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Securely Implementable Social Choice Functions in Divisible and Non-Excludable Public Good Economies with Quasi-Linear Utility Functions *

Katsuhiko Nishizaki †‡

March 30, 2016

Abstract

This paper studies the possibility of secure implementation (Saijo, T., T. Sjöström, and T. Yamato (2007) "Secure Implementation," *Theoretical Economics* 2, pp.203-229) in divisible and non-excludable public good economies with quasi-linear utility functions. Although Saijo, Sjöström, and Yamato (2007) showed that the Groves mechanisms (Groves, T. (1973) "Incentives in Teams," *Econometrica* 41, pp.617-631) are securely implementable in some of the economies, we have the following negative result: securely implementable social choice functions are dictatorial or constant in divisible and non-excludable public good economies with quasi-linear utility functions.

Keywords: Secure Implementation, Dominant Strategy Implementation, Nash Implementation, Strategy-Proofness, Non-Excludable Public Good.

JEL Classification: C72, D61, D63, D71, H41.

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1 Introduction

1.1 Background

This paper considers divisible and non-excludable public good economies in which $n \geq 2$ agents collectively decide (i) how much of the public good (e.g., seawalls, protection forests, and storm sewers) should be provided and (ii) how the cost should be shared among the agents. These decisions are made to achieve a goal characterized by a **social choice function** that associates an outcome with the agents' information. The agents' information is induced by a (direct) **mechanism** that associates an outcome with the agents' "revealed" information. In fact, a mechanism is equivalent to a social choice function. In public good economies, an outcome is defined as an allocation which is a profile of consumption bundles, where each consumption bundle consists of consumption of the public good and a cost share of the public good, and information as preferences defined over the set of consumption bundles. This paper assumes that each preference is represented by a quasi-linear utility function.

The manipulability of the mechanism is an important issue during its construction: some agent might reveal untruthful information to manipulate the outcome in the agent's favor. **Strategy-proofness** prevents such an untruthful revelation. This property requires that truthful revelation is a weakly dominant strategy for the agent. Many researchers have attempted to construct strategy-proof mechanisms with desirable properties in non-excludable public good economies. Groves (1973) introduced strategy-proof and decision-efficient mechanisms, called the Groves mechanisms. [1] Holmström (1979) showed that the Groves mechanisms are the only mechanisms that satisfy strategy-proofness and decision-efficiency in standard quasi-linear environments. In addition, Green and Laffont (1979) showed that the Groves mechanisms rarely satisfy budget-balancedness, that is, they rarely satisfy Pareto-efficiency. [2] On the basis of these findings, Moulin (1994) and Serizawa (1996, 1999) studied strategy-proof and budget-balanced mechanisms.

Although strategy-proofness is a desirable property, some experimental studies have questioned the performance of strategy-proof mechanisms. Because strategy-proofness does not require that truthful revelation is "strictly" dominant strategy for the agent, strategy-proof mechanisms might have multiple Nash equilibria that achieve non-optimal outcomes. [3] Attiyeh, Franciosi, and Isaac (2000), Kawagoe and Mori (2001), and Cason, Saijo, Sjöström, and Yamato (2006) observed that strategy-proof mechanisms with such "bad" Nash equilibria do not work well in laboratory experiments. [4] On the basis of these observations, Saijo, Sjöström, and Yamato (2007) introduced **secure implementation** that is defined as double implementation in dominant strategy equilibria and Nash equilibria. Cason, Saijo, Sjöström, and Yamato (2006) conducted experiments on secure implementation and suggested that it might be a

[1] Decision-efficiency requires that the consumption of the public good maximizes the sum of all the agents' benefits from the consumption. See Clarke (1971), Groves and Loeb (1975), Tideman and Tullock (1976), and Moulin (1986) for the Groves mechanisms in non-excludable public good economies.

[2] Budget-balancedness requires that the sum of cost shares of the public good is equal to the entire cost of providing the public good. In quasi-linear environments, the combination of decision-efficiency and budget-balancedness is equivalent to Pareto-efficiency. See Groves and Loeb (1975), Laffont and Maskin (1980), Tian (1996), and Liu and Tian (1999) for budget-balanced Groves mechanisms in non-excludable public good economies.

[3] See Saijo, Sjöström, and Yamato (2003) for examples of such Nash equilibria.

[4] See Chen (2008) for a survey of experimental studies on strategy-proof mechanisms in non-excludable public good economies.

benchmark for constructing a mechanism that works well in practice.

Saijo, Sjöström, and Yamato (2007) showed that the social choice function is **securely implementable** if and only if it satisfies strategy-proofness and the **rectangular property** (Saijo, Sjöström, and Yamato, 2007). [5] The rectangular property requires that the allocation does not change by changing all the agents' revelations, each of whom does not change the agent's utility. In addition, they showed that the rectangular property is in general equivalent to the combination of **strong non-bossiness** (Ritz, 1983) and the **outcome rectangular property** (Saijo, Sjöström, and Yamato, 2007). Neither strong non-bossiness nor the outcome rectangular property is equivalent to the rectangular property in the model presented here (see Examples 2 and 3). Strong non-bossiness requires that the agent cannot change the allocation by changing the agent's revelation while maintaining the agent's utility. This property is in general stronger than **non-bossiness** (Satterthwaite and Sonnenschein, 1981) requiring that the agent cannot change the allocation by changing the agent's revelation while maintaining the agent's consumption bundle. In addition, both properties are not equivalent in the model presented here (see Remark 11). The outcome rectangular property requires that the allocation does not change by changing all the agents' revelations, each of whom does not change the allocation. This property is independent of non-bossiness in the model presented here (see Remark 14). On the basis of these characterizations, some researchers have studied the possibility of secure implementation in several environments: voting environments (Saijo, Sjöström, and Yamato, 2007; Berga and Moreno, 2009), public good economies (Saijo, Sjöström, and Yamato, 2007; Nishizaki, 2011, 2013), pure exchange economies (Mizukami and Wakayama, 2005; Nishizaki, 2014), the problems of providing a divisible and private good with monetary transfers (Saijo, Sjöström, and Yamato, 2007; Kumar, 2013), the problems of allocating indivisible and private goods with monetary transfers (Fujinaka and Wakayama, 2008), queueing problems (Nishizaki, 2012), Shapley-Scarf housing markets (Fujinaka and Wakayama, 2011), and allotment economies with single-peaked preferences (Bochet and Sakai, 2010). [6] These studies illustrated the difficulty of finding securely implementable social choice functions with desirable properties.

1.2 Motivation

Investigating which environment has a non-trivial securely implementable social choice function is an interesting research topic because secure implementability might be a benchmark for constructing a mechanism that works well in practice, as stated in Subsection 1.1. This paper conducts such an investigation into divisible and non-excludable public good economies with quasi-linear utility functions. In some of the economies, Saijo, Sjöström, and Yamato (2007) showed that the Groves mechanisms are securely implementable.

1.3 Related Literature

This paper is closely related to those of Moulin (1994), Serizawa (1996, 1999), Saijo, Sjöström, and Yamato (2007), and Nishizaki (2013). The conservative equal cost sharing mechanism (Moulin, 1994)

[5] See Mizukami and Wakayama (2008) for an alternative characterization of securely implementable social choice functions in terms of a stronger version of Maskin monotonicity (Maskin, 1977).

[6] In addition, see Saijo, Sjöström, and Yamato (2003) for theoretical results on secure implementation.

is a strategy-proof and budget-balanced mechanism. [7] In non-excludable public good economies with classical preferences, Moulin (1994) characterized this mechanism by symmetry, individual rationality, and non-imposition in addition to group strategy-proofness and budget-balancedness. [8] Serizawa (1999) strengthened this characterization by replacing group strategy-proofness with strategy-proofness and dropping non-imposition. [9] In other directions, Serizawa (1996) characterized semi-convex cost sharing schemes determined by a minimum demand principle (Serizawa, 1996) by non-bossiness, individual rationality, and non-exploitation in addition to strategy-proofness and budget-balancedness. [10] Neither mechanism is securely implementable in the model presented here. [11] On the other hand, the Groves mechanisms are strategy-proof and decision-efficient mechanisms. Although Saijo, Sjöström, and Yamato (2007) showed that these mechanisms are securely implementable in certain divisible and non-excludable public good economies, they are not securely implementable in the model presented here (see Remark 15). In addition, Nishizaki (2013) showed a constancy result on secure implementation in discrete public good economies with quasi-linear utility functions. [12]

1.4 Overview of Results

This paper demonstrates that securely implementable social choice functions are dictatorial or constant in divisible and non-excludable public good economies with quasi-linear utility functions. This main result is compatible with the finding that the conservative equal cost sharing mechanism, semi-convex cost sharing schemes determined by a minimum demand principle, and the Groves mechanisms are not securely implementable in the model presented here. On the basis of the observations of Cason, Saijo, Sjöström, and Yamato (2006), the negative result suggests that non-trivial strategy-proof mechanisms actually do not work well in the economies except a limited number of the environments. In addition, this paper presents some technical results on secure implementation. These results contribute to studying the possibility of secure implementation in other environments.

The remainder of this paper is organized as follows. Section 2 introduces the model presented here and Section 3 the properties of social choice functions related to secure implementation. Section 4

[7] See Section 5 for a formal definition of the conservative equal cost sharing mechanism.

[8] Symmetry requires that the two agents with the same preference are treated equally in terms of their consumption bundles. Individual rationality requires that the agent is not worse off than at the status quo. Non-imposition requires the ontoness for the range of consumption of the public good. Group strategy-proofness is in general stronger than strategy-proofness and prevents any untruthful revelation by any group of agents, that changes the outcome in the agents' favor. Both properties are not equivalent in the model presented here because the Groves mechanisms satisfy strategy-proofness, but not group strategy-proofness.

[9] In addition, Serizawa (1999) characterized strategy-proof, budget-balanced, and anonymous social choice functions. Anonymity requires that the consumption bundles for the two agents are switched when their preferences are switched. See also Ohseto (1997) for strengthening the characterization of Moulin (1994).

[10] Non-exploitation requires that no agents are forced into monetary transfers to other agents in addition to sharing the entire cost of providing the public good. See Serizawa (1996) for a formal definition of a semi-convex cost sharing scheme and Deb and Ohseto (1999) for the characterization.

[11] See Section 5 for the conservative equal cost sharing mechanism and Remark 11 for semi-convex cost sharing schemes determined by a minimum demand principle.

[12] Specifically, this constancy is implied only by strategy-proofness and strong non-bossiness. In addition, Saijo, Sjöström, and Yamato (2007) showed the difficulty of secure implementation in discrete and non-excludable public good economies with quasi-linear utility functions.

demonstrates preliminary results on the properties and Section 5 the main result. Section 6 concludes this paper.

2 Model

This paper considers the problem of providing a divisible and non-excludable public good with the cost shares. Let $I \equiv \{1,\ldots,n\}$ be the set of **agents**, where $n \geq 2$. Let $Y \subseteq \mathbb{R}_+ \equiv \{r \in \mathbb{R} | r \geq 0\}$ be a convex set of **production levels of the public good** and $c \colon Y \to \mathbb{R}_+$ be the **cost function**. In the model presented here, a production level of the public good is equal to consumption of the public good for all the agents. For each $i \in I$, let $(y, x_i) \in Y \times \mathbb{R}_+$ be a **consumption bundle for agent *i***, where $x_i \in \mathbb{R}_+$ is a **cost share of the public good for agent *i***. Let (y, x) be an **allocation**, where $x \equiv (x_i)_{i \in I}$ is a profile of cost shares of the public good, and $Z \equiv \{(y,x) \in Y \times \mathbb{R}^n_+ | c(y) \leq \sum_{i \in I} x_i\}$ be the set of **feasible allocations**.

This paper assumes that an agent's preference is represented by a quasi-linear utility function. For each $i \in I$, let $u_i \colon Y \times \mathbb{R}_+ \to \mathbb{R}$ be an **utility function for agent *i*** such that there is $v_i \colon Y \to \mathbb{R}$, called a **valuation function of the public good for agent *i***, and for each $(y, x_i) \in Y \times \mathbb{R}_+$, $u_i(y, x_i) = v_i(y) - x_i$. For each $i \in I$, let V_i be the set of all valuation functions of the public good for agent i, that are strictly increasing and strictly concave. Let $v \equiv (v_k)_{k \in I}$ be a profile of valuation functions of the public good and $V \equiv \prod_{k \in I} V_k$ be the set of the profiles. For each $i \in I$, let $v_{-i} \equiv (v_k)_{k \in I \setminus \{i\}}$ be a profile of valuation functions of the public good other than agent i and $V_{-i} \equiv \prod_{k \in I \setminus \{i\}} V_k$ be the set of the profiles. For each $i, j \in I$, let $v_{-i,j} \equiv (v_k)_{k \in I \setminus \{i,j\}}$ be a profile of valuation functions of the public good other than agents i and j. For each $S, S', S'' \subseteq I$, where these sets are mutually disjoint and $S \cup S' \cup S'' = I$, and each $v, v', v'' \in V$, let $(v_S, v'_{S'}, v''_{S''})$ be the profile of valuation functions of the public good, where agent $i \in S$ has v_i, agent $i \in S'$ has v'_i, and agent $i \in S''$ has v''_i. For each $i \in I$, each $v_i \in V_i$, and each $(y, x_i) \in Y \times \mathbb{R}_+$, let $UC(y, x_i; v_i) \equiv \{(y', x'_i) \in Y \times \mathbb{R}_+ | v_i(y) - x_i \leq v_i(y') - x'_i\}$ be the **upper contour set for agent *i* with the valuation function of the public good v_i at the consumption bundle (y, x_i)**. In addition, let $M_i(y, x_i; v_i) \equiv \{v'_i \in V_i | UC(y, x_i; v'_i) \subseteq UC(y, x_i; v_i)\}$ be the **set of monotonic transformations for agent *i* with the valuation function of the public good v_i at the consumption bundle (y, x_i)** and $SM_i(y, x_i; v_i) \equiv \{v'_i \in M_i(y, x_i; v_i) | v_i(y) - x_i < v_i(y') - x'_i$ for each $(y', x'_i) \in UC(y, x_i; v'_i) \setminus \{(y, x_i)\}\}$ be the **set of strictly monotonic transformations for agent *i* with the valuation function of the public good v_i at the consumption bundle (y, x_i)**.

A social choice function associates an allocation with a profile of valuation functions of the public good. Let $f \colon V \to Z$ be a **social choice function**. For each $v \in V$, let $(y(v), x(v)) \in Z$ be the allocation under the social choice function f at the profile of valuation functions of the public good v and $(y(v), x_i(v))$ be the consumption bundle for agent $i \in I$ at the allocation $(y(v), x(v))$.

3 Properties of Social Choice Functions

This paper studies the possibility of secure implementation in divisible and non-excludable public good economies with quasi-linear utility functions. The social choice function is **securely implementable** if and only if there is a mechanism that simultaneously implements it in dominant strategy equilibria and in Nash equilibria. Saijo, Sjöström, and Yamato (2007, Theorem 1) characterized this social choice function by **strategy-proofness** and the **rectangular property** (Saijo, Sjöström, and Yamato, 2007). Strategy-proofness requires that truthful revelation is a weakly dominant strategy for the agent. The rectangular property requires that if each agent cannot change the agent's "utility" by changing the agent's revelation, then the allocation does not change by changing all the agents' revelations.

Definition 1. The social choice function f satisfies **strategy-proofness** if and only if for each $v, v' \in V$ and each $i \in I$, $v_i(y(v_i, v'_{-i})) - x_i(v_i, v'_{-i}) \geq v_i(y(v'_i, v'_{-i})) - x_i(v'_i, v'_{-i})$.

Definition 2. The social choice function f satisfies the **rectangular property** if and only if for each $v, v' \in V$, if $v_i(y(v_i, v'_{-i})) - x_i(v_i, v'_{-i}) = v_i(y(v'_i, v'_{-i})) - x_i(v'_i, v'_{-i})$ for each $i \in I$, then $(y(v), x(v)) = (y(v'), x(v'))$.

In addition, Saijo, Sjöström, and Yamato (2007, Proposition 3) showed that the rectangular property is in general equivalent to the combination of **strong non-bossiness** (Ritz, 1983) and the **outcome rectangular property** (Saijo, Sjöström, and Yamato, 2007). Strong non-bossiness requires that if the agent does not change the agent's "utility" by changing the agent's revelation, then the allocation also does not change by the change of the revelation. The outcome rectangular property requires that if each agent cannot change the "allocation" by changing the agent's revelation, then the allocation does not change by changing all the agents' revelations.

Definition 3. The social choice function f satisfies **strong non-bossiness** if and only if for each $v, v' \in V$ and each $i \in I$, if $v_i(y(v_i, v'_{-i})) - x_i(v_i, v'_{-i}) = v_i(y(v'_i, v'_{-i})) - x_i(v'_i, v'_{-i})$, then $(y(v_i, v'_{-i}), x(v_i, v'_{-i})) = (y(v'_i, v'_{-i}), x(v'_i, v'_{-i}))$.

Definition 4. The social choice function f satisfies the **outcome rectangular property** if and only if for each $v, v' \in V$, if $(y(v_i, v'_{-i}), x(v_i, v'_{-i})) = (y(v'_i, v'_{-i}), x(v'_i, v'_{-i}))$ for each $i \in I$, then $(y(v), x(v)) = (y(v'), x(v'))$.

Neither strong non-bossiness nor the outcome rectangular property is equivalent to the rectangular property in the model presented here (see Examples 2 and 3). In general, strong non-bossiness is stronger than **non-bossiness** (Satterthwaite and Sonnenschein, 1981) requiring that if the agent does not change the agent's "consumption bundle" by changing the agent's revelation, then the allocation also does not change by the change of the revelation. Both properties are not equivalent in the model presented here (see Remark 11).

Definition 5. The social choice function f satisfies **non-bossiness** if and only if for each $v, v' \in V$ and each $i \in I$, if $(y(v_i, v'_{-i}), x_i(v_i, v'_{-i})) = (y(v'_i, v'_{-i}), x_i(v'_i, v'_{-i}))$, then $(y(v_i, v'_{-i}), x(v_i, v'_{-i})) = (y(v'_i, v'_{-i}), x(v'_i, v'_{-i}))$.

Remark 1. Although the premise of the outcome rectangular property considers an allocation, that of non-bossiness considers a consumption bundle. In the model presented here, both properties are independent (see Remark 14).

4 Preliminary Results

This section demonstrates preliminary results on strategy-proofness, non-bossiness, strong non-bossiness, and the outcome rectangular property. These results specify the characteristics of the option sets, the cost shares of the public good, and the range of consumption of the public good under a securely implementable social choice function.

For each $i \in I$ and each $v'_{-i} \in V_{-i}$, let $O_i(v'_{-i}) \equiv \{y \in Y | \text{there is } v_i \in V_i \text{ such that } y(v_i, v'_{-i}) = y\}$ be the **option set for agent i at v_{-i} under the social choice function f**, that is, the set of consumption of the public good, that the agent can induce given f and v_{-i} and $O_i(V_{-i}) \equiv \cup_{v'_{-i} \in V_{-i}} O_i(v'_{-i})$. In addition, let $y(V) \equiv \{y \in Y | \text{there is } v \in V \text{ such that } y(v) = y\}$ be the **range of consumption of the public good under the social choice function f**, that is, the set of consumption of the public good, that all the agents can induce given f. By definition, $y(V) \supseteq O_i(V_{-i})$ for each $i \in I$. Lemma 1 shows that both sets are equivalent.

Lemma 1. *For each $i \in I$, $y(V) = O_i(V_{-i})$.*

Proof. Let $i \in I$. We show that $y(V) \subseteq O_i(V_{-i})$ because $y(V) \supseteq O_i(V_{-i})$ by definition. Let $y' \in y(V)$. This implies that there is $v' \in V$ such that $y(v'_i, v'_{-i}) = y'$ and $y(v'_i, v'_{-i}) \in O_i(v'_{-i}) \subseteq O_i(V_{-i})$ by definition. □

For each $i \in I$, let $t_i : Y \to \mathbb{R}_+$ be a **cost sharing function for agent i**.

Definition 6. The social choice function f is a **cost sharing scheme** if and only if there are cost sharing functions $t_1, \cdots, t_n$ such that for each $v \in V$ and each $i \in I$, $x_i(v) = t_i(y(v))$.

Definition 7. The cost sharing scheme f is

(a) **strictly increasing** if and only if for each $i \in I$ and each $v'_{-i} \in V_{-i}$, the cost sharing function t_i is strictly increasing on the option set $O_i(v'_{-i})$, that is, for each $y, y' \in O_i(v'_{-i})$, where $y < y'$, $t_i(y) < t_i(y')$,

(b) **lower semi-continuous** if and only if for each $i \in I$ and each $v'_{-i} \in V_{-i}$, the cost sharing function t_i is lower semi-continuous on the option set $O_i(v'_{-i})$, that is, for each $y \in O_i(v'_{-i})$ and each $\varepsilon > 0$, there is a neighborhood $U \subseteq O_i(v'_{-i})$ of y such that $t_i(y') \geq t_i(y) - \varepsilon$ for each $y' \in U$,

(c) **upper semi-continuous** if and only if for each $i \in I$ and each $v'_{-i} \in V_{-i}$, the cost sharing function t_i is upper semi-continuous on the option set $O_i(v'_{-i})$, that is, for each $y \in O_i(v'_{-i})$ and each $\varepsilon > 0$, there is a neighborhood $U \subseteq O_i(v'_{-i})$ of y such that $t_i(y') \leq t_i(y) + \varepsilon$ for each $y' \in U$,

(d) **continuous** if and only if for each $i \in I$ and each $v'_{-i} \in V_{-i}$, the cost sharing function t_i is continuous on the option set $O_i(v'_{-i})$, that is, t_i is upper semi-continuous and lower semi-continuous on $O_i(v'_{-i})$, and

(e) **convex** if and only if for each $i \in I$ and each $v'_{-i} \in V_{-i}$, the cost sharing function t_i is convex on the option set $O_i(v'_{-i})$, that is, for each $y, y' \in O_i(v'_{-i})$ and each $\lambda \in [0,1]$, $\lambda t_i(y) + (1-\lambda) t_i(y') \geq t_i(\lambda y + (1-\lambda) y')$.

Remark 2. The properties of a cost sharing scheme in Definition 7 are required on the option sets, but not on the set of consumption of the public good.

The remainder of this section demonstrates that (i) the option set is closed by strategy-proofness (Proposition 1) and convex by strategy-proofness and strong non-bossiness (Proposition 3), (ii) a social choice function satisfying strategy-proofness and non-bossiness is a cost sharing scheme (Corollary 2),

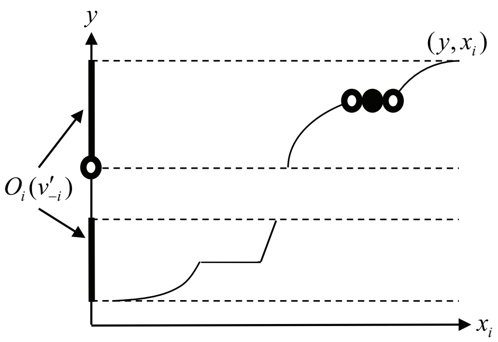

Figure 1: An implication of Lemma 2

(iii) the cost sharing scheme satisfying strategy-proofness is strictly increasing (Corollary 1) and lower semi-continuous (Lemma 3), and (iv) the cost sharing scheme satisfying strategy-proofness and strong non-bossiness is convex (Proposition 4). It further demonstrates that (v) the range of consumption of the public good is closed by strategy-proofness, non-bossiness, and the outcome rectangular property (Proposition 5) and convex by strategy-proofness, strong non-bossiness, and the outcome rectangular property (Proposition 6). On the basis of these results, we find the strict increasingness and continuity of cost sharing schemes satisfying strategy-proofness, strong non-bossiness, and the outcome rectangular property on the range of consumption of the public good (Remark 13).

4.1 Strategy-Proofness

Lemma 2 shows that the more the agent consumes the public good, the more the agent shares the cost of the public good on the agent's option set if the social choice function satisfies strategy-proofness (see Figure 1)and the agent's valuation function of the public good is strictly increasing (see Figure 1). This relationship among consumption bundles is called the diagonality (Barberà and Jackson, 1995). [13]

Lemma 2. *Suppose that the social choice function f satisfies* ***strategy-proofness****. For each $v, v' \in V$ and each $i \in I$, if $y(v_i, v'_{-i}) < y(v'_i, v'_{-i})$, then $x_i(v_i, v'_{-i}) < x_i(v'_i, v'_{-i})$.*

Proof. To the contrary, we suppose that there are $v, v' \in V$ and $i \in I$ such that $y(v_i, v'_{-i}) < y(v'_i, v'_{-i})$ and $x_i(v_i, v'_{-i}) \geq x_i(v'_i, v'_{-i})$. By the former and the strict increasingness of valuation functions of the public good, we find that $v_i(y(v_i, v'_{-i})) < v_i(y(v'_i, v'_{-i}))$. Together with the latter, this implies that $v_i(y(v_i, v'_{-i})) - x_i(v_i, v'_{-i}) < v_i(y(v'_i, v'_{-i})) - x_i(v'_i, v'_{-i})$ and contradicts strategy-proofness. □

By Lemma 2, we have Corollary 1 showing the strict increasingness of cost sharing schemes satisfying strategy-proofness. On the other hand, we have Lemma 3 showing the lower semi-continuity of cost sharing schemes satisfying strategy-proofness on the basis of the continuity of valuation functions of the public good.

Corollary 1. *If the cost sharing scheme satisfies* ***strategy-proofness****, then it is* ***strictly increasing****.*

[13] Barberà and Jackson (1995) considered the diagonality of the range of consumption of the public good under a social choice function in pure exchange economies.

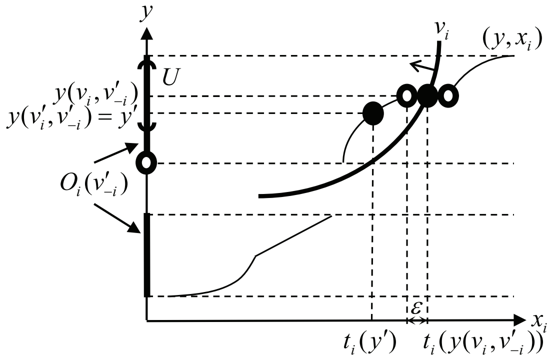

Figure 2: Proof of Lemma 3

Remark 3. The combination of Lemma 1 and Corollary 1 does not necessarily imply the strict increasingness of cost sharing schemes satisfying strategy-proofness on the range of consumption of the public good.

Lemma 3. *If the cost sharing scheme f satisfies **strategy-proofness**, then it is **lower semi-continuous**.*

Proof. To the contrary, we suppose that f is not lower semi-continuous. This implies that there are $i \in I$ and $v'_{-i} \in V_{-i}$ such that t_i is not lower semi-continuous on $O_i(v'_{-i})$. In addition, there is $v_i \in V_i$ such that t_i is not lower semi-continuous at $y(v_i, v'_{-i})$. This implies that there is $\varepsilon \in \mathbb{R}_+$ such that for each neighborhood $U \subseteq O_i(v'_{-i})$ of $y(v_i, v'_{-i})$,

$$t_i(y') < t_i(y(v_i, v'_{-i})) - \varepsilon \tag{1}$$

for some $y' \in U$. By the continuity of valuation functions of the public good, we can take the neighborhood to satisfy the following condition:

$$v_i(y(v_i, v'_{-i})) - v_i(y') < \varepsilon. \tag{2}$$

Because $U \subseteq O_i(v'_{-i})$, there is $v'_i \in V_i$ such that $y(v'_i, v'_{-i}) = y'$ and we find that $v_i(y(v_i, v'_{-i})) - v_i(y(v'_i, v'_{-i})) < \varepsilon < t_i(y(v_i, v'_{-i})) - t_i(y(v'_i, v'_{-i}))$ by (1) and (2). This implies that $v_i(y(v_i, v'_{-i})) - x_i(v_i, v'_{-i}) < v_i(y(v'_i, v'_{-i})) - x_i(v'_i, v'_{-i})$ and contradicts strategy-proofness (see Figure 2). □

Proposition 1 shows the closedness of the option sets under a cost sharing scheme satisfying strategy-proofness on the basis of Corollary 1 and the continuity and strict increasingness of valuation functions of the public good.

Proposition 1. *Suppose that the cost sharing scheme f satisfies **strategy-proofness**. For each $i \in I$ and each $v'_{-i} \in V_{-i}$, $O_i(v'_{-i})$ is **closed**.*

Proof. To the contrary, we suppose that there are $i \in I$ and $v'_{-i} \in V_{-i}$ such that $O_i(v'_{-i})$ is not closed. This implies that we can take $y \in \bar{O}(v'_{-i}) \setminus O_i(v'_{-i})$, where $\bar{O}(v'_{-i})$ is the closure of $O_i(v'_{-i})$. We have the following three situations according to the relationship between y and $O_i(v'_{-i})$.

Situation 1. $\boldsymbol{y = \inf O_i(v'_{-i})}$

Let $x_i^H \equiv \inf\{x_i \in \mathbb{R}_+ | \text{there is } v_i \in V_i \text{ such that } t_i(y(v_i, v'_{-i})) = x_i\}$. By Corollary 1, the definition of x_i^H, and the continuity and strict increasingness of valuation functions of the public good, we can take $v_i \in V_i$ such that $v_i(y) - v_i(y(v''_i, v'_{-i})) > x_i^H - t_i(y(v''_i, v'_{-i}))$ for each $v''_i \in V_i$. [14] This implies that $v_i(y(v_i, v'_{-i})) - t_i(y(v_i, v'_{-i})) < v_i(y) - x_i^H$. Together with the supposition of y and the definition of x_i^H, this implies that we can take $v'_i \in V_i$ such that $v_i(y(v_i, v'_{-i})) - t_i(y(v_i, v'_{-i})) < v_i(y(v'_i, v'_{-i})) - t_i(y(v'_i, v'_{-i}))$, that is, $v_i(y(v_i, v'_{-i})) - x_i(v_i, v'_{-i}) < v_i(y(v'_i, v'_{-i})) - x_i(v'_i, v'_{-i})$. This contradicts strategy-proofness.

Situation 2. $\boldsymbol{y = \sup O_i(v'_{-i})}$

Let $x_i^L \equiv \sup\{x_i \in \mathbb{R}_+ | \text{there is } v_i \in V_i \text{ such that } t_i(y(v_i, v'_{-i})) = x_i\}$. By Corollary 1, the definition of x_i^L, and the continuity and strict increasingness of valuation functions of the public good, we can take $v_i \in V_i$ such that $v_i(y) - v_i(y(v''_i, v'_{-i})) > x_i^L - t_i(y(v''_i, v'_{-i}))$ for each $v''_i \in V_i$. [15] This implies that $v_i(y(v_i, v'_{-i})) - t_i(y(v_i, v'_{-i})) < v_i(y) - x_i^L$. Together with the supposition of y and the definition of x_i^L, this implies that we can take $v'_i \in V_i$ such that $v_i(y(v_i, v'_{-i})) - t_i(y(v_i, v'_{-i})) < v_i(y(v'_i, v'_{-i})) - t_i(y(v'_i, v'_{-i}))$, that is, $v_i(y(v_i, v'_{-i})) - x_i(v_i, v'_{-i}) < v_i(y(v'_i, v'_{-i})) - x_i(v'_i, v'_{-i})$. This contradicts strategy-proofness.

Situation 3. Otherwise

Let $x_i^H \equiv \inf\{x_i \in \mathbb{R}_+ | \text{there is } v_i \in V_i \text{ such that } t_i(y(v_i, v'_{-i})) = x_i \text{ and } y(v_i, v'_{-i}) > y\}$ and $x_i^L \equiv \sup\{x_i \in \mathbb{R}_+ | \text{there is } v_i \in V_i \text{ such that } t_i(y(v_i, v'_{-i})) = x_i \text{ and } y(v_i, v'_{-i}) < y\}$. By the supposition of y, we have the following three cases according to whether x_i^H and x_i^L are induced by some valuation function of the public good or not: (i) there is $v_i^L \in V_i$ such that $t_i(y(v_i^L, v'_{-i})) = x_i^L$, but not $v_i^H \in V_i$ such that $t_i(y(v_i^H, v'_{-i})) = x_i^H$, (ii) there is $v_i^H \in V_i$ such that $t_i(y(v_i^H, v'_{-i})) = x_i^H$, but not $v_i^L \in V_i$ such that $t_i(y(v_i^L, v'_{-i})) = x_i^L$, and (iii) there are no $v_i^L, v_i^H \in V_i$ such that $t_i(y(v_i^L, v'_{-i})) = x_i^L$ and $t_i(y(v_i^H, v'_{-i})) = x_i^H$. In the case (i), we know that $y \neq y(v_i^L, v'_{-i})$. Together with Corollary 1, the definition of x_i^H, and the continuity and strict increasingness of valuation functions of the public good, this implies that we can take $v_i \in V_i$ such that $v_i(y(v_i, v'_{-i})) - t_i(y(v_i, v'_{-i})) < v_i(y) - x_i^H$ and have a contradiction by arguments similar to the situations 1 and 2 (see Figure 3). Similarly, we have a contradiction in the cases (ii) and (iii). □

Remark 4. The combination of Lemma 1 and Proposition 1 does not necessarily imply the closedness of the range of consumption of the public good under a cost sharing scheme satisfying strategy-proofness because the infinite union of closed sets is not necessarily closed.

Lemma 4 shows that the agent's cost share of the public good is uniquely determined according to the consumption of the public good on the agent's option set if the social choice function satisfies strategy-proofness. This is a well-known result on strategy-proofness.

Lemma 4. *Suppose that the social choice function f satisfies **strategy-proofness**. For each $v, v' \in V$ and each $i \in I$, if $y(v_i, v'_{-i}) = y(v'_i, v'_{-i})$, then $x_i(v_i, v'_{-i}) = x_i(v'_i, v'_{-i})$.*

[14] Note that we cannot take such a valuation function of the public good by the supposition of y and the strict increasingness of valuation functions of the public good if $x_i^H - t_i(y(v''_i, v'_{-i})) = 0$ for each $v''_i \in V_i$. By Corollary 1, we find that $x_i^H - t_i(y(v''_i, v'_{-i})) < 0$ for each $v''_i \in V_i$ because $y(v''_i, v'_{-i}) = y$ and we have a contradiction to the definition of y if $x_i^H = t_i(y(v''_i, v'_{-i}))$ for some $v''_i \in V_i$.

[15] Note that we can take such a valuation function of the public good even if $x_i^L - t_i(y(v''_i, v'_{-i})) = 0$ for each $v''_i \in V_i$ because $0 < v_i(y) - v_i(y(v''_i, v'_{-i}))$ for each $v''_i \in V_i$ by the supposition of y and the strict increasingness of valuation functions of the public good.

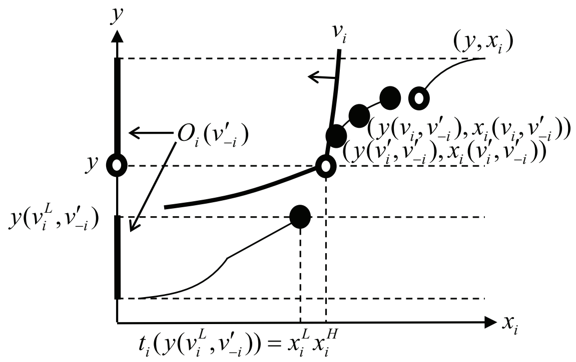

Figure 3: Proof of the case (i) of the situation 3 in Proposition 1

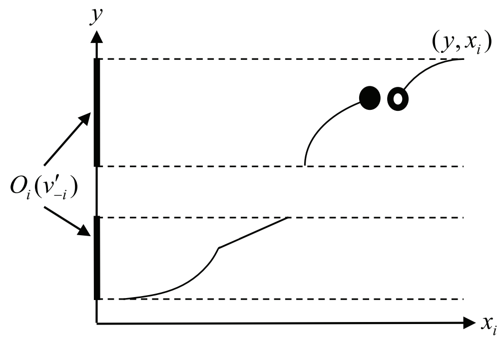

Figure 4: An example of the option set and the consumption bundles implied by strategy-proofness

Proof. To the contrary, we suppose that there are $v, v' \in V$ and $i \in I$ such that $y(v_i, v'_{-i}) = y(v'_i, v'_{-i})$ and $x_i(v_i, v'_{-i}) \neq x_i(v'_i, v'_{-i})$. If $x_i(v_i, v'_{-i}) > x_i(v'_i, v'_{-i})$, then we find that $v_i(y(v_i, v'_{-i})) - x_i(v_i, v'_{-i}) < v_i(y(v'_i, v'_{-i})) - x_i(v'_i, v'_{-i})$. This contradicts strategy-proofness. Similarly, we have a contradiction if $x_i(v_i, v'_{-i}) < x_i(v'_i, v'_{-i})$. □

On the basis of Lemmas 2 and 4 and Proposition 1, Figure 4 illustrates an example of the option set and the consumption bundles under a cost sharing scheme satisfying strategy-proofness. Lemma 5 is also a well-known result on strategy-proofness: each strictly monotonic transformation of the valuation function of the public good at the consumption bundle induces the same consumption of the public good as that of the consumption bundle if the social choice function satisfies strategy-proofness.

Lemma 5. *Suppose that the social choice function f satisfies* ***strategy-proofness****. For each $v, v' \in V$ and each $i \in I$, if $v'_i \in SM_i(y(v_i, v'_{-i}), x_i(v_i, v'_{-i}); v_i)$, then $y(v_i, v'_{-i}) = y(v'_i, v'_{-i})$.*

Proof. To the contrary, we suppose that there are $v, v' \in V$ and $i \in I$ such that $v'_i \in SM_i(y(v_i, v'_{-i}), x_i(v_i, v'_{-i}); v_i)$ and

$$y(v_i, v'_{-i}) \neq y(v'_i, v'_{-i}). \tag{3}$$

By the former, we know that

$$\begin{aligned} & v_i(y(v_i, v'_{-i})) - x_i(v_i, v'_{-i}) < v_i(y') - x'_i \\ & \qquad \text{for each } (y', x'_i) \in UC(y(v_i, v'_{-i}), x_i(v_i, v'_{-i}); v'_i) \setminus \{(y(v_i, v'_{-i}), x_i(v_i, v'_{-i}))\}. \end{aligned} \tag{4}$$

By strategy-proofness, we know that $(y(v'_i, v'_{-i}), x_i(v'_i, v'_{-i})) \in UC(y(v_i, v'_{-i}), x_i(v_i, v'_{-i}); v'_i)$. Together with (3), this implies that

$$(y(v'_i, v'_{-i}), x_i(v'_i, v'_{-i})) \in UC(y(v_i, v'_{-i}), x_i(v_i, v'_{-i}); v'_i) \setminus \{(y(v_i, v'_{-i}), x_i(v_i, v'_{-i}))\}. \tag{5}$$

By (4) and (5), we find that $v_i(y(v_i, v'_{-i})) - x_i(v_i, v'_{-i}) < v_i(y(v'_i, v'_{-i})) - x_i(v'_i, v'_{-i})$. This contradicts strategy-proofness. □

Lemma 6 shows that the two valuation functions of the public good, whose "peaks" on the option set are equal, induce the same consumption of the public good if the cost sharing scheme satisfies strategy-proofness.

Lemma 6. *Suppose that the cost sharing scheme f satisfies **strategy-proofness**. For each $v, v' \in V$ and each $i \in I$, if $v'_i(y(v''_i, v'_{-i})) - t_i(y(v''_i, v'_{-i})) < v'_i(y(v_i, v'_{-i})) - t_i(y(v_i, v'_{-i}))$ for each $y(v''_i, v'_{-i}) \in O_i(v'_{-i}) \setminus \{y(v_i, v'_{-i})\}$, then $y(v_i, v'_{-i}) = y(v'_i, v'_{-i})$.*

Proof. To the contrary, we suppose that there are $v, v' \in V$ and $i \in I$ such that $v'_i(y(v''_i, v'_{-i})) - t_i(y(v''_i, v'_{-i})) < v'_i(y(v_i, v'_{-i})) - t_i(y(v_i, v'_{-i}))$ for each $y(v''_i, v'_{-i}) \in O_i(v'_{-i}) \setminus \{y(v_i, v'_{-i})\}$ and $y(v_i, v'_{-i}) \neq y(v'_i, v'_{-i})$. The latter implies that $y(v'_i, v'_{-i}) \in O_i(v'_{-i}) \setminus \{y(v_i, v'_{-i})\}$. Together with the former, this implies that $v'_i(y(v'_i, v'_{-i})) - t_i(y(v'_i, v'_{-i})) < v'_i(y(v_i, v'_{-i})) - t_i(y(v_i, v'_{-i}))$, that is, $v'_i(y(v'_i, v'_{-i})) - x_i(v'_i, v'_{-i}) < v'_i(y(v_i, v'_{-i})) - x_i(v_i, v'_{-i})$. This contradicts strategy-proofness. □

4.2 Non-Bossiness

By Lemmas 4 and 5 and non-bossiness, we have Proposition 2 showing that all the agents' cost shares of the public good are uniquely determined according to the consumption of the public good on the range of consumption of the public good if the social choice function satisfies strategy-proofness and non-bossiness. In addition, we have Corollary 2 by Proposition 2.

Proposition 2. *Suppose that the social choice function f satisfies **strategy-proofness** and **non-bossiness**. For each $v, v' \in V$, if $y(v) = y(v')$, then $x(v) = x(v')$.*

Proof. Let $v, v' \in V$ be such that

$$y(v) = y(v'). \tag{6}$$

Let $i \in I$. By (6), we can take $v''_i \in V_i$ such that $v''_i \in SM_i(y(v), x_i(v); v_i) \cap SM_i(y(v'), x_i(v'); v'_i)$. Together with Lemma 5, this implies that $y(v_i, v_{-i}) = y(v''_i, v_{-i})$ and $y(v'_i, v'_{-i}) = y(v''_i, v'_{-i})$. Together with Lemma 4 and non-bossiness, this implies that

$$\begin{aligned} (y(v_i, v_{-i}), x(v_i, v_{-i})) &= (y(v''_i, v_{-i}), x(v''_i, v_{-i})), \\ (y(v'_i, v'_{-i}), x(v'_i, v'_{-i})) &= (y(v''_i, v'_{-i}), x(v''_i, v'_{-i})). \end{aligned} \tag{7}$$

Together with (6), this implies that

$$y(v''_i, v_{-i}) = y(v''_i, v'_{-i}). \tag{8}$$

Let $j \in I \setminus \{i\}$. By (8), we can take $v''_j \in V_j$ such that $v''_j \in SM_j(y(v''_i, v_{-i}), x_j(v''_i, v_{-i}); v_j) \cap SM_j(y(v''_i, v'_{-i}), x_j(v''_i, v'_{-i}); v'_j)$. Together with Lemma 5, this implies that $y(v''_i, v_j, v_{-i,j}) = y(v''_i, v''_j, v_{-i,j})$ and $y(v''_i, v'_j, v'_{-i,j}) =$

$y(v''_i, v''_j, v'_{-i,j})$. Together with Lemma 4 and non-bossiness, this implies that

$$\begin{aligned}(y(v''_i, v_j, v_{-i,j}), x(v''_i, v_j, v_{-i,j})) &= (y(v''_i, v''_j, v_{-i,j}), x(v''_i, v''_j, v_{-i,j})),\\ (y(v''_i, v'_j, v'_{-i,j}), x(v''_i, v'_j, v'_{-i,j})) &= (y(v''_i, v''_j, v'_{-i,j}), x(v''_i, v''_j, v'_{-i,j})).\end{aligned} \tag{9}$$

Together with (8), this implies that $y(v''_i, v''_j, v_{-i,j}) = y(v''_i, v''_j, v'_{-i,j})$.

By (7) and (9), we find that $x(v_i, v_j, v_{-i,j}) = x(v''_i, v''_j, v_{-i,j})$ and $x(v'_i, v'_j, v'_{-i,j}) = x(v''_i, v''_j, v'_{-i,j})$. By sequentially replacing v_k and v'_k by v''_k for each $k \in I \setminus \{i, j\}$ in the above manner, we find that $x(v) = x(v'') = x(v')$. □

Corollary 2. *If the social choice function satisfies **strategy-proofness** and **non-bossiness**, then it is a **cost sharing scheme**.*

Remark 5. In the model presented here, Serizawa (2006, Proposition 1 and Theorem 1) showed that if the social choice function satisfies effective pairwise strategy-proofness (Serizawa, 2006), then it satisfies non-bossiness and assigns an allocation similar to Proposition 2. [16] By definition, effective pairwise strategy-proofness is stronger than strategy-proofness. In addition, both properties are not equivalent in the model presented here because the Groves mechanisms (Groves, 1973) satisfy strategy-proofness, but not non-bossiness that is a necessary condition for effective pairwise strategy-proofness. These relationships imply that Proposition 2 strengthens the result of Serizawa (2006).

Remark 6. In the model presented here, Mizukami and Wakayama (2009, Theorem 3) showed that if the social choice function satisfies individual weak monotonicity (Mizukami and Wakayama, 2009) and non-bossiness, then it assigns an allocation similar to Proposition 2. [17] In general, individual weak monotonicity is weaker than strategy-proofness. In addition, both properties are not equivalent in the model presented here because the following social choice function satisfies individual weak monotonicity, but not strategy-proofness: there is a function $g\colon Y \to \mathbb{R}_+$, where g is convex on the set of production levels of the public good Y, and for each $v \in V$, $y(v) \in \operatorname{argmax}_{y \in Y}\{\sum_{k \in I} v_k(y) - g(y)\}$ and x_i is constant for each $i \in I$. These relationships imply that Proposition 2 is a corollary of the result of Mizukami and Wakayama (2009).

4.3 Strong Non-Bossiness

Lemma 7 shows the uniqueness of the agent's utility maximizer in the agent's option set under a social choice function satisfying strategy-proofness and strong non-bossiness.

Lemma 7. *Suppose that the social choice function f satisfies **strategy-proofness** and **strong non-bossiness**. For each $v, v' \in V$ and each $i \in I$, if $y(v_i, v'_{-i}) \neq y(v'_i, v'_{-i})$, then $v_i(y(v_i, v'_{-i})) - x_i(v_i, v'_{-i}) > v_i(y(v'_i, v'_{-i})) - x_i(v'_i, v'_{-i})$.*

Proof. To the contrary, we suppose that there are $v, v' \in V$ and $i \in I$ such that

$$y(v_i, v'_{-i}) \neq y(v'_i, v'_{-i}) \tag{10}$$

[16] See Serizawa (2006) for a formal definition of effective pairwise strategy-proofness.

[17] See Mizukami and Wakayama (2009) for a formal definition of individual weak monotonicity.

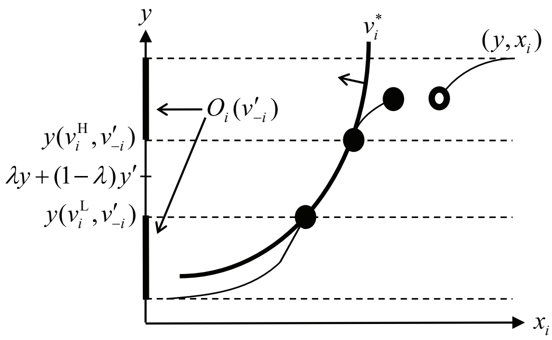

Figure 5: Proof of the subcase (iii-2) in Proposition 3

and $v_i(y(v_i, v'_{-i})) - x_i(v_i, v'_{-i}) \leq v_i(y(v'_i, v'_{-i})) - x_i(v'_i, v'_{-i})$. Together with strategy-proofness, this implies that $v_i(y(v_i, v'_{-i})) - x_i(v_i, v'_{-i}) = v_i(y(v'_i, v'_{-i})) - x_i(v'_i, v'_{-i})$. Together with strong non-bossiness, this implies that $y(v_i, v'_{-i}) = y(v'_i, v'_{-i})$. This contradicts (10). □

By Proposition 1, we know that strategy-proofness implies the closedness of the option sets under a cost sharing scheme. By imposing strong non-bossiness in addition to strategy-proofness, we also have Proposition 3 showing the convexity.

Proposition 3. *Suppose that the cost sharing scheme f satisfies **strategy-proofness** and **strong non-bossiness**. For each $i \in I$ and each $v'_{-i} \in V_{-i}$, $O_i(v'_{-i})$ is **convex**.*

Proof. Let $i \in I$ and $v'_{-i} \in V_{-i}$. In addition, let $y, y' \in O_i(v'_{-i})$ and $\lambda \in [0,1]$. We have the following three cases according to the value of λ: (i) $\lambda = 0$; (ii) $\lambda = 1$; and (iii) $\lambda \in (0,1)$. In the case (i), we know that $\lambda y + (1-\lambda) y' = y' \in O_i(v'_{-i})$. In the case (ii), we know that $\lambda y + (1-\lambda) y' = y \in O_i(v'_{-i})$. In the case (iii), we have the following two subcases according to the relationship between y and y': (iii-1) $y = y'$ and (iii-2) $y \neq y'$. In the subcase (iii-1), we know that $\lambda y + (1-\lambda) y' \in O_i(v'_{-i})$.

The remainder of this proof demonstrates that $\lambda y + (1-\lambda) y' \in O_i(v'_{-i})$ in the subcase (iii-2). To the contrary, we suppose that $\lambda y + (1-\lambda) y' \notin O_i(v'_{-i})$. Together with Proposition 1, this implies that we can take $v_i^{\mathrm{L}}, v_i^{\mathrm{H}} \in V_i$ such that

$$[y(v_i^{\mathrm{L}}, v'_{-i}), y(v_i^{\mathrm{H}}, v'_{-i})] \cap O_i(v'_{-i}) = \{y(v_i^{\mathrm{L}}, v'_{-i}), y(v_i^{\mathrm{H}}, v'_{-i})\}, \tag{11}$$

$$y(v_i^{\mathrm{L}}, v'_{-i}) < \lambda y + (1-\lambda) y' < y(v_i^{\mathrm{H}}, v'_{-i}). \tag{12}$$

By (11) and Corollary 1, we can take $v_i^* \in V_i$ such that (iii-2a) $v_i^*(y(v_i^{\mathrm{L}}, v'_{-i})) - v_i^*(y(v''_i, v'_{-i})) \geq t_i(y(v_i^{\mathrm{L}}, v'_{-i})) - t_i(y(v''_i, v'_{-i}))$ for each $v''_i \in V_i$, where $y(v''_i, v'_{-i}) \leq y(v_i^{\mathrm{L}}, v'_{-i})$, and (iii-2b) $v_i^*(y(v_i^{\mathrm{H}}, v'_{-i})) - v_i^*(y(v''_i, v'_{-i})) \geq t_i(y(v_i^{\mathrm{H}}, v'_{-i})) - t_i(y(v''_i, v'_{-i}))$ for each $v''_i \in V_i$, where $y(v''_i, v'_{-i}) \geq y(v_i^{\mathrm{H}}, v'_{-i})$ (see Figure 5). Together with strategy-proofness, these imply that $v_i^*(y(v_i^{\mathrm{L}}, v'_{-i})) - t_i(y(v_i^{\mathrm{L}}, v'_{-i})) = v_i^*(y(v_i^*, v'_{-i})) - t_i(y(v_i^*, v'_{-i})) = v_i^*(y(v_i^{\mathrm{H}}, v'_{-i})) - t_i(y(v_i^{\mathrm{H}}, v'_{-i}))$, that is, $v_i^*(y(v_i^{\mathrm{L}}, v'_{-i})) - x_i(v_i^{\mathrm{L}}, v'_{-i}) = v_i^*(y(v_i^*, v'_{-i})) - x_i(v_i^*, v'_{-i}) = v_i^*(y(v_i^{\mathrm{H}}, v'_{-i})) - x_i(v_i^{\mathrm{H}}, v'_{-i})$. Together with Lemma 7, this implies that $y(v_i^{\mathrm{L}}, v'_{-i}) = y(v_i^*, v'_{-i}) = y(v_i^{\mathrm{H}}, v'_{-i})$ and contradicts (12). □

Remark 7. The combination of Lemma 1 and Proposition 3 does not necessarily imply the convexity of the range of consumption of the public good under a social choice function satisfying strategy-proofness and strong non-bossiness.

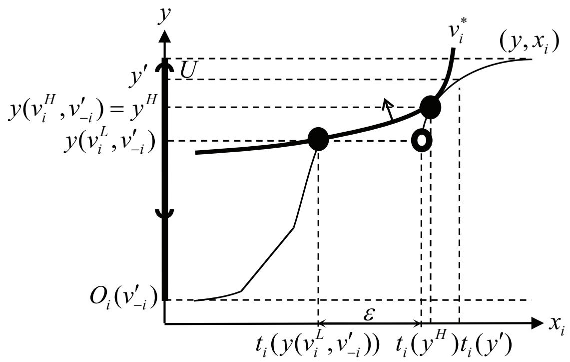

Figure 6: Proof of Lemma 8

By Lemma 3, we know that strategy-proofness implies the lower semi-continuity of the cost sharing scheme. By imposing strong non-bossiness in addition to strategy-proofness, we have Lemma 8 showing the continuity.

Lemma 8. *If the cost sharing scheme f satisfies **strategy-proofness** and **strong non-bossiness**, then it is **continuous**.*

Proof. To the contrary, we suppose that f is not continuous. This implies that there are $i \in I$ and $v'_{-i} \in V_{-i}$ such that t_i is not continuous on $O_i(v'_{-i})$. In addition, there is $v_i^L \in V_i$ such that t_i is not continuous at $y(v_i^L, v'_{-i})$. Together with Lemma 3, this implies that t_i is not upper semi-continuous at $y(v_i^L, v'_{-i})$ and there is $\varepsilon \in \mathbb{R}_+$ such that for each neighborhood $U \subseteq O_i(v'_{-i})$ of $y(v_i^L, v'_{-i})$, $t_i(y') > t_i(y(v_i^L, v'_{-i})) + \varepsilon$ for some $y' \in U$. Because $U \subseteq O_i(v'_{-i})$, this implies that $y(v_i^L, v'_{-i}) < y'$ by Corollary 1. On the basis of the above argument, let $y^H \in (y(v_i^L, v'_{-i}), y')$ be such that we can take $v_i \in V_i$ which satisfies the following condition: $v_i(y(v_i^L, v'_{-i})) - v_i(\lambda y(v_i^L, v'_{-i}) + (1-\lambda)y^H) > t_i(y(v_i^L, v'_{-i})) - t_i(\lambda y(v_i^L, v'_{-i}) + (1-\lambda)y^H)$ for each $\lambda \in (0,1)$. [18] Because $(y(v_i^L, v'_{-i}), y') \subseteq O_i(v'_{-i})$, there is $v_i^H \in V_i$ such that $y(v_i^H, v'_{-i}) = y^H$ and we find that

$$y(v_i^L, v'_{-i}) < y(v_i^H, v'_{-i}) \tag{13}$$

by the definition of y^H. On the basis of the definition of y^H and the continuity and strict increasingness of valuation functions of the public good, we can take $v_i^* \in V_i$ such that (a) $v_i^*(y(v_i^L, v'_{-i})) - v_i^*(y(v''_i, v'_{-i})) \geq t_i(y(v_i^L, v'_{-i})) - t_i(y(v''_i, v'_{-i}))$ for each $v''_i \in V_i$, where $y(v''_i, v'_{-i}) \leq y(v_i^L, v'_{-i})$, (b) $v_i^*(y(v_i^H, v'_{-i})) - v_i^*(y(v''_i, v'_{-i})) \geq t_i(y(v_i^H, v'_{-i})) - t_i(y(v''_i, v'_{-i}))$ for each $v''_i \in V_i$, where $y(v''_i, v'_{-i}) \geq y(v_i^H, v'_{-i})$, and (c) $v_i^*(y(v_i^L, v'_{-i})) - v_i^*(\lambda y(v_i^L, v'_{-i}) + (1-\lambda)y(v_i^H, v'_{-i})) > t_i(y(v_i^L, v'_{-i})) - t_i(\lambda y(v_i^L, v'_{-i}) + (1-\lambda)y(v_i^H, v'_{-i}))$ for each $\lambda \in (0,1)$ (see Figure 6). Together with strategy-proofness, these imply that $v_i^*(y(v_i^L, v'_{-i})) - t_i(y(v_i^L, v'_{-i})) = v_i^*(y(v_i^*, v'_{-i})) - t_i(y(v_i^*, v'_{-i})) = v_i^*(y(v_i^H, v'_{-i})) - t_i(y(v_i^H, v'_{-i}))$, that is, $v_i^*(y(v_i^L, v'_{-i})) - x_i(v_i^L, v'_{-i}) = v_i^*(y(v_i^*, v'_{-i})) - x_i(v_i^*, v'_{-i}) = v_i^*(y(v_i^H, v'_{-i})) - x_i(v_i^H, v'_{-i})$. Together with Lemma 7, this implies that $y(v_i^L, v'_{-i}) = y(v_i^*, v'_{-i}) = y(v_i^H, v'_{-i})$ and contradicts (13). $\square$

On the basis of Lemma 8, we have Proposition 4 showing the convexity of the cost sharing scheme satisfying strategy-proofness and strong non-bossiness.

[18] Note that we can take such a valuation function of the public good by letting y^H be sufficiently close to $y(v_i^L, v'_{-i})$. This requirement is introduced to respect the strict concavity of valuation functions of the public good.

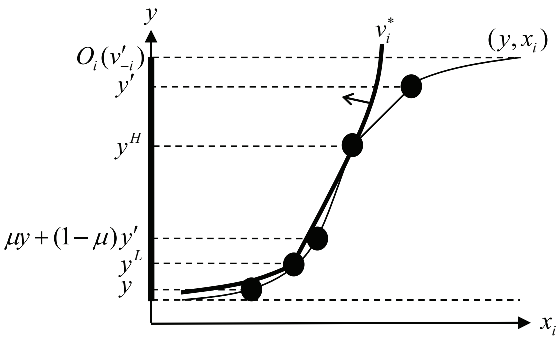

Figure 7: Proof of Proposition 4

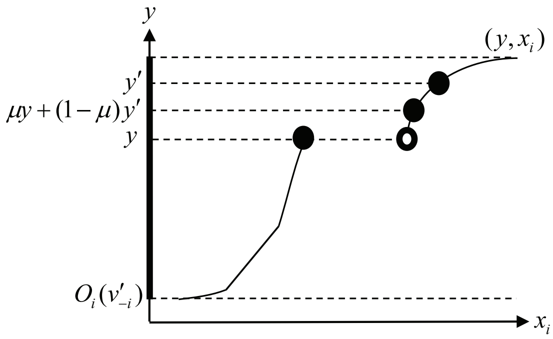

Figure 8: Necessity of Lemma 8 in the proof of Proposition 4

Proposition 4. *If the cost sharing scheme f satisfies* ***strategy-proofness*** *and* ***strong non-bossiness****, then it is* ***convex****.*

Proof. To the contrary, we suppose that f is not convex. This implies that there are $i \in I$ and $v'_{-i} \in V_{-i}$ such that t_i is not convex on $O_i(v'_{-i})$. In addition, there are $y, y' \in O_i(v'_{-i})$ and $\mu \in [0,1]$ such that $\mu t_i(y) + (1-\mu)t_i(y') < t_i(\mu y + (1-\mu)y')$. Without loss of generality, we suppose that $y < y'$. On the basis of the above argument and Lemma 8, let $y^L, y^H \in (y, y')$, where $y^L < y^H$, be such that $[y^L, y^H] \subseteq (y, y')$ and we can take $v_i \in V_i$ which satisfies the following condition: $v_i(y^L) - v_i(\lambda y^L + (1-\lambda)y^H) > t_i(y^L) - t_i(\lambda y^L + (1-\lambda)y^H)$ for each $\lambda \in (0,1)$ (see Figure 7). Because $(y, y') \subseteq O_i(v'_{-i})$, this implies that we have a contradiction by an argument similar to Lemma 8. □

Remark 8. If Lemma 8 is not established, then we cannot necessarily take $y^L, y^H \in (y, y')$ in the proof of Proposition 4 (see Figure 8).

Remark 9. As noted by Serizawa (1999), Proposition 4 is established even if the cost function is not convex because the convexity is required only on the agent's option set.

Remark 10. The combination of Lemma 1 and Proposition 4 does not necessarily imply the convexity of cost sharing schemes satisfying strategy-proofness and strong non-bossiness on the range of consumption of the public good even if the range is convex. (see Figure 9). Serizawa (1999, Theorem 2) showed the convexity by strategy-proofness, budget-balancedness, and anonymity. [19]

[19] See Step 1 in the proof of Theorem 2 of Serizawa (1999).

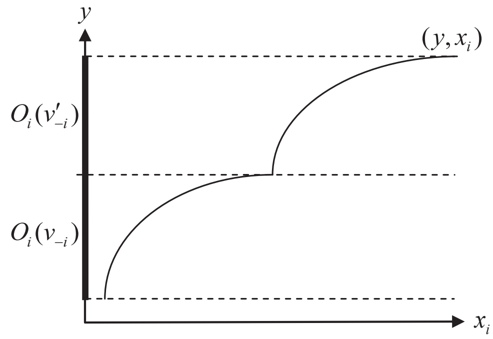

Figure 9: An example of the non-convexity of a cost sharing scheme satisfying strategy-proofness and strong non-bossiness on the range of consumption of the public good

Remark 11. Serizawa (1996) introduced semi-convex cost sharing schemes determined by a minimum demand principle. They satisfy strategy-proofness and non-bossiness, but not strong non-bossiness when the cost function is not convex by Proposition 4 because a certain agent shares the entire cost up to some production level of the public good. This implies that they are not securely implementable. In addition, we find that strong non-bossiness is not equivalent to non-bossiness in the model presented here.

4.4 Outcome Rectangular Property

As stated in Lemma 9, the outcome rectangular property requires that if each agent cannot change the agent's consumption bundle by changing the agent's revelation, then the allocation does not change by changing all the agents' revelations under a social choice function satisfying non-bossiess.

Lemma 9. *Suppose that the social choice function f satisfies **non-bossiness** and the **outcome rectangular property**. For each $v, v' \in V$, if $(y(v_i, v'_{-i}), x_i(v_i, v'_{-i})) = (y(v'_i, v'_{-i}), x_i(v'_i, v'_{-i}))$ for each $i \in I$, then $(y(v), x(v)) = (y(v'), x(v'))$.*

Proof. Let $v, v' \in V$ be such that $(y(v_i, v'_{-i}), x_i(v_i, v'_{-i})) = (y(v'_i, v'_{-i}), x_i(v'_i, v'_{-i}))$ for each $i \in I$. Together with non-bossiness, this implies that $(y(v_i, v'_{-i}), x(v_i, v'_{-i})) = (y(v'_i, v'_{-i}), x(v'_i, v'_{-i}))$ for each $i \in I$. Together with the outcome rectangular property, this implies that $(y(v), x(v)) = (y(v'), x(v'))$. □

By Lemmas 4 and 9, we have Lemma 10 showing that the outcome rectangular property can be considered within the range of consumption of the public good under a social choice function satisfying strategy-proofness and non-bossiness.

Lemma 10. *Suppose that the social choice function f satisfies **strategy-proofness**, **non-bossiness**, and the **outcome rectangular property**. For each $v, v' \in V$, if $y(v_i, v'_{-i}) = y(v'_i, v'_{-i})$ for each $i \in I$, then $y(v) = y(v')$.*

Proof. Let $v, v' \in V$ be such that $y(v_i, v'_{-i}) = y(v'_i, v'_{-i})$ for each $i \in I$. Together with Lemma 4, this implies that $x_i(v_i, v'_{-i}) = x_i(v'_i, v'_{-i})$ for each $i \in I$. Together with Lemma 9, these imply that $y(v) = y(v')$. □

As stated in Remark 4, the combination of Lemma 1 and Proposition 1 does not necessarily imply the closedness of the range of consumption of the public good under a cost sharing scheme satisfying strategy-proofness. By imposing non-bossiness and the outcome rectangular property in addition to strategy-proofness, we have Proposition 5 showing the closedness.

Proposition 5. *If the cost sharing scheme f satisfies **strategy-proofness**, **non-bossiness**, and the **outcome rectangular property**, then $y(V)$ is **closed**.*

Proof. To the contrary, we suppose that $y(V)$ is not closed. This implies that we can take $y \in \bar{y}(V) \setminus y(V)$, where $\bar{y}(V)$ is the closure of $y(V)$. We have the following three situations according to the relationship between y and $y(V)$.

Situation 1. $y = \inf y(V)$

By Corollary 1, Proposition 1, and strategy-proofness, we can take $v \in V$ such that $y(v_i, v_{-i}) = \min O_i(v_{-i})$ for each $i \in I$. [20] In addition, we can take $v' \in V$ such that

$$y < y(v') < y(v) \tag{14}$$

by the supposition of y. For each $i \in I$, we have the following two cases according to the position of $y(v')$ in $O_i(v'_{-i})$ by Proposition 1: (i) $y(v'_i, v'_{-i}) = \max O_i(v'_{-i})$ and (ii) $y(v'_i, v'_{-i}) < \max O_i(v'_{-i})$. In addition, we consider the following two subcases of the case (ii) according to the relationship between $y(v_i, v_{-i})$ and $y(v'_i, v_{-i})$ on the basis of Lemma 6 and the definition of $y(v_i, v_{-i})$: (ii-1) $y(v_i, v_{-i}) = y(v'_i, v_{-i})$ and (ii-2) $y(v_i, v_{-i}) < y(v'_i, v_{-i})$. Let $I_{\text{(i)}} \subseteq I$ be the set of agents belonging to the case (i), $I_{\text{(ii-1)}} \subseteq I$ be the set of agents belonging to the subcase (ii-1), and $I_{\text{(ii-2)}} \subseteq I$ be the set of agents belonging to the subcase (ii-2).

For each $i \in I_{\text{(i)}}$, we can take $v_i^* \in V_i$ such that $y(v_i, v_{-i}) = y(v_i^*, v_{-i})$ and $y(v'_i, v'_{-i}) = y(v_i^*, v'_{-i})$ by Lemma 6 and an argument similar to Proposition 3 because $y(v_i, v_{-i}) = \min O_i(v_{-i})$ (see the left hand side of Figure 10). For each $i \in I_{\text{(ii-1)}}$, we know that $y(v_i, v_{-i}) = y(v'_i, v_{-i})$ by definition. For each $i \in I_{\text{(ii-2)}}$, we can take $v_i^{**} \in V_i$ such that $y(v_i, v_{-i}) = y(v_i^{**}, v_{-i})$ and $y(v'_i, v'_{-i}) = y(v_i^{**}, v'_{-i})$ by Lemma 6 and an argument similar to Proposition 3 because $y(v_i, v_{-i}) = \min O_i(v_{-i})$ (see the right hand side of Figure 10). Let $v'' \equiv (v''_{I_{\text{(i)}}}, v''_{I_{\text{(ii-1)}}}, v''_{I_{\text{(ii-2)}}})$ be such that $(v''_{I_{\text{(i)}}}, v''_{I_{\text{(ii-1)}}}, v''_{I_{\text{(ii-2)}}}) = (v^*_{I_{\text{(i)}}}, v'_{I_{\text{(ii-1)}}}, v^{**}_{I_{\text{(ii-2)}}})$. These imply that $y(v_i, v_{-i}) = y(v''_i, v_{-i})$ and $y(v'_i, v'_{-i}) = y(v''_i, v'_{-i})$ for each $i \in I$. Together with Lemma 10, this implies that $y(v) = y(v'') = y(v')$ and contradicts (14).

Situation 2. $y = \sup y(V)$

By Corollary 1, Proposition 1, and strategy-proofness, we can take $v \in V$ such that $y(v_i, v_{-i}) = \max O_i(v_{-i})$ for each $i \in I$. [21] In addition, we can take $v' \in V$ such that $y(v) < y(v') < y$ by the supposition of y. For each $i \in I$, we have the following two cases according to the position of $y(v')$ in $O_i(v'_{-i})$ by Proposition 1: (i) $y(v'_i, v'_{-i}) = \min O_i(v'_{-i})$ and (ii) $y(v'_i, v'_{-i}) > \min O_i(v'_{-i})$. In addition, we consider the

[20] Note that we can take such a profile of valuation functions of the public good by letting the slope of v_i on $y(V)$ be sufficiently low for each $i \in I$. To the contrary, we suppose that we cannot take $v \in V$ such that $y(v_i, v_{-i}) = \min O_i(v_{-i})$ for each $i \in I$. This implies that there is $i \in I$ such that $y(v_i, v_{-i}) > \min O_i(v_{-i})$ for each $v \in V$. Let v_i be such that the slope on $y(V)$ is sufficiently low. By supposition, there is $y(v'_i, v_{-i}) \in O_i(v_{-i})$ such that $y(v_i, v_{-i}) > y(v'_i, v_{-i}) > \min O_i(v_{-i})$ and we find that $v_i(y(v_i, v'_{-i})) - x_i(v_i, v'_{-i}) < v_i(y(v'_i, v'_{-i})) - x_i(v'_i, v'_{-i})$ by Corollary 1 and the definition of v_i. This contradicts strategy-proofness.

[21] Note that we can take such a profile of valuation functions of the public good by letting the slope of v_i on $y(V)$ be sufficiently high for each $i \in I$ by an argument similar to the situation 1.

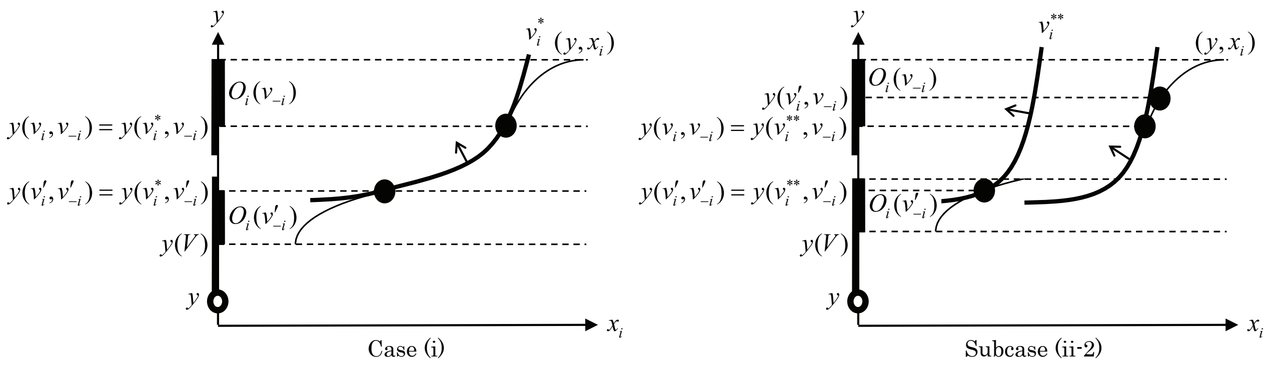

Figure 10: Proof of the sisuation 1 in Proposition 5

following two subcases of the case (ii) according to the relationship between $y(v_i, v_{-i})$ and $y(v'_i, v_{-i})$ on the basis of Lemma 6 and the definition of $y(v_i, v_{-i})$: (ii-1) $y(v_i, v_{-i}) = y(v'_i, v_{-i})$ and (ii-2) $y(v_i, v_{-i}) > y(v'_i, v_{-i})$. By an argument similar to the situation 1, we have a contradiction.

Situation 3. Otherwise

Let $U \subseteq Y$ be a neighborhood of y such that $U \cap y(V)$ is convex. This implies that there are the following two cases according the relationship between y and consumption of the public good in $U \cap y(V)$: (i) $y < y''$ for each $y'' \in U \cap y(V)$ and (ii) $y > y''$ for each $y'' \in U \cap y(V)$.

In the case (i), we can take $v \in V$ such that $y(v) \in U$ and $y(v_i, v_{-i}) = \min O_i(v_{-i})$ for each $i \in I$ by Proposition 1. In addition, we can take $v' \in V$ such that $y(v') \in U$ and $y < y(v') < y(v)$ by the supposition of y. By an argument similar to the situation 1, we have a contradiction.

In the case (ii), we can take $v \in V$ such that $y(v) \in U$ and $y(v_i, v_{-i}) = \max O_i(v_{-i})$ for each $i \in I$ by Proposition 1. In addition, we can take $v' \in V$ such that $y(v') \in U$ and $y(v) < y(v') < y$ by the supposition of y. By an argument similar to the situation 2, we have a contradiction. □

Remark 12. In non-excludable public good economies with classical preferences, Serizawa (1996, Lemma) showed the closedness of the range of consumption of the public good under a social choice function satisfying strategy-proofness, non-bossiness, individually rationality, budget-balancedness, and non-exploitation. In other directions, Serizawa (1999, Fact 1) showed it by strategy-proofness, symmetry, and budget-balancedness.

As stated in Remark 7, the combination of Lemma 1 and Proposition 3 does not necessarily imply the convexity of the range of consumption of the public good under a cost sharing scheme satisfying strategy-proofness and strong non-bossiness. By imposing the outcome rectangular property in addition to strategy-proofness and strong non-bossiness, we have Proposition 6 showing the convexity.

Proposition 6. *If the cost sharing scheme f satisfies **strategy-proofness**, **strong non-bossiness**, and the **outcome rectangular property**, then $y(V)$ is **convex**.*

Proof. Let $y, y' \in y(V)$ and $\lambda \in [0,1]$. We have the following three cases according to the value of λ: (i) $\lambda = 0$; (ii) $\lambda = 1$; and (iii) $\lambda \in (0,1)$. In the case (i), we know that $\lambda y + (1-\lambda)y' = y' \in y(V)$. In the case (ii), we know that $\lambda y + (1-\lambda)y' = y \in y(V)$. In the case (iii), we have the following two subcases

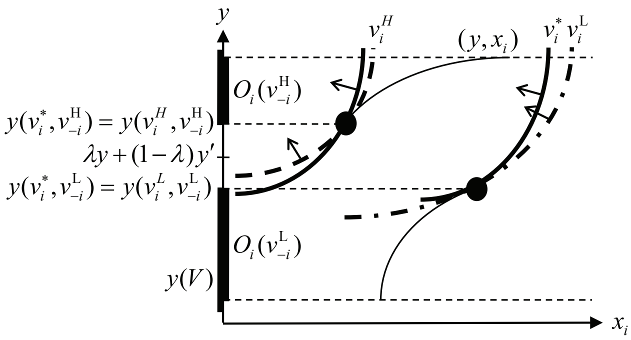

Figure 11: Proof of Proposition 6

according to the relationship between y and y': (iii-1) $y = y'$ and (iii-2) $y \neq y'$. In the subcase (iii-1), we know that $\lambda y + (1-\lambda)y' \in y(V)$.

The remainder of this proof demonstrates that $\lambda y + (1-\lambda)y' \in y(V)$ in the subcase (iii-2). To the contrary, we suppose that $\lambda y + (1-\lambda)y' \notin y(V)$. On the basis of Proposition 5, we can take $v^{\mathrm{L}}, v^{\mathrm{H}} \in V$ such that for each $i \in I$,

$$[y(v^{\mathrm{L}}), y(v^{\mathrm{H}})] \cap O_i(v_{-i}^{\mathrm{L}}) = \{y(v^{\mathrm{L}})\}, \tag{15}$$

$$[y(v^{\mathrm{L}}), y(v^{\mathrm{H}})] \cap O_i(v_{-i}^{\mathrm{H}}) = \{y(v^{\mathrm{H}})\}, \tag{16}$$

$$y(v^{\mathrm{L}}) < \lambda y + (1-\lambda)y' < y(v^{\mathrm{H}}). \tag{17}$$

By (15), (16), and Corollary 1, we can take $v^* \in V$ such that for each $i \in I$, (iii-2a) $v_i^*(y(v_i^{\mathrm{L}}, v_{-i}^{\mathrm{L}})) - v_i^*(y(v_i'', v_{-i}^{\mathrm{L}})) > t_i(y(v_i^{\mathrm{L}}, v_{-i}^{\mathrm{L}})) - t_i(y(v_i'', v_{-i}^{\mathrm{L}}))$ for each $y(v_i'', v_{-i}^{\mathrm{L}}) \in O_i(v_{-i}^{\mathrm{L}}) \setminus \{y(v_i^{\mathrm{L}}, v_{-i}^{\mathrm{L}})\}$ and (iii-2b) $v_i^*(y(v_i^{\mathrm{H}}, v_{-i}^{\mathrm{H}})) - v_i^*(y(v_i'', v_{-i}^{\mathrm{H}})) > t_i(y(v_i^{\mathrm{H}}, v_{-i}^{\mathrm{H}})) - t_i(y(v_i'', v_{-i}^{\mathrm{H}}))$ for each $y(v_i'', v_{-i}^{\mathrm{H}}) \in O_i(v_{-i}^{\mathrm{H}}) \setminus \{y(v_i^{\mathrm{H}}, v_{-i}^{\mathrm{H}})\}$ (see Figure 11). Together with Lemma 6, this implies that $y(v_i^*, v_{-i}^{\mathrm{L}}) = y(v_i^{\mathrm{L}}, v_{-i}^{\mathrm{L}})$ and $y(v_i^*, v_{-i}^{\mathrm{H}}) = y(v_i^{\mathrm{H}}, v_{-i}^{\mathrm{H}})$ for each $i \in I$. Together with Lemma 10, this implies that $y(v^{\mathrm{L}}) = y(v^*) = y(v^{\mathrm{H}})$ and contradicts (17). $\square$

Remark 13. The combination of Lemma 1, Corollary 1, and Proposition 6 implies the strict increasingness of a cost sharing scheme satisfying strategy-proofness, strong non-bossiness, and the outcome rectangular property on the range of consumption of the public good. In addition, the combination of Lemmas 1 and 8 and Proposition 6 implies the continuity of a cost sharing scheme satisfying strategy-proofness, strong non-bossiness, and the outcome rectangular property on the range of consumption of the public good.

5 Main Result

In the model presented here, the conservative equal cost sharing mechanism (Moulin, 1994) is a social choice function satisfying strategy-proofness. Given cost sharing functions $t_1, \cdots, t_n$, for each $v \in V$ and each $i \in I$, let $B_i(v_i, t_i, y(V)) \equiv \{y \in y(V) | v_i(y) - t_i(y) \geq v_i(y') - t_i(y')$ for each $y' \in y(V)\}$ be the set of utility maximizers for agent i in the range of consumption of the public good $y(V)$ at the profile of valuation functions of the public good v and $b_i(v_i, t_i, y(V)) \equiv \max B_i(v_i, t_i, y(V))$. The social choice

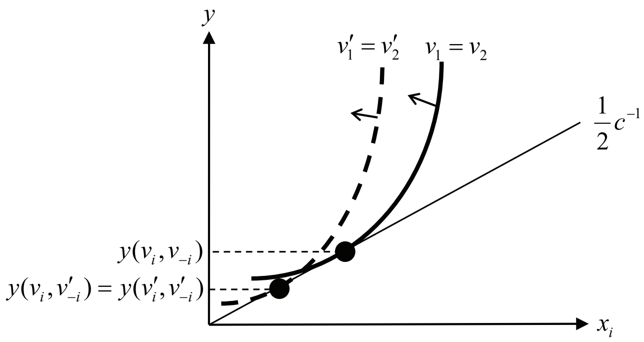

Figure 12: A violation of the outcome rectangular property under the conservative equal cost sharing mechanism when $n = 2$ and c is a liner cost function

function f is the **conservative equal cost sharing mechanism** if and only if f is a cost cost sharing scheme such that for each $v \in V$, $y(v) = \min\{b_i(v_i, t_i, y(V))\}_{i \in I}$ and $x_i(v) = c(y(v))/n$ for each $i \in I$. If the cost function c is convex, then the conservative equal cost sharing mechanism also satisfies strong non-bossiness, but not the outcome rectangular property (see Figure 12). [22] Together with the result of Saijo, Sjöström, and Yamato (2007), this implies that the conservative equal cost sharing mechanism is not securely implementable. The main result of this paper is compatible with this finding, as stated in the following theorem and corollary.

The social choice function f is **dictatorial** if and only if there is $i \in I$ such that for each $v, v' \in V$, $v_i(y(v)) - x_i(v) \geq v_i(y(v')) - x_i(v')$. [23] The social choice function f is **constant** if and only if for each $v, v' \in V$, $(y(v), x(v)) = (y(v'), x(v'))$.

Theorem. *If the social choice function f satisfies **strategy-proofness**, **strong non-bossiness**, and the **outcome rectangular property**, then f is **dictatorial** or **constant**.*

Proof. By Corollary 2, we know that f is a cost sharing scheme. By Propositions 5 and 6, we know that the range of consumption of the public good is closed and convex. These imply that the problem of providing a divisible and non-excludable public good with the cost shares is reduced to a voting environment in which the set of alternatives is equivalent to the range of consumption of the public good, which is a closed interval. In addition, we know the continuity of f on the range of consumption of the public good, as stated in Remark 13. This implies that each utility function induces a continuous preference defined over the range of consumption of the public good. Together with the result of Barberà and Peleg (1990, Theorem 3.1), these imply that f is dictatorial if the range of consumption of the public good contains at least three alternatives. If not, then f is constant because the range of consumption of the public good is closed and convex. □

The above theorem is tight. Example 1 shows the necessity of strategy-proofness, Example 2 the necessity of strong non-bossiness, and Example 3 the necessity of the outcome rectangular property.

[22] In this figure, $((y(v), x(v)))$ is the allocation induced by the unique weakly dominant strategy equilibrium and $((y(v'), x(v')))$ is an allocation induced by a "bad" Nash equilibrium.

[23] Note that this dictatorship is required on the range of the social choice function, but not on the set of all feasible allocations.

Example 1. Let f be the following social choice function: there is $y \in Y$ such that for each $v \in V$, $y(v) = y$ and $x_i(v) = -\{\sum_{k \in I} v_k(y(v)) - c(y(v))\}$ for each $i \in I$. We find that f satisfies strong non-bossiness and the outcome rectangular property. In addition, we find that f does not satisfy strategy-proofness because the agent benefits from untruthful revelation that changes the agent's cost share of the public good in the agent's favor.

Example 2. Let f be the following social choice function: there is $y \in Y$ such that for each $v \in V$, $y(v) = y$ and $x_i(v) = -\{\sum_{k \in I \setminus \{i\}} v_k(y(v)) - c(y(v))\}$ for each $i \in I$. We find that f satisfies strategy-proofness and the outcome rectangular property. In addition, we find that f does not satisfy strong non-bossiness because the agent can change other agents' cost shares of the public good by changing the agent's revelation while maintaining the agent's utility.

Example 3. Let f be the following social choice function: there are cost sharing functions $t_1, \cdots, t_n$, where t_i is convex on the range of consumption of the public good for each $i \in I$, and for each $v \in V$, $y(v) = \min\{b_i(v_i, t_i, y(V))\}_{i \in I}$ and $x_i(v) = t_i(y(v))$ for each $i \in I$. By an argument similar to the case of the conservative equal cost sharing mechanism, we find that f satisfies strategy-proofness and strong non-bossiness, but not the outcome rectangular property.

Remark 14. The social choice function in Example 2 does not even satisfy non-bossiness. This implies that the outcome rectangular property is not stronger than non-bossiness. In addition, we know that non-bossiness is not stronger than the outcome rectangular property by Example 3. These imply that both properties are independent in the model presented here.

Remark 15. Although the cost shares of the public good under the social choice function in Example 2 are contained in those of the Groves mechanisms, the consumption of the public good does not maximize the sum of all the agents' benefits from the consumption. If the consumption maximizes it, then the social choice function does not satisfy the outcome rectangular property as well as strong non-bossiness, that is, the Groves mechanisms are not securely implementable in the model presented here. On the other hand, Saijo, Sjöström, and Yamato (2007, Example 1) showed that the Groves mechanisms are securely implementable in a part of the model presented here, in which the form of valuation functions of the public good is fixed and each agent is identified with a parameter.

Together with the result of Saijo, Sjöström, and Yamato (2007), the above theorem implies the following negative result on secure implementation in divisible and non-excludable public good economies with quasi-linear utility functions.

Corollary 3. *If the social choice function is **securely implementable**, then it is **dictatorial** or **constant**.*

6 Conclusion

This paper studies the possibility of secure implementation in divisible and non-excludable public good economies with quasi-linear utility functions. Although Saijo, Sjöström, and Yamato (2007) showed that the Groves mechanisms are securely implementable in some of the economies, the results presented here showed that securely implementable social choice functions are dictatorial or constant in divisible and non-excludable public good economies with quasi-linear utility functions. On the basis of the observations of Cason, Saijo, Sjöström, and Yamato (2006), this negative result suggests that non-trivial strategy-proof mechanisms actually do not work well in the economies except a limited number of the environments.

The results presented here also contributed to studying the possibility of secure implementation in other environments including divisible and "excludable" public good economies in which it is open to study the possibility. Investigating securely implementable social choice functions in the economies is an interesting research topic because there are non-trivial ones (e.g. a convex cost sharing mechanism under which the convexity of the cost sharing functions is established on the range of consumption of the public good and each agent is assigned the consumption bundle that maximizes the agent's utility according to the agent's cost sharing function) although the serial cost sharing mechanism (Molin, 1994) is not securely implementable. [24] In addition, Saijo, Sjöström, and Yamato (2007) and Kumar (2013) showed a positive result on secure implementation in the problems of providing a divisible private good with monetary transfers and Nishizaki (2014) in pure exchange economies with Leontief utility functions. These environments suggest our future research on secure implementation.

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