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ON THE GROWTHS OF MEROMORPHIC FUNCTION GENERATED WRONSKIANS FROM THE VIEW POINT OF SLOWLY CHANGING FUNCTIONS

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Abstract. In the paper we extend and sometimes improve few results on the comparative growth properties of composite entire or meromorphic functions of [3], [4] and [5] using m-th generalized ${}_pL^*$-order and the m-th generalized ${}_pL^*$-lower order and Wronskians generated by one of the factors where m and p are any two positive integers.

Keywords: Transcendental entire function, transcendental meromorphic function, composition, growth, m-th generalized ${}_pL^*$-order and the m-th generalized ${}_pL^*$-lower order, Wronskian.

1. Introduction, Definitions and Notations

We denote by $\mathbb{C}$ a set of all finite complex numbers. Let f be an entire function defined on $\mathbb{C}$. The maximum modulus function corresponding to the entire f is defined as $M(r, f) = \max\{|f(z)| : |z| = r\}$. When f is meromorphic, $M(r, f)$ cannot be defined as f is not analytic. In this case one may define another function $T(r, f)$ known as Nevanlinna's Characteristic function of f, playing the same role as the maximum modulus function in the following manner:

$$T(r, f) = N(r, f) + m(r, f),$$

where the functions $N(r, f)$ and $m(r, f)$ are respectively the enumerative function and the proximity function corresponding to f. For further details one may see [6]. If f is an entire function, then the Nevanlinna's Characteristic $T(r, f)$ of f reduces to $m(r, f)$.

The following definitions are well known:

Definition 1.1. A meromorphic function $a \equiv a(z)$ is called small with respect to f if $T(r, a) = S(r, f)$.

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Definition 1.2. Let $a_1, a_2, ...a_k$ be linearly independent meromorphic functions and small with respect to f. We denote by $L(f) = W(a_1, a_2, ...a_k, f)$ the Wronskian determinant of $a_1, a_2, ..., a_k, f$ i.e.,

$$L(f) = \begin{vmatrix} a_1 & a_2 & . & . & . & a_k & f \\ a_1' & a_2' & . & . & . & a_k' & f' \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_1^{(k)} & a_2^{(k)} & . & . & . & a_k^{(k)} & f^{(k)} \end{vmatrix} .$$

Definition 1.3. If $a \in \mathbb{C} \cup \{\infty\}$, the quantity

$$\begin{aligned} \delta(a; f) &= 1 - \limsup_{r\to\infty} \frac{N(r, a; f)}{T(r, f)} \\ &= \liminf_{r\to\infty} \frac{m(r, a; f)}{T(r, f)} \end{aligned}$$

is called the Nevanlinna deficiency of the value 'a'.

From the second fundamental theorem it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a; f) > 0$ is countable and $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) \leq 2$ (cf.[[6], *p*.43]). If in particular, $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$, we say that f has the maximum deficiency sum.

Let $L \equiv L(r)$ be a positive continuous function increasing slowly *i.e.*, $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. *Singh and Barker* [9] defined it in the following way:

Definition 1.4. [9] A positive continuous function $L(r)$ is called a slowly changing function if for $\varepsilon (> 0)$,

$$\frac{1}{k^{\varepsilon}} \leq \frac{L(kr)}{L(r)} \leq k^{\varepsilon} \ for\ r \geq r(\varepsilon) \text{ and}$$

uniformly for $k (\geq 1)$.

Somasundaram and Thamizharasi [10] introduced the notions of L-order and L-lower order for entire function where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e.,$L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant 'a'. The more generalized concept for L-order and L-lower order for the entire function are L^*-order and L^*-lower order. Their definitions are as follows:

Definition 1.5. [10] The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of an entire function f are defined as

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log\left[re^{L(r)}\right]}$$

where $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ for $k = 1, 2, 3, ...$ and $\log^{[0]} x = x$.

When f is meromorphic, the above definition reduces to

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}\ .$$

In the line of Somasundaram and Thamizharasi [10] , for any two positive integers m and p, Datta and Biswas [2] introduced the following definition:

Definition 1.6. [2] The m-th generalized ${}_pL^*$-order with rate p denoted by ${}_{(p)}^{(m)}\rho_f^{L^*}$ and the m-th generalized ${}_pL^*$-lower order with rate p denoted as ${}_{(p)}^{(m)}\lambda_f^{L^*}$ of an entire function f are defined in the following way:

$${}_{(p)}^{(m)}\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[m+1]} M(r,f)}{\log\left[r\exp^{[p]} L(r)\right]} \text{ and } {}_{(p)}^{(m)}\lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[m+1]} M(r,f)}{\log\left[r\exp^{[p]} L(r)\right]},$$

where both m and p are positive integers.

When f is meromorphic, it can be easily verified that

$${}_{(p)}^{(m)}\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[m]} T(r,f)}{\log\left[r\exp^{[p]} L(r)\right]} \text{ and } {}_{(p)}^{(m)}\lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[m]} T(r,f)}{\log\left[r\exp^{[p]} L(r)\right]},$$

where both m and p are positive integers.

Since the natural extension of a derivative is a differential polynomial, in this paper we prove our results for a special type of linear differential polynomials viz. the Wronskians. In the paper we establish some new results depending on the comparative growth properties of composite entire or meromorphic functions using m-th generalized ${}_pL^*$-order with rate p (respectively m-th generalized ${}_pL^*$-lower order with rate p) where m and p are any two positive integers and Wronskians generated by one of the factors which extend and sometimes improve earlier results of [3], [4] and [5]. We have used the standard notations and definitions in the theory of entire and meromorphic functions which are available in [6] and [11].

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [1] *If f be meromorphic and g be entire then for all sufficiently large values of r,*

$$T(r, f\circ g) \leq \{1 + o(1)\} \frac{T(r,g)}{\log M(r,g)} T(M(r,g), f)\ .$$

Lemma 2.2. [8] *Let f and g be any two entire functions. Then for all $r>0$,*

$$T(r, f\circ g) \geq \frac{1}{3}\log M\left\{\frac{1}{8}M\left(\frac{r}{4},g\right)+o(1),f\right\}.$$

Lemma 2.3. [7] *Let f be a transcendental meromorphic function having the maximum deficiency sum. Then*

$$\lim_{r\to\infty}\frac{T(r,L(f))}{T(r,f)} = 1+k-k\delta(\infty;f).$$

Lemma 2.4. *Let f be a transcendental meromorphic function having the maximum deficiency sum and m and p are any two positive integers.Then the m-th generalized ${}_pL^*$-order with rate p (the m-th generalized ${}_pL^*$-lower order with rate p) of $L(f)$ and that of f are same.*

Proof. By Lemma 2.3, $\lim\limits_{r\to\infty}\frac{\log^{[m]}T(r,L(f))}{\log^{[m]}T(r,f)}$ exists and is equal to 1 for $m\geq 1$. Now

$$\begin{aligned}
{}_{(p)}^{(m)}\rho_{L(f)}^{L^*} &= \limsup_{r\to\infty}\frac{\log^{[m]}T(r,L(f))}{\log\left[r\exp^{[p]}L(r)\right]}\\
&= \lim_{r\to\infty}\frac{\log^{[m]}T(r,L(f))}{\log^{[m]}T(r,f)}\cdot\limsup_{r\to\infty}\frac{\log^{[m]}T(r,f)}{\log\left[r\exp^{[p]}L(r)\right]}\\
&= {}_{(p)}^{(m)}\rho_{f}^{L^*}.
\end{aligned}$$

In a similar manner, ${}_{(p)}^{(m)}\lambda_{L(f)}^{L^*} = {}_{(p)}^{(m)}\lambda_{f}^{L^*}$.
This proves the lemma. □

3. Theorems

In this section we present the main results of the paper.

Theorem 3.1. *Let f be a transcendental meromorphic function having the maximum deficiency sum and g be entire such that ${}_{(p)}^{(n)}\rho_{g}^{L^*} < {}_{(p)}^{(m)}\rho_{f}^{L^*} < \infty$ where m, n and p are any three positive integers . Then*

$$\liminf_{r\to\infty}\frac{\log^{[m]}T(r,f\circ g)+\log^{[n+1]}M(r,g)}{\log^{[m-1]}T(r,L(f))\cdot K(r,g;L)} = 0,$$

where $K(r,g;L) = \begin{cases} 1 \text{ if } \exp^{[p-1]}L(M(r,g)) = \\ o\left\{\exp^{[m-1]}\left[r\exp^{[p]}L(r)\right]^{\alpha}\right\} \text{ as } r\to\infty \\ \qquad \text{and for some } \alpha < {}_{(p)}^{(m)}\rho_{f}^{L^*} \\ \exp^{[p-1]}L(M(r,g)) \text{ otherwise.}\end{cases}$

Proof. In view of the inequality $T(r,g) \leq \log^+ M(r,g)$ and by Lemma 2.1, we get for all sufficiently large positive numbers of r that

$$\begin{aligned} & i.e., \ \log^{[m]} T(r, f \circ g) \\ \leq \ & \left({}^{(m)}_{(p)}\rho^{L^*}_f + \varepsilon \right) \left(\log M(r,g) + \exp^{[p-1]} L(M(r,g)) \right) \\ & + O(1) \, . \end{aligned} \tag{3.1}$$

Now from the definition of ${}^{(n)}_{(p)}\rho^{L^*}_g$, we obtain for all sufficiently large positive numbers of r that

$$\log^{[n+1]} M(r,g) \leq \left({}^{(n)}_{(p)}\rho^{L^*}_g + \varepsilon \right) \left[\log r + \exp^{[p-1]} L(r) \right] \, . \tag{3.2}$$

Therefore from (3.1) and in view of (3.2), we get for all sufficiently large positive numbers of r that

$$\begin{aligned} & \log^{[m]} T(r, f \circ g) \\ \leq \ & O(1) + \left({}^{(m)}_{(p)}\rho^{L^*}_f + \varepsilon \right) \cdot \\ & \left[\exp^{[n-1]} \left(r \exp^{[p]} L(r) \right)^{\left({}^{(n)}_{(p)}\rho^{L^*}_g + \varepsilon \right)} + \exp^{[p-1]} L(M(r,g)) \right] \, . \end{aligned} \tag{3.3}$$

Now from (3.2) and (3.3), it follows for all sufficiently large positive numbers of r that

$$\begin{aligned} & \log^{[m]} T(r, f \circ g) + \log^{[n+1]} M(r,g) \\ \leq & \left({}^{(m)}_{(p)}\rho^{L^*}_f + \varepsilon \right) \left[\exp^{[n-1]} \left(r \exp^{[p]} L(r) \right)^{\left({}^{(n)}_{(p)}\rho^{L^*}_g + \varepsilon \right)} + \exp^{[p-1]} L(M(r,g)) \right] \\ & O(1) + \left({}^{(n)}_{(p)}\rho^{L^*}_g + \varepsilon \right) \left[\log r + \exp^{[p-1]} L(r) \right] \, . \end{aligned} \tag{3.4}$$

Also in view of Lemma 2.4, we obtain for a sequence of positive numbers of r tending to infinity that

$$\log^{[m]} T(r, L(f)) \geq \left({}^{(m)}_{(p)}\rho^{L^*}_{L(f)} - \varepsilon \right) \log \left[r \exp^{[p]} L(r) \right]$$

$$i.e., \ \log^{[m]} T(r, L(f)) \geq \left({}^{(m)}_{(p)}\rho^{L^*}_f - \varepsilon \right) \log \left[r \exp^{[p]} L(r) \right]$$

$$i.e., \ \log^{[m-1]} T(r, L(f)) \geq \left[r \exp^{[p]} L(r) \right]^{\left({}^{(m)}_{(p)}\rho^{L^*}_f - \varepsilon \right)} \, .$$

Now from (3.4) and (3.), we get for a sequence of positive numbers of r tending to infinity that

$$\frac{\log^{[m]} T(r, f\circ g) + \log^{[n+1]} M(r,g)}{\log^{[m-1]} T(r, L(f))}$$

$$\leq \frac{O(1) + \left({}^{(n)}_{(p)}\rho_g^{L^*} + \varepsilon\right)\left[\log r + \exp^{[p-1]} L(r)\right]}{\log^{[m-1]} T(r, L(f))}$$

$$+\frac{\left({}^{(m)}_{(p)}\rho_f^{L^*} + \varepsilon\right)\left[\exp^{[n-1]}\left(r\exp^{[p]} L(r)\right)^{\left({}^{(n)}_{(p)}\rho_g^{L^*}+\varepsilon\right)} + \exp^{[p-1]} L(M(r,g))\right]}{\left[r\exp^{[p]} L(r)\right]^{\left({}^{(m)}_{(p)}\rho_f^{L^*}-\varepsilon\right)}} . \tag{3.5}$$

Since ${}^{(n)}_{(p)}\rho_g^{L^*} < {}^{(m)}_{(p)}\rho_f^{L^*}$, we can choose $\varepsilon\,(>0)$ in such a way that

$$ {}^{(n)}_{(p)}\rho_g^{L^*} + \varepsilon < {}^{(m)}_{(p)}\rho_f^{L^*} - \varepsilon . \tag{3.6}$$

Case I. Let $\exp^{[p-1]} L(M(r,g)) = o\left\{\exp^{[m-1]}\left[r\exp^{[p]} L(r)\right]^{\alpha}\right\}$ as $r\to\infty$ and for some $\alpha < {}^{(m)}_{(p)}\rho_f^{L^*}$.

As $\alpha < {}^{(m)}_{(p)}\rho_f^{L^*}$, we can choose $\varepsilon\,(>0)$ in such a way that

$$\alpha < {}^{(m)}_{(p)}\rho_f^{L^*} - \varepsilon . \tag{3.7}$$

Since $\exp^{[p-1]} L(M(r,g)) = o\left\{\exp^{[m-1]}\left[r\exp^{[p]} L(r)\right]^{\alpha}\right\}$ as $r\to\infty$ we get on using (3.7) that

$$\frac{\exp^{[p-1]} L(M(r,g))}{\exp^{[m-1]}\left[r\exp^{[p]} L(r)\right]^{\alpha}} \to 0 \text{ as } r\to\infty$$

$$\text{i.e.,}\quad \frac{\exp^{[p-1]} L(M(r,g))}{\exp^{[m-1]}\left[r\exp^{[p]} L(r)\right]^{\left({}^{(m)}_{(p)}\rho_f^{L^*}-\varepsilon\right)}} \to 0 \text{ as } r\to\infty .$$

Now in view of (3.5), (3.6) and (3.) we get that

$$\liminf_{r\to\infty}\frac{\log^{[m]} T(r, f\circ g) + \log^{[n+1]} M(r,g)}{\log^{[m-1]} T(r, L(f))} = 0 . \tag{3.8}$$

Case II. If $\exp^{[p-1]} L(M(r,g)) \neq o\left\{\exp^{[m-1]}\left[r\exp^{[p]} L(r)\right]^{\alpha}\right\}$ as $r\to\infty$ and for some $\alpha < {}^{(m)}_{(p)}\rho_f^{L^*}$ then we get from (3.5) for a sequence of positive numbers of r tending to infinity that

$$\frac{\log^{[m]} T(r, f\circ g) + \log^{[n+1]} M(r,g)}{\log^{[m-1]} T(r, L(f)) \exp^{[p-1]} L(M(r,g))}$$
$$\leq \frac{O(1) + \left({}^{(n)}_{(p)}\rho_g^{L^*} + \varepsilon\right)\left[\log r \exp^{[p]} L(r)\right]}{\left[r\exp^{[p]} L(r)\right]^{\left({}^{(m)}_{(p)}\rho_f^{L^*} - \varepsilon\right)} \cdot \exp^{[p-1]} L(M(r,g))}$$

$$+ \frac{\left({}^{(m)}_{(p)}\rho_f^{L^*} + \varepsilon\right)\left[\exp^{[n-1]}\left(r\exp^{[p]} L(r)\right)^{\left({}^{(n)}_{(p)}\rho_g^{L^*}+\varepsilon\right)} + \exp^{[p-1]} L(M(r,g))\right]}{\left[r\exp^{[p]} L(r)\right]^{\left({}^{(m)}_{(p)}\rho_f^{L^*} - \varepsilon\right)} \cdot \exp^{[p-1]} L(M(r,g))}. \tag{3.9}$$

Now using (3.6), it follows from (3.9) that

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g) + \log^{[n+1]} M(r,g)}{\log^{[m-1]} T(r, L(f)) \exp^{[p-1]} L(M(r,g))} = 0. \tag{3.10}$$

Combining (3.8) and (3.10) we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g) + \log^{[n+1]} M(r,g)}{\log^{[m-1]} T(r, L(f)) \cdot K(r,g;L)} = 0,$$

where $K(r,g;L) = \begin{cases} 1 \text{ if } \exp^{[p-1]} L(M(r,g)) = \\ o\left\{\exp^{[m-1]}\left[r\exp^{[p]} L(r)\right]^{\alpha}\right\} \text{ as } r\to\infty \\ \qquad \text{and for some } \alpha < {}^{(m)}_{(p)}\rho_f^{L^*} \\ \exp^{[p-1]} L(M(r,g)) \text{ otherwise.} \end{cases}$

Thus the theorem is established. □

Theorem 3.2. *Let f be a transcendental meromorphic function having the maximum deficiency sum and g be entire with ${}^{(n)}_{(p)}\rho_g^{L^*} < {}^{(m)}_{(p)}\lambda_f^{L^*} < \infty$ where m, n and p are any three positive integers. Then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g) + \log^{[n+1]} M(r,g)}{\log^{[m-1]} T(r, L(f)) \cdot K(r,g;L)} = 0,$$

where $K(r,g;L) = \begin{cases} 1 \textit{ if } \exp^{[p-1]} L(M(r,g)) = \\ o\left\{\exp^{[m-1]}\left[r\exp^{[p]} L(r)\right]^{\alpha}\right\} \textit{ as } r\to\infty \\ \qquad \textit{and for some } \alpha < {}^{(m)}_{(p)}\lambda_f^{L^*} \\ \exp^{[p-1]} L(M(r,g)) \textit{ otherwise.} \end{cases}$

Theorem 3.3. *Let f be a transcendental meromorphic function having the maximum deficiency sum and g be entire such that ${}^{(n)}_{(p)}\lambda^{L^*}_{g} < {}^{(m)}_{(p)}\lambda^{L^*}_{f} \leq {}^{(m)}_{(p)}\rho^{L^*}_{f} < \infty$ and ${}^{(n)}_{(p)}\rho^{L^*}_{g} < \infty$ where m, n and p are any three positive integers. Then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g) + \log^{[n+1]} M(r,g)}{\log^{[m-1]} T(r, L(f)) \cdot K(r,g;L)} = 0\,,$$

$$\text{where } K(r,g;L) = \begin{cases} 1 \text{ if } \exp^{[p-1]} L(M(r,g)) = \\ o\left\{\exp^{[m-1]}\left[r\exp^{[p]} L(r)\right]^{\alpha}\right\} \text{ as } r\to\infty \\ \qquad\qquad \text{and for some } \alpha < {}^{(m)}_{(p)}\lambda^{L^*}_{f} \\ \exp^{[p-1]} L(M(r,g)) \text{ otherwise.} \end{cases}$$

Theorem 3.4. *Let f be a transcendental meromorphic function having the maximum deficiency sum and g be entire with ${}^{(n)}_{(p)}\rho^{L^*}_{g} < {}^{(m)}_{(p)}\lambda^{L^*}_{f} \leq {}^{(m)}_{(p)}\rho^{L^*}_{f} < \infty$ where m, n and p are any three positive integers. Then*

$$\lim_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g) + \log^{[n+1]} M(r,g)}{\log^{[m-1]} T(r, L(f)) \cdot K(r,g;L)} = 0\,,$$

$$\text{where } K(r,g;L) = \begin{cases} 1 \text{ if } \exp^{[p-1]} L(M(r,g)) = \\ o\left\{\exp^{[m-1]}\left[r\exp^{[p]} L(r)\right]^{\alpha}\right\} \text{ as } r\to\infty \\ \qquad\qquad \text{and for some } \alpha < {}^{(m)}_{(p)}\lambda^{L^*}_{f} \\ \exp^{[p-1]} L(M(r,g)) \text{ otherwise.} \end{cases}$$

The proof of Theorem 3.2, Theorem 3.3 and Theorem 3.4 are omitted because those can be carried out in the line of Theorem 3.1.

Theorem 3.5. *Let f a transcendental entire function having maximum deficiency sum and g be an entire function such that $0 < {}^{(m)}_{(p)}\lambda^{L^*}_{f} \leq {}^{(m)}_{(p)}\rho^{L^*}_{f} < \infty$, $0 < {}^{(n)}_{(p)}\lambda^{L^*}_{g} \leq {}^{(n)}_{(p)}\rho^{L^*}_{g} < \infty$ where m, n and p are any three positive integers. Then for every constant A and for any real number x,*

$$\lim_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\left\{\log^{[m]} T(r^A, L(f))\right\}^{1+x}} = \infty\,.$$

Proof. If x is such that $1 + x \leq 0$, then the theorem is obvious. So we suppose that $1 + x > 0$.

Now in view of Lemma 2.2, we have for all sufficiently large positive numbers of r that

$$T(r, f\circ g) \geq \frac{1}{3}\log M\left\{\frac{1}{8}M\left(\frac{r}{4}, g\right) + o(1), f\right\}$$

$$i.e.,\ \log^{[m]} T(r, f\circ g) \geq o(1) + \log^{[m+1]} M\left\{\frac{1}{8}M\left(\frac{r}{4}, g\right) + o(1), f\right\}\,.$$

$$i.e.,\ \log^{[m]} T(r, f\circ g) \geq o(1) + \left({}^{(m)}_{(p)}\lambda_f^{L^*} - \varepsilon\right)\left[\log\left\{\frac{1}{8}M\left(\frac{r}{4}, g\right) + o(1)\right\} + \exp^{[p-1]} L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right)\right]$$

$$i.e.,\ \log^{[m]} T(r, f\circ g) \geq o(1) + \left({}^{(m)}_{(p)}\lambda_f^{L^*} - \varepsilon\right)\left[\log M\left(\frac{r}{4}, g\right) + o(1) + \exp^{[p-1]} L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right)\right]$$

$$\begin{aligned} &i.e.,\ \log^{[m]} T(r, f\circ g) \\ \geq\ & o(1) + \left({}^{(m)}_{(p)}\lambda_f^{L^*} - \varepsilon\right)\left[\exp^{[n-1]}\left[\left(\frac{r}{4}\right)\exp^{[p-1]} L(r)\right]^{{}^{(n)}_{(p)}\lambda_g^{L^*} - \varepsilon} + o(1)\right. \\ & \left. + \exp^{[p-1]} L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right)\right] \qquad (3.11) \end{aligned}$$

where we choose $0 < \varepsilon < \min\left\{{}^{(m)}_{(p)}\lambda_f^{L^*}, {}^{(n)}_{(p)}\lambda_g^{L^*}\right\}$.
Also for all sufficiently large positive numbers of r, we get from Lemma 2.4 that

$$\begin{aligned} \log^{[m]} T\left(r^A, L(f)\right) &\leq \left({}^{(m)}_{(p)}\rho_{L(f)}^{L^*} + \varepsilon\right)\log\left[r^A \exp^{[p]} L\left(r^A\right)\right] \\ i.e.,\ \log^{[m]} T\left(r^A, L(f)\right) &\leq \left(\rho^{(m)}_{(p)}\rho_f^{L^*} + \varepsilon\right)\log\left[r^A \exp^{[p]} L\left(r^A\right)\right] \end{aligned}$$

$$\begin{aligned} &i.e.,\ \left\{\log^{[m]} T\left(r^A, L(f)\right)\right\}^{1+x} \\ \leq\ & \left({}^{(m)}_{(p)}\rho_f^{L^*} + \varepsilon\right)^{1+x}\left(\log\left[r^A \exp^{[p]} L\left(r^A\right)\right]\right)^{1+x}. \qquad (3.12) \end{aligned}$$

Therefore from (3.11) and (3.12) it follows for all sufficiently large positive numbers of r that

$$\frac{\log^{[m]} T(r, f\circ g)}{\left\{\log^{[m]} T(r^A, L(f))\right\}^{1+x}}$$

$$\geq \frac{o(1) + \left({}^{(m)}_{(p)}\lambda_f^{L^*} - \varepsilon\right)\left[\exp^{[n-1]}\left[\left(\frac{r}{4}\right)\exp^{[p-1]} L(r)\right]^{{}^{(n)}_{(p)}\lambda_g^{L^*} - \varepsilon} + \exp^{[p-1]} L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right)\right]}{\left({}^{(m)}_{(p)}\rho_f^{L^*} + \varepsilon\right)^{1+x}\left(\log\left[r^A \exp^{[p]} L(r^A)\right]\right)^{1+x}}$$

Thus from the above the theorem follows. □

Theorem 3.6. *Let f be an entire function and g be a transcendental entire function with $0 < {}^{(m)}_{(p)}\lambda_f^{L^*} \leq {}^{(m)}_{(p)}\rho_f^{L^*} < \infty$, $0 < {}^{(n)}_{(p)}\lambda_g^{L^*} \leq {}^{(n)}_{(p)}\rho_g^{L^*} < \infty$ and $\sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2$ where m, n and p are any three positive integers. Then for every constant A and for any real number x,*

$$\lim_{r \to \infty} \frac{\log^{[m]} T(r, f \circ g)}{\{\log T(r^A, L(g))\}^{1+x}} = \infty .$$

The proof of Theorem 3.6 is omitted as it can be carried out in the line of Theorem 3.5.

Theorem 3.7. *Let f be transcendental meromorphic having maximum deficiency sum and g be entire satisfying the conditions that (i) ${}^{(m)}_{(p)}\rho_f^{L^*}, {}^{(n)}_{(p)}\rho_g^{L^*}$ are both finite and (ii) ${}^{(m)}_{(p)}\rho_f^{L^*}$ is positive where m, n and p are any three positive integers. Then for each $\alpha \in (-\infty, \infty)$,*

$$\liminf_{r \to \infty} \frac{\left\{\log^{[m]} T(r, f \circ g)\right\}^{1+\alpha}}{\log^{[m]} T(\exp(r^A), L(f))} = 0$$

where $A > (1+\alpha) \cdot {}^{(n)}_{(p)}\rho_g^{L^*}$.

Proof. If $1+\alpha < 0$, then the theorem is trivial. So we take $1+\alpha > 0$. Now from (3.3) we obtain for all sufficiently large positive numbers of r that

$$\log^{[m]} T(r, f \circ g) \leq \exp^{[n-1]}\left(r \exp^{[p]} L(r)\right)^{\left({}^{(n)}_{(p)}\rho_g^{L^*}+\varepsilon\right)} \cdot \left({}^{(m)}_{(p)}\rho_f^{L^*} + \varepsilon\right)$$
$$O(1) + \left({}^{(m)}_{(p)}\rho_f^{L^*} + \varepsilon\right) \cdot \exp^{[p-1]} L(M(r, g))$$

$$\text{i.e.,}\ \left\{\log^{[m]} T(r, f \circ g)\right\}^{1+\alpha}$$
$$\leq \left[\exp^{[n-1]}\left(r \exp^{[p]} L(r)\right)^{\left({}^{(n)}_{(p)}\rho_g^{L^*}+\varepsilon\right)} \cdot \left\{\left({}^{(m)}_{(p)}\rho_f^{L^*} + \varepsilon\right) + O(1)\right\}\right.$$
$$\left. + \left({}^{(m)}_{(p)}\rho_f^{L^*} + \varepsilon\right) \cdot \exp^{[p-1]} L(M(r, g))\right]^{1+\alpha} . \tag{3.13}$$

Again in view of Lemma 2.4 we have for a sequence of positive numbers of r tending to infinity and for $\varepsilon (> 0)$,

$$\log^{[m]} T\left(\exp\left(r^A\right), L(f)\right) \geq \left({}^{(m)}_{(p)}\rho_{L(f)}^{L^*} - \varepsilon\right) \log\left[\exp\left(r^A\right) \exp^{[p]}\left\{L\left(\exp\left(r^A\right)\right)\right\}\right]$$

$$\text{i.e.,}\ \log^{[m]} T\left(\exp\left(r^A\right), L(f)\right)$$
$$\geq \left({}^{(m)}_{(p)}\rho_f^{L^*} - \varepsilon\right)\left[r^A + \exp^{[p-1]} L\left(\exp\left(r^A\right)\right)\right]. \tag{3.14}$$

Now let

$$\left({}^{(m)}_{(p)}\rho_f^{L^*}+\varepsilon\right)+O(1)=k_1,\left({}^{(m)}_{(p)}\rho_f^{L^*}+\varepsilon\right)\cdot\exp^{[p-1]}L\left(M\left(r,g\right)\right)=k_2,$$

$$\left({}^{(m)}_{(p)}\rho_f^{L^*}-\varepsilon\right)=k_3 \text{ and } \left({}^{(m)}_{(p)}\rho_f^{L^*}-\varepsilon\right)\exp^{[p-1]}L\left(\exp\left(r^A\right)\right)=k_4.$$

Then from (3.13), (3.14) and above we get for a sequence of positive numbers of r tending to infinity that

$$\frac{\left\{\log^{[m]}T\left(r,f\circ g\right)\right\}^{1+\alpha}}{\log^{[m]}T\left(\exp\left(r^A\right),L(f)\right)}\leq\frac{\left[\exp^{[n-1]}\left(r\exp^{[p]}L\left(r\right)\right)^{\left({}^{(n)}_{(p)}\rho_g^{L^*}+\varepsilon\right)}k_1+k_2\right]^{1+\alpha}}{k_3r^A+k_4}$$

$$i.e.,\ \frac{\left\{\log^{[m]}T\left(r,f\circ g\right)\right\}^{1+\alpha}}{\log^{[m]}T\left(\exp\left(r^A\right),L(f)\right)}$$

$$\leq\frac{\exp^{[n-1]}\left(r\exp^{[p]}L\left(r\right)\right)^{\left({}^{(n)}_{(p)}\rho_g^{L^*}+\varepsilon\right)(1+\alpha)}\left[k_1+\frac{k_2}{\exp^{[n-1]}\left(r\exp^{[p]}L(r)\right)^{\left({}^{(n)}_{(p)}\rho_g^{L^*}+\varepsilon\right)}}\right]^{1+\alpha}}{k_3r^A+k_4}$$

where k_1,k_2,k_3 and k_4 are all finite.

Since$\left({}^{(n)}_{(p)}\rho_g^{L^*}+\varepsilon\right)(1+\alpha)<A$, we obtain from the above

$$\liminf_{r\to\infty}\frac{\left\{\log^{[m]}T\left(r,f\circ g\right)\right\}^{1+\alpha}}{\log^{[m]}T\left(\exp\left(r^A\right),L(f)\right)}=0$$

where we choose $\varepsilon(>0)$ in such a way that

$$0<\varepsilon<\min\left\{{}^{(m)}_{(p)}\rho_f^{L^*},\frac{A}{1+\alpha}-{}^{(n)}_{(p)}\rho_g^{L^*}\right\}.$$

This proves the theorem. □

Remark 3.1. The condition $A>(1+\alpha)\cdot{}^{(n)}_{(p)}\rho_g^{L^*}$ is essential in Theorem 3.7 as we see in the following example.

Example 3.1. Let $f=g=\exp z$, $m=n=p=1$, $A=1$, $\alpha=0$ and $L\left(r\right)=\frac{1}{l}\exp\left(\frac{1}{r}\right)$ where l is any positive real number.
Then

$$\lambda_f^{L^*}=\rho_f^{L^*}=\lambda_g^{L^*}=\rho_g^{L^*}=1,\ f\circ g=\exp^{[2]}z \text{ and } \sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2.$$

Also taking $a_1 = 1$ and $a_2 = ... = a_k = 0$, we get that

$$L(f) = \begin{vmatrix} a_1 & f \\ a_1' & f' \end{vmatrix} = \begin{vmatrix} 1 & \exp z \\ 0 & \exp z \end{vmatrix} = \exp z.$$

Now

$$\log T(r, f \circ g) \sim \log\left\{\frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}\right\} \quad (r \to \infty)$$
$$= r - \frac{1}{2}\log r + O(1) \quad (r \to \infty) .$$

Therefore

$$\liminf_{r\to\infty} \frac{\left\{\log^{[m]} T(r, f \circ g)\right\}^{1+\alpha}}{\log^{[m]} T(\exp(r^A), L(f))} = \liminf_{r\to\infty} \frac{r - \frac{1}{2}\log r + O(1)}{r + O(1)} = 1,$$

which is contrary to Theorem 3.7.

In the line of Theorem 3.7, the following theorem may be proved and therefore its proof is omitted:

Theorem 3.8. *Let f be transcendental meromorphic with $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$ and g be entire satisfying the conditions (i) $0 < {}^{(m)}_{(p)}\lambda_f^{L^*} \leq {}^{(m)}_{(p)}\rho_f^{L^*} < \infty$ and (ii) ${}^{(n)}_{(p)}\rho_g^{L^*}$ is finite where m, n and p are any three positive integers. Then for each $\alpha \in (-\infty, \infty)$,*

$$\lim_{r\to\infty} \frac{\left\{\log^{[m]} T(r, f \circ g)\right\}^{1+\alpha}}{\log^{[m]} T(\exp(r^A), L(f))} = 0$$

where $A > (1 + \alpha) \cdot {}^{(n)}_{(p)}\rho_g^{L^*}$.

Theorem 3.9. *Let f be meromorphic and g be transcendental entire such that $0 < {}^{(n)}_{(p)}\lambda_g^{L^*} \leq {}^{(n)}_{(p)}\rho_g^{L^*} < \infty$ and ${}^{(m)}_{(p)}\rho_f^{L^*} < \infty$ where m, n and p are any three positive integers. Also let $\sum_{a\neq\infty} \delta(a; g) + \delta(\infty; g) = 2$.Then for each $\alpha \in (-\infty, \infty)$,*

$$\lim_{r\to\infty} \frac{\left\{\log^{[m]} T(r, f \circ g)\right\}^{1+\alpha}}{\log^{[n]} T(\exp(r^A), L(g))} = 0 \text{ if } A > (1 + \alpha) \cdot {}^{(n)}_{(p)}\rho_g^{L^*} .$$

Theorem 3.10. *Let f be meromorphic and g be transcendental entire with ${}^{(m)}_{(p)}\rho_f^{L^*} < \infty$ and $0 < {}^{(n)}_{(p)}\rho_g^{L^*} < \infty$ where m, n and p are any three positive integers. Also let and $\sum_{a\neq\infty} \delta(a; g) + \delta(\infty; g) = 2$. Then for each $\alpha \in (-\infty, \infty)$,*

$$\liminf_{r\to\infty} \frac{\left\{\log^{[m]} T(r, f \circ g)\right\}^{1+\alpha}}{\log^{[n]} T(\exp(r^A), L(g))} = 0 \text{ where } A > (1 + \alpha) \cdot {}^{(n)}_{(p)}\rho_g^{L^*} .$$

The proof of Theorem 3.9 and Theorem 3.10 are omitted because those can be carried out in the line of Theorem 3.8 and Theorem 3.7 respectively.

Remark 3.2. Considering $f = g = \exp z$, $m = n = p = 1$, $A = 1$, $\alpha = 0$ and $L(r) = \frac{1}{l}\exp\left(\frac{1}{r}\right)$ for any positive real number l, one can easily verify that the condition $A > (1+\alpha)\cdot {}_{(p)}^{(n)}\rho_g^{L^*}$ is essential in Theorem 3.8, Theorem 3.9 and Theorem 3.10.

Theorem 3.11. *Let f be a transcendental meromorphic function having maximum deficiency sum and g be an entire function such that ${}_{(p)}^{(n)}\rho_f^{L^*} < \infty$ and ${}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} = \infty$ where m, n and p are any three positive integers. Then*

$$\lim_{r\to\infty}\frac{\log^{[m]} T(r, f\circ g)}{\log^{[n]} T(r, L(f))} = \infty.$$

Proof. Let us suppose that the conclusion of the theorem does not hold.Then we can find a constant $\beta > 0$ such that for a sequence of positive numbers of r tending to infinity

$$\log^{[m]} T(r, f\circ g) \le \beta \log^{[n]} T(r, L(f))\,. \tag{3.15}$$

Again from the definition of ${}_{(p)}^{(n)}\rho_{L(f)}^{L^*}$, it follows that for all sufficiently large positive numbers of r and in view of Lemma 2.4

$$\begin{aligned}\log^{[n]} T(r, L(f)) &\le \left({}_{(p)}^{(n)}\rho_{L(f)}^{L^*} + \varepsilon\right)\log\left[r\exp^{[p]} L(r)\right]\\ i.e.,\ \log^{[n]} T(r, L(f)) &\le \left({}_{(p)}^{(n)}\rho_{f}^{L^*} + \varepsilon\right)\log\left[r\exp^{[p]} L(r)\right]\,. \end{aligned}\tag{3.16}$$

Thus from (3.15) and (3.16) , we have for a sequence of positive numbers of r tending to infinity that

$$\begin{aligned}\log^{[m]} T(r, f\circ g) &\le \beta\left({}_{(p)}^{(n)}\rho_{f}^{L^*} + \varepsilon\right)\log\left[r\exp^{[p]} L(r)\right]\\ i.e.,\ \frac{\log^{[m]} T(r, f\circ g)}{\log\left[r\exp^{[p]} L(r)\right]} &\le \frac{\beta\left({}_{(p)}^{(n)}\rho_{f}^{L^*} + \varepsilon\right)\log\left[r\exp^{[p]} L(r)\right]}{\log\left[r\exp^{[p]} L(r)\right]}\\ i.e.,\ \liminf_{r\to\infty}\frac{\log^{[m]} T(r, f\circ g)}{\log\left[r\exp^{[p]} L(r)\right]} &= {}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} < \infty\,.\end{aligned}$$

This is a contradiction.

This proves the theorem. □

Remark 3.3. Theorem 3.11 is also valid with "limit superior" instead of "limit" if ${}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} = \infty$ is replaced by ${}_{(p)}^{(m)}\rho_{f\circ g}^{L^*} = \infty$ and the other conditions remaining the same.

Corollary 3.1. *Under the assumptions of Theorem 3.11 and Remark 3.3,*

$$\lim_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{\log^{[n-1]}T(r,L(f))}=\infty \ \textit{and} \ \limsup_{r\to\infty}\frac{\log^{[m-1]}T(r,f\circ g)}{\log^{[n-1]}T(r,L(f))}=\infty$$

respectively holds.

Proof. From Theorem 3.11 we obtain for all sufficiently large positive numbers of r and for $K>1$,

$$\begin{aligned}\log^{[m]}T(r,f\circ g) &> K\log^{[n]}T(r,L(f))\\ \textit{i.e.,}\ \log^{[m-1]}T(r,f\circ g) &> \left\{\log^{[n-1]}T(r,L(f))\right\}^{K},\end{aligned}$$

from which the first part of the corollary follows.

Similarly, using Remark 3.3, we obtain the second part of the corollary. □

Remark 3.4. The condition ${}^{(m)}_{(p)}\lambda^{L^*}_{f\circ g}=\infty$ in Theorem 3.11 and Corollary 3.1 is necessary which is evident from the following example.

Example 3.2. Let $f=\exp z$, $g=z$, $m=n=p=1$ and $L(r)=\frac{1}{l}\exp\left(\frac{1}{r}\right)$ where l is any positive real number. Then

$$\rho^{L^*}_{f}=1<\infty,\ \lambda^{L^*}_{f\circ g}=1<\infty \text{ and } \sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2.$$

Now taking $a_1=1$ and $a_2=...=a_k=0$, we obtain that

$$L(f)=\begin{vmatrix} a_1 & f\\ a_1' & f'\end{vmatrix}=\begin{vmatrix} 1 & \exp z\\ 0 & \exp z\end{vmatrix}=\exp z.$$

Now

$$T(r,f\circ g)=T(r,L(f))=\frac{r}{\pi}.$$

Hence

$$\begin{aligned}\lim_{r\to\infty}\frac{\log T(r,f\circ g)}{\log T(r,L(f))} &= \lim_{r\to\infty}\frac{\log r+O(1)}{\log r+O(1)}=1\\ \text{and }\lim_{r\to\infty}\frac{T(r,f\circ g)}{T(r,L(f))} &= \lim_{r\to\infty}\frac{\left(\frac{r}{\pi}\right)}{\left(\frac{r}{\pi}\right)}=1,\end{aligned}$$

which is a contradiction.

Remark 3.5. Choosing $f=\exp z$, $g=z$, $m=n=p=1$, $L(r)=\frac{1}{l}\exp\left(\frac{1}{r}\right)$ for any positive real number l and taking $a_1=1$ and $a_2=...=a_k=0$ in Definition 1.2, one can easily verify that the condition ${}^{(m)}_{(p)}\rho^{L^*}_{f\circ g}=\infty$ in Remark 3.3 and Corollary 3.1 is essential.

Theorem 3.12. *Let f be a meromorphic function and g be transcendental entire having the maximum deficiency sum with ${}_{(p)}^{(n)}\rho_g^{L^*} < \infty$ and ${}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} = \infty$ where m, n and p are any three positive integers. Then*

$$\lim_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{\log^{[n]} T(r, L(g))} = \infty.$$

We omit the proof of Theorem 3.12 because it can be carried out in the line of Theorem 3.11.

Remark 3.6. Theorem 3.12 is also valid with "limit superior" instead of "limit" if ${}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} = \infty$ is replaced by ${}_{(p)}^{(m)}\rho_{f\circ g}^{L^*} = \infty$ and the other conditions remaining the same.

In the line of Corollary 3.1, one can easily verify the following corollary:

Corollary 3.2. *Under the assumptions of Theorem 3.12 and Remark 3.6,*

$$\lim_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{\log^{[n-1]} T(r, L(g))} = \infty \quad \text{and} \quad \limsup_{r\to\infty} \frac{\log^{[m-1]} T(r, f\circ g)}{\log^{[n-1]} T(r, L(g))} = \infty$$

respectively hold.

Remark 3.7. Considering $f = \exp z$, $g = z$, $m = n = p = 1$, $L(r) = \frac{1}{l}\exp\left(\frac{1}{r}\right)$ for any positive real number l and taking $a_1 = 1$ and $a_2 = ... = a_k = 0$ in Definition 1.2, we may establish the necessity of the conditions ${}_{(p)}^{(m)}\lambda_{f\circ g}^{L^*} = \infty$ and ${}_{(p)}^{(m)}\rho_{f\circ g}^{L^*} = \infty$ respectively, in Theorem 3.12 Remark 3.6 and Corollary 3.2.

Theorem 3.13. *Let f be meromorphic and g be transcendental entire such that ${}_{(p)}^{(m)}\rho_f^{L^*} < \infty$, $0 < {}_{(p)}^{(n)}\lambda_g^{L^*} \le {}_{(p)}^{(n)}\rho_g^{L^*} < \infty$ and $\sum_{a\neq\infty} \delta(a; g) + \delta(\infty; g) = 2$ where m, n and p are any three positive integers. Then*
(a) if $\exp^{[p-1]} L(M(r, g)) = o\left\{\log^{[n]} T(r, L(g))\right\}$ then

$$\limsup_{r\to\infty} \frac{\log^{[m+n]} T(r, f\circ g)}{\log^{[n]} T(r, L(g)) + \exp^{[p-1]} L(M(r, g))} \le \frac{{}_{(p)}^{(n)}\rho_g^{L^*}}{{}_{(p)}^{(n)}\lambda_g^{L^*}}$$

and (b) if $\log^{[n]} T(r, L(g)) = o\left\{\exp^{[p-1]} L(M(r, g))\right\}$ then

$$\lim_{r\to\infty} \frac{\log^{[m+n]} T(r, f\circ g)}{\log^{[n]} T(r, L(g)) + \exp^{[p-1]} L(M(r, g))} = 0\,.$$

Proof. Using $\log\left[1+\frac{\exp^{[p-1]}L(M(r,g))+O(1)}{\log M(r,g)}\right]\sim\frac{\exp^{[p-1]}L(M(r,g))+O(1)}{\log M(r,g)}$, we obtain from (3.1) for all sufficiently large positive numbers of r that

$$\log^{[m]}T(r,f\circ g)\leq\left({}_{(p)}^{(m)}\rho_f^{L^*}+\varepsilon\right)\log M(r,g)\left[1+\frac{\exp^{[p-1]}L(M(r,g))+O(1)}{\log M(r,g)}\right]$$

i.e.,

$$\log^{[m+1]}T(r,f\circ g)\leq\log\left({}_{(p)}^{(m)}\rho_f^{L^*}+\varepsilon\right)+\log^{[2]}M(r,g)$$
$$+\log\left[1+\frac{\exp^{[p-1]}L(M(r,g))+O(1)}{\log M(r,g)}\right]$$

i.e.,

$$\log^{[m+1]}T(r,f\circ g)\leq\log\left({}_{(p)}^{(m)}\rho_f^{L^*}+\varepsilon\right)+\log^{[2]}M(r,g)$$
$$+\frac{\exp^{[p-1]}L(M(r,g))+O(1)}{\log M(r,g)}$$

i.e.,

$$\log^{[m+1]}T(r,f\circ g)\leq$$
$$\log^{[2]}M(r,g)\left[1+\frac{\exp^{[p-1]}L(M(r,g))+O(1)+\log M(r,g)\cdot\log\left({}_{(p)}^{(m)}\rho_f^{L^*}+\varepsilon\right)}{\log M(r,g)\cdot\log^{[2]}M(r,g)}\right].$$

Further using $\log(1+x)\sim x$ for $x=\frac{\exp^{[p-1]}L(M(r,g))+O(1)+\log M(r,g)\cdot\log\left({}_{(p)}^{(m)}\rho_f^{L^*}+\varepsilon\right)}{\prod_{k=1}^{n}\log^{[k]}M(r,g)}$ we get from the above for all sufficiently large positive numbers of r that

$$\log^{[m+n]}T(r,f\circ g)\leq\log^{[n+1]}M(r,g)$$
$$+\log\left[1+\frac{\exp^{[p-1]}L(M(r,g))+O(1)+\log M(r,g)\cdot\log\left({}_{(p)}^{(m)}\rho_f^{L^*}+\varepsilon\right)}{\prod_{k=1}^{n}\log^{[k]}M(r,g)}\right].$$

i.e.,

$$\log^{[m+n]}T(r,f\circ g)\leq\left({}_{(p)}^{(n)}\rho_g^{L^*}+\varepsilon\right)\log\left[r\exp^{[p]}L(r)\right]$$

$$+\frac{\exp^{[p-1]}L(M(r,g))+O(1)+\log M(r,g)\cdot\log\left({}_{(p)}^{(m)}\rho_f^{L^*}+\varepsilon\right)}{\prod_{k=1}^{n}\log^{[k]}M(r,g)}.\tag{3.17}$$

Again in view of Lemma 2.4, we get from the definition of L^*−lower order for all sufficiently large positive numbers of r that

$$\log^{[n]} T(r, L(g)) \geq \left({}^{(n)}_{(p)}\lambda^{L^*}_{L(g)} - \varepsilon\right) \log\left[r \exp^{[p]} L(r)\right]$$

$$i.e., \;\; \log^{[n]} T(r, L(g)) \geq \left({}^{(n)}_{(p)}\lambda^{L^*}_{g} - \varepsilon\right) \log\left[r \exp^{[p]} L(r)\right]$$

$$i.e., \;\; \log^{[n]} T(r, L(g)) \geq \left({}^{(n)}_{(p)}\lambda^{L^*}_{g} - \varepsilon\right) \log\left[r \exp^{[p]} L(r)\right]$$

$$i.e., \;\; \log\left[r \exp^{[p]} L(r)\right] \leq \frac{\log^{[n]} T(r, L(g))}{\left({}^{(n)}_{(p)}\lambda^{L^*}_{g} - \varepsilon\right)} .$$

Hence from (3.17) and (3.), it follows for all sufficiently large positive numbers of r that

$$\log^{[m+n]} T(r, f \circ g) \leq \left(\frac{{}^{(n)}_{(p)}\rho^{L^*}_{g} + \varepsilon}{{}^{(n)}_{(p)}\lambda^{L^*}_{g} - \varepsilon}\right) \cdot \log^{[n]} T(r, L(g))$$
$$+ \frac{\exp^{[p-1]} L(M(r, g)) + O(1) + \log M(r, g) \cdot \log\left({}^{(m)}_{(p)}\rho^{L^*}_{f} + \varepsilon\right)}{\prod_{k=1}^{n} \log^{[k]} M(r, g)}$$

$$i.e, \;\; \frac{\log^{[m+n]} T(r, f \circ g)}{\log^{[n]} T(r, L(g)) + \exp^{[p-1]} L(M(r, g))}$$
$$\leq \left(\frac{{}^{(n)}_{(p)}\rho^{L^*}_{g} + \varepsilon}{{}^{(n)}_{(p)}\lambda^{L^*}_{g} - \varepsilon}\right) \cdot \frac{\log^{[n]} T(r, L(g))}{\log^{[n]} T(r, L(g)) + \exp^{[p-1]} L(M(r, g))}$$
$$+ \frac{\exp^{[p-1]} L(M(r, g)) + O(1) + \log M(r, g) \cdot \log\left({}^{(m)}_{(p)}\rho^{L^*}_{f} + \varepsilon\right)}{\left(\log^{[n]} T(r, L(g)) + \exp^{[p-1]} L(M(r, g))\right) \cdot \prod_{k=1}^{n} \log^{[k]} M(r, g)}$$

$$i.e, \;\; \frac{\log^{[m+n]} T(r, f \circ g)}{\log^{[n]} T(r, L(g)) + \exp^{[p-1]} L(M(r, g))} \leq \frac{\left(\frac{{}^{(n)}_{(p)}\rho^{L^*}_{g} + \varepsilon}{{}^{(n)}_{(p)}\lambda^{L^*}_{g} - \varepsilon}\right)}{1 + \frac{\exp^{[p-1]} L(M(r,g))}{\log^{[n]} T(r, L(g))}}$$

$$+ \frac{1 + \frac{O(1) + \log M(r,g) \cdot \log\left({}^{(m)}_{(p)}\rho^{L^*}_{f} + \varepsilon\right)}{\exp^{[p-1]} L(M(r,g))}}{\left[1 + \frac{\log^{[n]} T(r,L(g))}{\exp^{[p-1]} L(M(r,g))}\right] \cdot \prod_{k=1}^{n} \log^{[k]} M(r, g)} . \tag{3.18}$$

Since $\exp^{[p-1]} L(M(r,g)) = o\left\{\log^{[n]} T(r, L(g))\right\}$ as $r \to \infty$ and $\varepsilon\,(>0)$, is arbitrary we obtain from (3.18) that

$$\limsup_{r\to\infty} \frac{\log^{[m+n]} T(r, f\circ g)}{\log^{[n]} T(r, L(g)) + \exp^{[p-1]} L(M(r,g))} \le \frac{{}^{(n)}_{(p)}\rho_g^{L^*}}{{}^{(n)}_{(p)}\lambda_g^{L^*}}. \tag{3.19}$$

Again if $\log^{[n]} T(r, L(g)) = o\left\{\exp^{[p-1]} L(M(r,g))\right\}$ then from (3.18) we get that

$$\lim_{r\to\infty} \frac{\log^{[m+n]} T(r, f\circ g)}{\log^{[n]} T(r, L(g)) + \exp^{[p-1]} L(M(r,g))} = 0. \tag{3.20}$$

Thus from (3.19) and (3.20) the theorem is established. □

In the line of Theorem 3.13 the following theorem may be proved and therefore its proof is omitted:

Theorem 3.14. *Let f be meromorphic with ${}^{(m)}_{(p)}\rho_f^{L^*} < \infty$ and g be transcendental entire such that either $0 < {}^{(n)}_{(p)}\rho_g^{L^*} < \infty$ or $0 < {}^{(n)}_{(p)}\lambda_g^{L^*} < \infty$ holds where m, n and p are any three positive integers. Further let $\sum_{a\neq\infty} \delta(a; g) + \delta(\infty; g) = 2$. Then*
(a) if $\exp^{[p-1]} L(M(r,g)) = o\left\{\log^{[n]} T(r, L(g))\right\}$ then

$$\liminf_{r\to\infty} \frac{\log^{[m+n]} T(r, f\circ g)}{\log^{[n]} T(r, L(g)) + \exp^{[p-1]} L(M(r,g))} \le 1$$

and (b) if $\log^{[n]} T(r, L(g)) = o\left\{\exp^{[p-1]} L(M(r,g))\right\}$ then

$$\liminf_{r\to\infty} \frac{\log^{[m+n]} T(r, f\circ g)}{\log^{[n]} T(r, L(g)) + \exp^{[p-1]} L(M(r,g))} = 0.$$

Remark 3.8. The equality sign in Theorem 3.13 and Theorem 3.14 cannot be removed as we see in the following example:

Example 3.3. Let $f = g = \exp z$, $m = n = p = 1$ and $L(r) = \frac{1}{l}\exp\left(\frac{1}{r}\right)$ where l is any positive real number.Then

$$\lambda_f^{L^*} = \rho_f^{L^*} = \lambda_g^{L^*} = \rho_g^{L^*} = 1,\ f\circ g = \exp^{[2]} z \text{ and } \sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2.$$

Also choosing $a_1 = 1$ and $a_2 = \ldots = a_k = 0$, it follows that

$$L(f) = \begin{vmatrix} a_1 & f \\ a_1' & f' \end{vmatrix} = \begin{vmatrix} 1 & \exp z \\ 0 & \exp z \end{vmatrix} = \exp z.$$

Now

$$T(r, f \circ g) \sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}} \ (r \to \infty), \ T(r, g) = \frac{r}{\pi}$$
$$\text{and } M(r, g) = \exp r.$$

So

$$L(M(r, g)) = L(\exp r) = \frac{1}{l} \exp\left(\frac{1}{\exp r}\right).$$

Hence

$$\begin{aligned} & \liminf_{r\to\infty} \frac{\log^{[2]} T(r, f \circ g)}{\log T(r, L(g)) + L(M(r, g))} \\ = & \limsup_{r\to\infty} \frac{\log^{[2]} T(r, f \circ g)}{\log T(r, L(g)) + L(M(r, g))} \\ = & \limsup_{r\to\infty} \frac{\log\left[r - \frac{1}{2}\log r + O(1)\right]}{\log r + O(1) + \frac{1}{l}\exp\left(\frac{1}{\exp r}\right)} = 1. \end{aligned}$$

Now we state the following three theorems without their proofs as those can be carried out in the line of Theorem 3.13 and Theorem 3.14:

Theorem 3.15. *Let f be transcendental meromorphic with the maximum deficiency sum and g be entire with $0 < {}^{(m)}_{(p)}\lambda_f^{L^*} \leq {}^{(m)}_{(p)}\rho_f^{L^*} < \infty$ and ${}^{(n)}_{(p)}\rho_g^{L^*} < \infty$ where m, n and p are any three positive integers. Then*
(a) if $\exp^{[p-1]} L(M(r, g)) = o\left\{\log^{[m]} T(r, L(f))\right\}$ then

$$\limsup_{r\to\infty} \frac{\log^{[m+n]} T(r, f \circ g)}{\log^{[m]} T(r, L(f)) + \exp^{[p-1]} L(M(r, g))} \leq \frac{{}^{(n)}_{(p)}\rho_g^{L^*}}{{}^{(m)}_{(p)}\lambda_f^{L^*}}$$

and (b) if $\log^{[m]} T(r, L(f)) = o\left\{\exp^{[p-1]} L(M(r, g))\right\}$ then

$$\lim_{r\to\infty} \frac{\log^{[m+n]} T(r, f \circ g)}{\log^{[m]} T(r, L(f)) + \exp^{[p-1]} L(M(r, g))} = 0.$$

Theorem 3.16. *Let f be transcendental meromorphic with the maximum deficiency sum and g be entire such that $0 < {}^{(m)}_{(p)}\rho_f^{L^*} < \infty$ and ${}^{(n)}_{(p)}\rho_g^{L^*} < \infty$ where m, n and p are any three positive integers. Then*
(a) if $\exp^{[p-1]} L(M(r, g)) = o\left\{\log^{[m]} T(r, L(f))\right\}$ then

$$\liminf_{r\to\infty} \frac{\log^{[m+n]} T(r, f \circ g)}{\log^{[m]} T(r, L(f)) + \exp^{[p-1]} L(M(r, g))} \leq \frac{{}^{(n)}_{(p)}\rho_g^{L^*}}{{}^{(m)}_{(p)}\rho_f^{L^*}}$$

and (b) *if* $\log^{[m]} T(r, L(f)) = o\left\{\exp^{[p-1]} L(M(r,g))\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[m+n]} T(r, f\circ g)}{\log^{[m]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))} = 0\,.$$

Theorem 3.17. *Let f be transcendental meromorphic with the maximum deficiency sum and g be entire with $0 < {}^{(m)}_{(p)}\lambda_f^{L^*} \le {}^{(m)}_{(p)}\rho_f^{L^*} < \infty$ and ${}^{(n)}_{(p)}\lambda_g^{L^*} < \infty$ where m, n and p are any three positive integers. Then*
(a) *if* $\exp^{[p-1]} L(M(r,g)) = o\left\{\log^{[m]} T(r, L(f))\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[m+n]} T(r, f\circ g)}{\log^{[m]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))} \le \frac{{}^{(n)}_{(p)}\lambda_g^{L^*}}{{}^{(m)}_{(p)}\lambda_f^{L^*}}$$

and (b) *if* $\log^{[m]} T(r, L(f)) = o\left\{\exp^{[p-1]} L(M(r,g))\right\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[m+n]} T(r, f\circ g)}{\log^{[m]} T(r, L(f)) + \exp^{[p-1]} L(M(r,g))} = 0\,.$$

Remark 3.9. Taking $f = g = \exp z$, $m = n = p = 1$, $L(r) = \frac{1}{l}\exp\left(\frac{1}{r}\right)$ for any positive real number l and taking $a_1 = 1$ and $a_2 = \dots = a_k = 0$ in Definition 1.2, one can easily verify that the equality sign in Theorem 3.15, Theorem 3.16 and Theorem 3.17 cannot be removed

Theorem 3.18. *Let f be transcendental entire and g be an entire function such that $0 < {}^{(m)}_{(p)}\lambda_f^{L^*} \le {}^{(m)}_{(p)}\rho_f^{L^*} < \infty$, ${}^{(n)}_{(p)}\rho_g^{L^*} > 0$ and $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$ where m, n and p are any three positive integers. Then*

$$\limsup_{r\to\infty} \frac{\log^{[m+n]} T(r, f\circ g)}{\log^{[m]} T(r, L(f)) + \exp^{[p-1]} L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right)} \ge \frac{{}^{(n)}_{(p)}\rho_g^{L^*}}{{}^{(m)}_{(p)}\rho_f^{L^*}}\,.$$

Proof. Now from (3.) , we have for all sufficiently large positive numbers of r that

$$\log^{[m]} T(r, f\circ g) \ge o(1) + \left({}^{(m)}_{(p)}\lambda_f^{L^*} - \varepsilon\right)\left[\log\left\{\frac{1}{8}M\left(\frac{r}{4},g\right) + o(1)\right\} + \exp^{[p-1]} L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right)\right]$$

i.e.,

$$\log^{[m]} T(r, f\circ g) \ge o(1) + \left({}^{(m)}_{(p)}\lambda_f^{L^*} - \varepsilon\right) \left[\log\left\{\frac{1}{8}M\left(\frac{r}{4},g\right)\left(1 + \frac{o(1)}{\frac{1}{8}M\left(\frac{r}{4},g\right)}\right)\right\} + \exp^{[p-1]} L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right)\right]$$

i.e.,

$$\log^{[m]} T(r, f \circ g) \geq \left({}^{(m)}_{(p)}\lambda_f^{L^*} - \varepsilon\right) \log M\left(\frac{r}{4}, g\right) \cdot \left\{\frac{\log M\left(\frac{r}{4}, g\right) + \log\left(1 + \frac{o(1)}{\frac{1}{8}M\left(\frac{r}{4}, g\right)}\right) + \exp^{[p-1]} L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right)}{\log M\left(\frac{r}{4}, g\right)}\right\}$$

i.e.,

$$\log^{[m+1]} T(r, f \circ g) \geq \log^{[2]} M\left(\frac{r}{4}, g\right) + \log\left\{\frac{\log M\left(\frac{r}{4}, g\right) + \exp^{[p-1]} L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right) + o(1)}{\exp^{[p-1]} L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right) \cdot \log M\left(\frac{r}{4}, g\right)}\right\}$$

i.e.,

$$\log^{[m+1]} T(r, f \circ g) \geq \log^{[2]} M\left(\frac{r}{4}, g\right) \cdot \left\{\frac{\log^{[2]} M\left(\frac{r}{4}, g\right) + \log\left\{\frac{\log M\left(\frac{r}{4}, g\right) + \exp^{[p-1]} L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right) + o(1)}{\exp^{[p-1]} L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right) \cdot \log M\left(\frac{r}{4}, g\right)}\right\}}{\log^{[2]} M\left(\frac{r}{4}, g\right)}\right\}$$

i.e.,

$$\log^{[m+2]} T(r, f \circ g) \geq \log^{[3]} M\left(\frac{r}{4}, g\right) + \log\left\{\frac{\log^{[2]} M\left(\frac{r}{4}, g\right) + \log\left\{\frac{\log M\left(\frac{r}{4}, g\right) + \exp^{[p-1]} L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right) + o(1)}{\exp^{[p-1]} L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right) \cdot \log M\left(\frac{r}{4}, g\right)}\right\}}{\exp^{[p-1]} L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right) \cdot \log^{[2]} M\left(\frac{r}{4}, g\right)}\right\}.$$

....

i.e.,
$$\log^{[m+n]} T(r, f \circ g) \geq \log^{[n+1]} M\left(\frac{r}{4}, g\right) + \left(\frac{{}^{(n)}_{(p)}\rho_g^{L^*} - \varepsilon}{{}^{(m)}_{(p)}\rho_f^{L^*} + \varepsilon}\right) \exp^{[p-1]} L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right) - \log\left[\exp\left\{\left(\frac{{}^{(n)}_{(p)}\rho_g^{L^*} - \varepsilon}{{}^{(m)}_{(p)}\rho_f^{L^*} + \varepsilon}\right) \cdot \exp^{[p-1]} L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right)\right\}\right]$$

$$+\log\left\{\frac{\log^{[n]} M\left(\frac{r}{4},g\right)+\log\left\{\frac{\log^{[n-1]} M\left(\frac{r}{4},g\right)+\log\left\{\ldots\log\left\{\frac{\log M\left(\frac{r}{4},g\right)+\exp^{[p-1]} L\left(\frac{1}{8} M\left(\frac{r}{4},g\right)\right)+o(1)}{\exp^{[p-1]} L\left(\frac{1}{8} M\left(\frac{r}{4},g\right)\right)\cdot\log M\left(\frac{r}{4},g\right)}\right\}\right\}}{\exp^{[p-1]} L\left(\frac{1}{8} M\left(\frac{r}{4},g\right)\right)\cdot\log^{[n-1]} M\left(\frac{r}{4},g\right)}\right\}}{\log^{[n]} M\left(\frac{r}{4},g\right)}\right\}$$

$$\textit{i.e.,}\quad \log^{[m+n]} T(r, f\circ g) \geq \log^{[n+1]} M\left(\frac{r}{4},g\right)+$$

$$\left(\frac{{}^{(n)}_{(p)}\rho_g^{L^*}-\varepsilon}{{}^{(m)}_{(p)}\rho_f^{L^*}+\varepsilon}\right)\exp^{[p-1]} L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right)$$

$$+\log\left\{\frac{\log^{[n]} M\left(\frac{r}{4},g\right)+\log\left\{\frac{\log^{[n-1]} M\left(\frac{r}{4},g\right)+\log\left\{\ldots\ldots\log\left\{\frac{\log M\left(\frac{r}{4},g\right)+\exp^{[p-1]} L\left(\frac{1}{8} M\left(\frac{r}{4},g\right)\right)+o(1)}{\exp^{[p-1]} L\left(\frac{1}{8} M\left(\frac{r}{4},g\right)\right)\cdot\log M\left(\frac{r}{4},g\right)}\right\}\right\}}{\exp^{[p-1]} L\left(\frac{1}{8} M\left(\frac{r}{4},g\right)\right)\cdot\log^{[n-1]} M\left(\frac{r}{4},g\right)}\right\}}{\exp\left\{\left(\frac{{}^{(n)}_{(p)}\rho_g^{L^*}-\varepsilon}{{}^{(m)}_{(p)}\rho_f^{L^*}+\varepsilon}\right)\cdot\exp^{[p-1]} L\left(\frac{1}{8} M\left(\frac{r}{4},g\right)\right)\right\}\cdot\log^{[n]} M\left(\frac{r}{4},g\right)}\right\}$$

and later

$$\log^{[m+n]} T(r, f\circ g) \geq \log^{[n+1]} M\left(\frac{r}{4},g\right)+$$

$$\left(\frac{{}^{(n)}_{(p)}\rho_g^{L^*}-\varepsilon}{{}^{(m)}_{(p)}\rho_f^{L^*}+\varepsilon}\right)\exp^{[p-1]} L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right).$$

Now from the above it follows for a sequence of positive numbers of r tending to infinity that

$$\log^{[m+n]} T(r, f\circ g) \geq \left({}^{(n)}_{(p)}\rho_g^{L^*}-\varepsilon\right)\log\left[\frac{r}{4}\exp^{[p]} L\left(\frac{r}{4}\right)\right]$$
$$+\left(\frac{{}^{(n)}_{(p)}\rho_g^{L^*}-\varepsilon}{{}^{(m)}_{(p)}\rho_f^{L^*}+\varepsilon}\right)\exp^{[p-1]} L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right).$$

In view of Lemma 2.4, we get for all sufficiently large positive numbers of r that

$$\log^{[m]} T(r, L(f)) \leq \left({}^{(m)}_{(p)}\rho_{L(f)}^{L^*}+\varepsilon\right)\log\left[r\exp^{[p]} L(r)\right]$$

which further implies

$$\begin{aligned}&\log^{[m]} T(r, L(f))\\ \leq\ &\left({}^{(m)}_{(p)}\rho_f^{L^*}+\varepsilon\right)\log\left[\frac{r}{4}\exp^{[p]} L\left(\frac{r}{4}\right)\right]+\log 4.\end{aligned}\qquad(3.21)$$

Hence from (3.21) and (3.21), it follows for a sequence of positive numbers of r tending to infinity that

$$i.e., \ \log^{[m+n]} T(r, f \circ g) \geq \left(\frac{{}^{(n)}_{(p)}\rho_g^{L^*} - \varepsilon}{{}^{(m)}_{(p)}\rho_f^{L^*} + \varepsilon} \right) \left(\log^{[m]} T(r, L(f)) - \log 4 \right) + \left(\frac{{}^{(n)}_{(p)}\rho_g^{L^*} - \varepsilon}{{}^{(m)}_{(p)}\rho_f^{L^*} + \varepsilon} \right) \exp^{[p-1]} L\left(\frac{1}{8} M\left(\frac{r}{4}, g \right) \right)$$

$$i.e., \ \log^{[m+n]} T(r, f \circ g) \geq \left(\frac{{}^{(n)}_{(p)}\rho_g^{L^*} - \varepsilon}{{}^{(m)}_{(p)}\rho_f^{L^*} + \varepsilon} \right) \left[\log^{[m]} T(r, L(f)) + \exp^{[p-1]} L\left(\frac{1}{8} M\left(\frac{r}{4}, g \right) \right) \right] - \left(\frac{{}^{(n)}_{(p)}\rho_g^{L^*} - \varepsilon}{{}^{(m)}_{(p)}\rho_f^{L^*} + \varepsilon} \right) \log 4$$

Finally,

$$\frac{\log^{[m+n]} T(r, f \circ g)}{\log^{[m]} T(r, L(f)) + \exp^{[p-1]} L\left(\frac{1}{8} M\left(\frac{r}{4}, g \right) \right)} \geq \left(\frac{{}^{(n)}_{(p)}\rho_g^{L^*} - \varepsilon}{{}^{(m)}_{(p)}\rho_f^{L^*} + \varepsilon} \right) - \frac{\left(\frac{{}^{(n)}_{(p)}\rho_g^{L^*} - \varepsilon}{{}^{(m)}_{(p)}\rho_f^{L^*} + \varepsilon} \right) \log 4}{\log^{[m]} T(r, L(f)) + \exp^{[p-1]} L\left(\frac{1}{8} M\left(\frac{r}{4}, g \right) \right)}.$$

Since $\varepsilon\ (> 0)$ is arbitrary, it follows from above that

$$\limsup_{r \to \infty} \frac{\log^{[m+n]} T(r, f \circ g)}{\log^{[m]} T(r, L(f)) + \exp^{[p-1]} L\left(\frac{1}{8} M\left(\frac{r}{4}, g \right) \right)} \geq \frac{{}^{(n)}_{(p)}\rho_g^{L^*}}{{}^{(m)}_{(p)}\rho_f^{L^*}}.$$

This proves the theorem. □

In the line of Theorem 3.18, the following two theorems may be proved and therefore their proofs are omitted:

Theorem 3.19. *Let f be transcendental entire and g be an entire function with* $0 < {}^{(m)}_{(p)}\lambda_f^{L^*} < \infty$, ${}^{(n)}_{(p)}\lambda_g^{L^*} > 0$ *and* $\sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2$ *where m, n and p are any three positive* *integers. Then*

$$\limsup_{r\to\infty} \frac{\log^{[m+n]} T(r, f\circ g)}{\log^{[m]} T(r, L(f)) + \exp^{[p-1]} L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right)} \geq \frac{{}^{(n)}_{(p)}\lambda_g^{L^*}}{{}^{(m)}_{(p)}\lambda_f^{L^*}} .$$

Theorem 3.20. *Let f be transcendental entire having the maximum deficiency sum and g be an entire function such that $0 < {}^{(m)}_{(p)}\lambda_f^{L^*} \leq {}^{(m)}_{(p)}\rho_f^{L^*} < \infty$ and ${}^{(n)}_{(p)}\lambda_g^{L^*} > 0$ where m, n and p are any three positive integers. Then*

$$\liminf_{r\to\infty} \frac{\log^{[m+n]} T(r, f\circ g)}{\log^{[m]} T(r, L(f)) + \exp^{[p-1]} L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right)} \geq \frac{{}^{(n)}_{(p)}\lambda_g^{L^*}}{{}^{(m)}_{(p)}\rho_f^{L^*}} .$$

Now we state the following two theorems without their proofs as those can be carried out in the line of Theorem 3.18 and Theorem 3.20:

Theorem 3.21. *Let f be an entire function with ${}^{(m)}_{(p)}\lambda_f^{L^*} > 0$ and g be transcendental entire such that either $0 < {}^{(n)}_{(p)}\lambda_g^{L^*} < \infty$ or $0 < {}^{(n)}_{(p)}\rho_g^{L^*} < \infty$ holds where m, n and p are any three positive integers. Further let $\sum_{a\neq\infty} \delta(a; g) + \delta(\infty; g) = 2$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[m+n]} T(r, f\circ g)}{\log^{[m]} T(r, L(g)) + \exp^{[p-1]} L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right)} \geq 1 .$$

Theorem 3.22. *Let f be an entire and g be a transcendental entire function with ${}^{(m)}_{(p)}\lambda_f^{L^*} > 0$, $0 < {}^{(n)}_{(p)}\lambda_g^{L^*} \leq {}^{(n)}_{(p)}\rho_g^{L^*} < \infty$ and $\sum_{a\neq\infty} \delta(a; g) + \delta(\infty; g) = 2$ where m, n and p are any three positive integers. Then*

$$\liminf_{r\to\infty} \frac{\log^{[m+n]} T(r, f\circ g)}{\log^{[m]} T(r, L(g)) + \exp^{[p-1]} L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right)} \geq \frac{{}^{(n)}_{(p)}\lambda_g^{L^*}}{{}^{(n)}_{(p)}\rho_g^{L^*}} .$$

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