ |
| $T(3)$ | $(x^4 + (2-\beta)x^3 + (-5-2\beta)x^2 + (-6+4\beta)x + (3+5\beta))^2$ |
| $T(5)$ | $x^8 + (-26+9\beta)x^6 + (260-152\beta)x^4 + (-944+656\beta)x^2 + (612-432\beta)$ |

where $\beta = \sqrt{2}$.

(4.f) $q = 47$, $\mu = 6$ (contained in $S_{\mathrm{III}}$)

| | |
|---|---|
| $T(2)$ | $(x^5 - 8x^3 + 8x + 3)^2$ |
| $T(3)$ | $(x^5 - 2x^4 - 7x^3 + 12x^2 + 7x - 8)^2$ |
| $T(5)$ | $x^{10} - 36x^8 + 460x^6 - 2416x^4 + 4432x^2 - 2592$ |

Table (D)

Table (D) is the table of $\Phi_{\tilde{f},n}(x)$ where $f(z)$ belongs to the space $\mathfrak{U}^{\frac{q-1}{2}}$.

1.$q = 23$, $h(-23) = 3$ (contained in $S_{\mathrm{III}}$)

| | |
|---|---|
| $T(2)$ | $x^3 - 6x - 3$ |
| $T(3)$ | $x^3 - 9x - 2$ |
| $T(5)$ | $x^3$ |

2. $q = 31$, $h(-31) = 3$ (contained in $S_{\mathrm{III}}$)

| | |
|---|---|
| $T(2)$ | $x^3 - 6x - 1$ |
| $T(3)$ | $x^3$ |
| $T(5)$ | $x^3 - 15x - 2$ |

3. $q = 43$, $h(-43) = 1$ (contained in $S_{\mathrm{I}}$)

| | |
|---|---|
| $T(2)$ | $x \cdot (x^2 - 6)$ |
| $T(11)$ | $(x + 1 \cdot (x + 1)^2$ |
| $T(13)$ | $(x - 3) \cdot (x + 3)^2$ |

4. $q = 47$ , $h(-47) = 5$ (contained in $S_{\mathrm{III}}$)

| | |
|---|---|
| $T(2)$ | $x^5 - 10x^3 + 20x - 9$ |
| $T(3)$ | $x^5 - 15x^3 + 45x - 28$ |
| $T(5)$ | $x^5$ |

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