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THE PRACTICAL TURN IN PHILOSOPHY OF MATHEMATICS: A PORTRAIT OF A YOUNG DISCIPLINE

abstract

In the present article, the current situation of the so-called philosophy of mathematical practice is discussed. First, its emergence is evaluated in relation to the "practical" turn in philosophy of science and in philosophy of mathematics. Second, the variety of approaches concerned with the practice of mathematics and the new topics being now object of research are introduced. Third, the possible replies to the question about what counts as mathematical practice are taken into account. Finally, some of the problems that are still open in the philosophy of mathematical practice are presented and some possible new directions of research considered.

keywords

philosophy of mathematics, practical turn, philosophy of mathematical practice, mathematical practice

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1. Introduction: The Practical Turn, From Science to Mathematics

Starting from the 1970s, philosophy of science has shown a tendency to move more and more away from the definition of abstract and general theories about scientific knowledge towards the investigation of the concrete work that scientists as practitioners in a particular scientific field are engaged in everyday. This change of perspective has commonly been labeled as the "practical" or "practice" turn:[1] science is not conceived anymore as true and justified belief that has to be examined, so to speak, *in vitro*, but as a mingle of different practices that should be considered *in vivo*, looking at the behaviors and the habits characterizing the people involved in them.

There were reasons for this new perspective to emerge. First, in reaction to views of science that were too disembodied: science cannot be science *from nowhere*; second, in opposition to the "rational reconstructions" that were typical of the philosophy of science of the beginning of the 20th century: science cannot either be science *from anywhere*. According to this new line of thought, the scientific enterprise is analogous to other *human* practices: it is historically and culturally situated, and the task of philosophy should be to clarify its specificities.

However, the consequences of such a practical "revolution" are still under discussion. First of all, was it really a revolution? And if this is the case, in which way was it revolutionary? What are the new objects of research? What is its methodology? In other words, where exactly has the practical turn led us?[2] These questions and analogous ones have been at the center of the recent debate, starting from famous proposals such as Kuhn (1962)'s notion of paradigm up to naturalistic approaches, which totally deny the possibility of *a priori* knowledge and embrace some form of empiricism or pragmatism.

Many of these issues go back to a crucial ambiguity concerning in general "practice-based" approaches. How is 'practice' defined? Is there just one practice or are there many? And if there are many, what are their common features?[3] As Salanskis (2014) has very conveniently summarized,

> if the very program of practice turn is to be significant, it must recommend looking at science in a specific way, which can then be contrasted with other ways. This absolutely

1 For details about the introduction of this term, see Soler *et al*. (2014, p. 39, n. 1).

2 For an in-depth discussion of these topics, see Soler *et al.* (2014).

3 Incidentally, analogous difficulties may rise for the notion of "paradigm" as introduced by Kuhn (1962).

> requires that we have at our disposal a non universally encompassing notion of practice (p. 44).

Compared to the case of philosophy of science, a practical turn in philosophy of mathematics has happened later and only partially.[4] As Hersh (2005) puts it, in the 1970s philosophy of science was already in its renaissance, while most of philosophy of mathematics rather looked as "foundationalist ping-pong" (p. vii). An evident exception was Lakatos' *Proofs and Refutations* (1975), a book which, according to Hersh, is surely fascinating but was however virtually unknown at the time. Moreover, despite Lakatos' approach finds inspiration in Polya's previous work on problem-solving and is undoubtedly new and original, his book ends up being a sort of hybrid, caught between the before and after the practical turn. In fact, the dialogue between the characters in the book still offers a rational reconstruction of the Theorem of Descartes-Euler, even if it is a reconstruction of the *process* leading to its mathematical proof and not of its mathematical proof itself. However, in the footnotes there happens to be another book, which tells us the story of the development of the same theorem. The result is somewhat schizophrenic: the theorem is an historical product but it can still be discussed "from anywhere".

The situation did not change in the following decades, with some exceptions, such as the collection edited by Tymoczko (Ed.) (1986; rev. ed. 1998), which contained articles criticizing the contemporary philosophy of mathematics, or the one edited by Aspray and Kitcher (Eds.) (1988), which had a strong interdisciplinary line of attack to modern mathematics. In the 1990s, van Bendegem (1993) writes as follows: "if science is what scientists do, as it has become fashionable to claim, should it then not be the case that mathematics is what mathematicians do?" (p. 263). Kuhn himself would be reluctant to reply unhesitatingly 'yes' to such a question, since in his view mathematics has a special status that seems to elude the typical complications having to do with change and scientific development; by contrast, Lakatos' answer would undeniably be positive. However, as van Bendegem points out, differently from philosophy of science, in the philosophy of mathematics of the 1990s there still existed something like a "received view": mathematics tended to be considered always – and wrongly, according to him – as "the exact science".

A sharp distinction between the research in philosophy of science and in philosophy of mathematics is however very surprising: on the one hand, mathematics is a scientific discipline; on the other hand, science makes use of many mathematical tools.[5] However, because of its supposed special status compared to other scientific disciplines, mathematics has also built strong connections with formal logic, a discipline that did not exist before the work of Frege. In fact, a drastic change took place even before Frege: if up to the 19th century mathematics was conceived as a discipline that, in one way or another, aimed at describing the real world, it later became an independent corpus of ideas that are proper to mathematics and mathematics only. This conception brought to new questions relative to the nature of mathematics that are very familiar today: if mathematics does not speak about the real world, what is its object of study? It also led to the search for stable foundations for mathematics. For these reasons, at the turn of the 20th century, philosophy of mathematics was specifically concerned with topics such as justification and formal mathematics, leaving

4 This is even more striking for philosophy of logic, but for reasons of space I won't expand on this issue here.

5 The argument of the indispensability of mathematics for the sciences has also brought about reasons to believe in the existence of mathematical entities. For a recent debate on the subject, see for reference Panza and Sereni (2013, chapters 6 and 7).

aside more epistemological issues.[6] Recently however, an interest toward philosophical questions about the practice of mathematics has finally risen, undoubtedly in reaction to this neglect. Mancosu, in his introduction to a collection of essays published less than 10 years ago entitled *The Philosophy of Mathematical Practice*, claims that the works in the book represent "the first steps in a very difficult area and we hope that our efforts might stimulate others to do better" (p. 20). In 2009, the *Association for the Philosophy of Mathematical Practice* (APMP) was created.[7]

The picture of the philosophy of mathematical practice emerging today is very varied and heterogeneous. In Section 2, I will present some possible ways of sorting out the different approaches that have been proposed so far in this domain. In Section 3, I will briefly go back to the problem of how to define a mathematical practice. Finally, in Section 4, I will sketch a picture of the philosophy of mathematical practice today and draw some tentative conclusions about the most crucial issues that remain open.

2. Not Just One Philosophy of Mathematical Practice (and more)

One evident feature of the philosophy of mathematical practice in 2017 is that it is characterized by a variety of different proposals and different methodologies, which only in some cases overlap or happen to be complementary. Recently, some authors have tried to identify the distinct orientations. I will focus in particular on two proposals.

Van Bendegem (2014) presents a list of eight disciplines that look at mathematics from the point of view of its practice: (1) the *Lakatosian* approach, namely the "maverick" tradition; (2) the *descriptive analytical naturalizing* approach; (3) the *normative analytical naturalizing* approach; (4) the *sociology of mathematics* approach; (5) the *mathematics educationalist* approach; (6) the *ethno-mathematical* approach; (7) the *evolutionary biology* of mathematics; and (8) the *cognitive psychology* of mathematics. For him, only the first three perspectives have a distinct philosophical nature, which means that philosophy is not alone in the enterprise; this establishes already a difference with the received view of mathematics as the exact science.

The Lakatosian approach goes back to *Proofs and Refutations*: to make sense of the development of mathematics, it is essential to understand discovery processes. This is in tension with a second tradition that mainly refers to Kitcher's work.[8] According to this tradition, which van Bendegem labels "analytical and naturalizing", the object of research is the final version of the proof and the *desideratum* is to find out what is needed to justify the claim that such a final version is indeed a proof. Differently from the first approach, such a justification is here totally independent of the process that has brought to the proof. In fact, the analytical naturalizing approach is related to the more general agenda in analytic philosophy and in particular to Quine's program of naturalizing epistemology, and breaks down into the *descriptive* and the *normative* analytic naturalizing approaches. For the descriptive approach, the methodology is simply to relate what mathematicians think about their everyday work, for example about what counts for them as a proof; for the normative approach, the methodology is to examine the nature of the proofs that are put forward by mathematicians and to establish whether they are genuine proofs, no matter what the mathematicians' beliefs are. Proposals from (4) to (8)

6 In Agazzi and Heinzmann (2015)'s reconstruction, it is precisely to overcome the foundational crisis that affected the exact sciences, mathematics and physics, at the end of the 19th and at the beginning of the 20th century, that new trends and ideas in philosophy were produced and philosophy of science in its contemporary sense was born. Such a crisis challenged "the pervasive positivist view that had attributed to science the monopole of secure knowledge and the role of being the ground of human progress" (p. 8).

7 In 2017, the APMP will celebrate its fourth international meeting. See for information http://institucional.us.es/apmp/.

8 See for reference Kitcher (1984). We will go back to Kitcher's views in the remainder of the article.

are related to disciplines other than philosophy that are however interested in the practice of mathematics.[9]

From the emerging picture, it is evident that a unity of these distinct approaches is questionable and as a consequence the possible unity of the study of mathematical practice with traditional philosophy of mathematics is even more problematic.[10] Van Bendegem (2014)'s charitable suggestion is to keep an open mind and accept "to work in different 'registers' where reading texts in the field (or that I, at least, consider to be relevant)" (p. 221). In a forthcoming paper, Carter (forthcoming) identifies three different, in some cases overlapping, "strands" in the philosophy of mathematical practice: (1) the *agent based* strand, (2) the *historical* strand, and (3) the *epistemological* strand. Differently from van Bendegem's proposal, all these strands mainly ask philosophical questions, but in a clear interdisciplinary fashion. A crucial point is that according to Carter the philosophy of mathematical practice has changed in the most recent years: in fact, it does not develop anymore in contrast with a more traditional philosophy of mathematics, but on the contrary is aimed to complement it. The agent based strand emerges from the belief that philosophy has to take into account the human beings who are doing mathematics. In her reconstruction, this strand develops along two lines, the first having strong interconnection with sociology – mathematics is a *social* activity – and the second following the views of philosophers such as Peirce, Dewey and Putnam – the so-called *pragmatic orientation*. The historical strand focuses on the *products* of the activity of doing mathematics and on how such products shape across time. History of mathematics is of course an old discipline that has already provided interesting results; however, as Carter points out, the possible relationship between philosophy and history is still matter of discussion, ranging from positions considering history as philosophically laden to others defending the independence of history from philosophy. The third strand that Carter calls epistemological "for lack of a better term" is closer to traditional philosophy of mathematics. However, differently from it, it does not consider epistemology as a view from nowhere but demands that new topics emerging from everyday mathematics be considered as philosophically relevant. Other possible names for this strand could be the *extension-of-topics* strand, the *phenomenological* strand or the *philosophy of real mathematics* strand.

Van Bendegem claims that if the aim is to consider mathematical practice, then philosophy should collaborate with other disciplines; Carter seems instead to argue that philosophy itself should be open to change its nature in view of contributions coming from other disciplines. Moreover, for both of them, the listed approaches are not exclusive, as some scholars happen to endorse more than one. The picture of the philosophy of mathematical practice that results from these two proposals is evidently complex and in progress.

Instead of presenting a new possible catalog of the available views, I will introduce here briefly three categories of new questions about mathematics that have clearly emerged. As Soler and Jullien (2014) claim,

> *dynamic*, *genetic*, and *heuristic* aspects were largely ignored from the pre-practice turn philosophy of science (including mathematics), and the simple fact to take them as an object of study has often worked as a sufficient reason to classify an author as an actor of the practice turn (p. 232).

9 Löwe (2016) explores the interplay between philosophy and other disciplines and its effect on the further development of the field.

10 See the comment to van Bendegem's contribution by Soler and Jullien (2014).

I will follow their hint and focus on these three concerns in turn.

1. Dynamic aspects. One question formulated by Lakatos himself has not been answered yet: can we talk about 'progress' in mathematics? Does mathematics evolve? And if this is the case, are there constraints in its evolution? These issues were among the first to be considered by pioneering works going beyond traditional worries about mathematics. In particular, two collections focused on "revolutions" in mathematics – Gillies (Ed.), 1992 – and on the "growth" of mathematical knowledge – Grosholz & Breger (Eds.), 2000. However, in more recent years, the attention has moved away from questions about change in mathematics to new emerging topics in this category. One relevant subject is the consideration of *values* in mathematics that might influence the direction of research, for example when a certain framework is recognized as interesting, fruitful, providing explanations, or containing promises of solution. What do all these expressions mean? On this subject, it is not clear yet if and how sociological elements may be taken into account.

2. Genetic aspects. When it comes to the genetic aspects of mathematics, history of mathematics on the one hand and cognitive science on the other become relevant. Many authors have acknowledged the importance of an historical perspective to investigate mathematical practice. Corfield (2003) points out that in the course of the 20th century, philosophy moved its attention away from the real mathematical progresses because of the "foundationalist filter" that is the "unhappy" idea behind all forms of neo-logicism. In his view, the job of the philosopher is to dismantle such filter: mathematics is a human activity, and therefore it is situated in time. An interdisciplinary investigation may be of help, "in the process demonstrating that philosophers, historians and sociologists working on pre-1900 mathematics are contributing to our understanding of mathematical thought, rather than acting as chroniclers of proto-rigorous mathematics" (p. 8). The interest in looking at the research in cognitive science for a philosophy of mathematical practice is instead more controversial. Some scholars have explicitly discussed cognitive science research, for example Giaquinto (2006) and Ferreiros (2015). However, a shared intuition is that much work still need to be done to understand what exactly the views about the cognitive foundations of mathematics (Butterworth, 1999; Dehaene, 2007) or the role of conceptual metaphors and conceptual blending in mathematics (Lakoff & Nunez, 2001) might bring to the consideration in particular of the practice of *advanced* mathematics.[11]

3. Heuristic aspects. This category of problems goes back to typical themes from the work of Polya (1945) about the methods to put in place to solve mathematical problems. To give an example, one subject that has been very extensively addressed in the most recent years is the use of diagrams in mathematics.[12] However, thinking in terms of heuristics might be misleading, since heuristics pertains traditionally to the context of discovery but is not part of the context of justification, where proofs are "syntactic objects consisting only of sentences arranged in a finite and inspectable way"[13]. However, a crucial point about the consideration of the role of representations, notations and other kinds of cognitive tools is the evaluation of the influence that they might have on understanding and even on creating mathematics.

11 Some of these issues are discussed in Schlimm (2013) and Giardino (2014).
12 For a survey of the studies about diagrammatic reasoning in mathematics, see Giardino (2017).
13 This passage is quoted from Tennant in Barwise and Etchemendy (1996, p. 3) as expressing the "dogma" of logocentricity that they want to challenge.

It is possible to argue that some proofs, because of their format, have both a heuristic and a justificatory role. For these reasons, the distinction between a context of discovery and a context of justification has become more and more precarious.
In the next section, I will discuss how this heterogeneity of topics for the philosophy of the mathematical practice is reflected in the heterogeneity of possible answers that are given to the question of what a mathematical practice is.

3. Not Just One (Mathematical) Practice

As for philosophy of science, a major difficulty for the philosophy of mathematical practice is how to intend 'mathematical practice'.[14] What do we talk about when we talk about a mathematical practice? Under which conditions is an agent recognized as a practitioner? Shall we talk about one practice or *more* practices? In other words, do several practices exist within one same practice?
On the website of the APMP, the philosophy of mathematical practice is defined as "a broad outward-looking approach" to the study of mathematics "which engages with mathematics in practice (including issues in history of mathematics, the applications of mathematics, cognitive science, etc.)". This definition is indeed far-reaching, most likely to the aim of being as inclusive as possible given the current state of the domain.
Famously, Kitcher (1984) defined a mathematical practice as the quintuple *<L, M, S, R, Q>* composed by *L*anguage, *M*etamathematical views, accepted *S*tatements, *R*easoning methods and *Q*uestions (chapters 7 and 8). Some authors thought of extending or reconsidering this quintuple. For example, Ferreiros (2015, chapter 3) has recently pointed out that Kitcher's quintuple is misleading because it is still based on an analysis of the production of scientific knowledge that depends mainly on linguistic knowledge. If the approach is instead intended to be agent-based and practice-oriented, non-linguistic elements, such as for example some forms of tacit knowledge, become relevant. For this reason, Ferreiros argues that it is necessary to think in terms of the couple Framework *plus* Agent, to whom the metamathematical views belong. Moreover, frameworks are of two kinds: *theoretical* and *symbolic*. However, the Framework-Agent pair is not identified with mathematical practice but is at the core of practice and of the production and reproduction of knowledge. As it is evident, the picture gets more and more complex.[15]
Of course, scholars who are interested in the philosophy of mathematical practice seem to share some notion of practice, but this happens on the surface, while the devil is in the details. In fact, the variety of philosophies of mathematical practice described in the previous section is indeed reflected in the variety of possible views about mathematical practice. For this reason, I propose here to identify (at least) four replies that philosophers of the mathematical practice might give to the question about mathematical practice.
I will call the first reply (1) the *situated* reply. Mathematical practice is a historically situated human activity, and therefore the aim of the philosophy of mathematical practice is to reconstruct the history of the different practices. If mathematics is truly what mathematicians do, then this target has considerably varied over times and places; as a consequence, mathematical practice must be understood in a way that would include this variance. A static view of mathematics considering it as merely a collection of theories,

14 To be true, also the very definition of a practice *in general* is problematic, and therefore it is not clear how to intend the claim that mathematics is analogous to *other* human practices.
15 Another extension of Kitcher's model was proposed by van Kerkhove & van Bendegem (2004), who generalized it and arrived at a seventuple *<M, P, F, PM, C, AM, PS>*, containing a mathematical *community M* of individual mathematicians, a *research program P*, a *formal language F*, a set *PM* of *proof methods*, a set *C* of *concepts*, a set *AM* of *argumentative methods* and a set *PS* of *proof strategies*. Also in this case, the model gets more convoluted.

independent of human activities, is misleading, and it is necessary to move towards a *dynamic* view of mathematics. Of course, Lakatos' lesson is always in the background. The target of the philosophical investigation is the subject matter of mathematics at each time: mathematical theories, theorems, and proofs. Of course, these objects may be presented in different formats and media, and there is an interest in considering their development.

The second reply is (2) the *semiotic* reply. Mathematical practice is a human activity implying the use of many different tools, more importantly several kinds of texts, which are the target of the philosophical research. Mathematicians in their everyday work write drafts, inscriptions, publications; they draw objects, calculate on paper, and write demonstrations. These are all elements of the practice of mathematics, and they can be analyzed one by one without leaving aside their mutual relations. The mathematical practice has then to do with the *traces* of mathematics that are left in sketchbooks, textbooks, essays, and proofs.

The third reply is (3) the *epistemological* reply. Mathematical practice is the construction of theories, but this does not imply endorsing more dogmatic points of view, for example the claim that such theories are necessarily formal systems. As Mancosu (2008) claims,

> the epistemology of mathematics needs to be extended well beyond its present confines to address epistemological issues having to do with fruitfulness, evidence, visualization, diagrammatic reasoning, understanding, explanation and other aspects of mathematical epistemology which are orthogonal to the problem of access to 'abstract objects' (pp. 1-2).

Despite the fact that case studies are necessary precisely because certain areas of mathematics can provide useful tools for addressing important philosophical problems, such approach is not meant to be simply a description of the mathematical theories and of their growth.[16]

The fourth reply is (4) the *pragmatist* reply. For example, for Ferreiros (2015), mathematical practice is what the community of mathematicians does when they employ resources such as frameworks and other tools to the aim of solving problems, proving theorems, and in some cases elaborating new theories and frameworks. Moreover, the choice of these tools is constrained by their cognitive abilities. The study of mathematical practice broadens the scope of philosophical and historical studies by considering carefully the contexts from which mathematical theories and proofs emerge, and issues such as understanding beyond mere logical reconstruction become crucial. Ferreiros' approach is agent-based: practice can be understood only by focusing on practitioners; moreover, it is pragmatist and historically oriented.

As for the approaches in the previous section, these groups of replies are of course not disconnected from each other. For example, the semiotic reply has clearly epistemological interests as well;[17] or, for both the situated and the epistemological reply, the construction of theories and the notion of proof are still two crucial issues in the investigation of mathematical practice.

16 I will come back to this issue in the last section.

17 For example Chemla (2009), discussing the importance of learning to read mathematical texts, explicitly refers to the notion of "epistemological cultures" as introduced by Fox Keller (2002) to define the primordial character of the specific epistemological choices that are made and shared by the agents of cultures that are far from ours.

4. Conclusions: A Pluralism of Approaches

The emerging picture of the philosophy of mathematical practice seems thus to describe a domain of research, mathematics, which has lost its unity because it is now partitioned in many different case studies. More importantly, there is neither unity in the philosophy of mathematical practice itself, since each of the distinct approaches may endorse different points of view on each of these case studies. With this worry in mind, in this last section I will discuss some open problems.

First, the relation between the philosophy of mathematical practice and traditional philosophy of mathematics is not easy to evaluate. Many scholars believe that there is a true need for an extension of the theoretical inquiry that would address topics ignored by the foundationalist tradition (because of what Corfield called the "foundationalist filter"). As Mancosu (2008) highlights, the philosophical literature has extensively pursued the Benecerraf's ontological and epistemological problems (are there abstract objects? and if there are, how can we access them?) and without this extension, it risks being drastically impoverished. However, this does not mean that the work in traditional philosophy of mathematics has to be forgotten or considered as irrelevant. On the contrary, the tools that it has provided can be extended as well to new areas of research that have been previously largely neglected. As he sums up, philosophers today are less ambitious and at the same time more ambitious than before. They are less ambitious because differently from scholars such as Lakatos or Kitcher, they are not concerned anymore with metaphilosophical issues; however, they are more ambitious because they want to cover "a broad spectrum of case studies arising from mathematical practice" that are subject to analytic investigation (p. 14). In a motto: less metaphysical questions, more topics addressed.

Second, it is not clear what the purpose of philosophy should be in considering the practice of mathematics. Some approaches aim at maintaining a normative role for philosophy while others consider that the research in philosophy should provide an attentive description of the situated practice. This might create some tension. In fact, a potential risk is that too much focus on practice will end up dispelling philosophy. As Maddy (1997) already argued in *Naturalism in Mathematics*:

> if our philosophical account of mathematics comes into conflict with successful mathematical practice, it is the philosophy that must give. [...] Similar sentiments appear in the writings of many philosophers of mathematics who hold that the goal of philosophy of mathematics is to account for mathematics as it is practiced, not to recommend reform (p. 161).

However, many philosophers of the mathematical practice today would not subscribe to Maddy's naturalistic claim (in her specified meaning of this term).

Third, another issue not settled yet is the autonomy of mathematics from the natural sciences. In his book, Ferreiros (2015) emphasizes the interplay between mathematics and other kinds of practices. In his view, the problem of the "applicability" of mathematics should not be considered as external to mathematical knowledge but on the contrary as *internal* to its analysis. There is no opposition between "pure" and "applied" mathematics, since to some extent all frameworks are designed to be applicable.

Fourth, for many of the different views that have been described so far, practice has to do with some form of *action*. However, an appropriate analysis of this feature in the practice is still lacking.[18] In this spirit, a promising direction of research will be to explore the view of mathematical knowledge as a *knowing-how*, as practical and/or tacit knowledge, in contrast

18 One attempt in this direction is made in Salanskis (2014).

with or more modestly in addition to the standard view of mathematical knowledge as a knowing-that.
To quote again van Bandegem (2014),

> if all of this looks rather sketchy, it is important to realize, [...] that we are looking, in comparison with developments in the philosophy of science, at a *very young discipline*. Nevertheless, I do think it is important, right from the start, to look for collaborations and not exclusions. Exclusions are only to be accepted when everything else fails, and this is definitely not the case at the present moment (p. 224, emphasis added).

Moreover, I would argue that this lack of unity in the philosophy of mathematical practice is to some extent what is really revolutionary about it: after the practical turn, the territory of philosophical inquiry has radically changed. Enterprises such as the identification of criteria of validity for what counts as a mathematical proof have become local enterprises, which may vary in their methodology and in their results depending on the particular practice and on the particular case study that are taken at each time into account. It would then seem that mathematics, which was considered as a stable, static, certain, exact science not subject to change or development, has finally exploded into pieces and it will be impossible for philosophy ever again to provide a unitary account for it. Alternatively, an improved philosophy of mathematics will consider this as an occasion to specify new questions and take really into account the actual richness of its domain of interest in all its complexity. Will the philosophy of mathematical practice ever become an adult discipline? More time is needed to reply to this question.

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