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$(\psi,\gamma,2)$-CHEREDNIK-OPDAM LIPSCHITZ FUNCTIONS IN THE SPACE $L^2_{\alpha,\beta}(\mathbb{R})$

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Abstract. In this paper, using a generalized translation operator, we obtain an analog of Younis Theorem 5.2 in [3] for the Cherednik-Opdam transform for functions satisfying the $(\psi,\gamma,2)$-Cherednik-Opdam Lipschitz condition in the space $L^2_{\alpha,\beta}(\mathbb{R})$.

Keywords: Cherednik-Opdam operator; Cherednik-Opdam transform; generalized translation.

1. Introduction and Preliminaries

Various investigators such as V.N. Mishra and L.N. Mishra [7], Mishra and al. [5, 6] have determined the degree of approximation of 2π-periodic signals (functions) belonging to various classes $Lip\alpha$, $Lip(\alpha,r)$, $Lip(\xi(t),r)$ and $W(L_r,\xi(t))$, $(r\geq 1)$, of functions through trigonometric Fourier approximation using different summability matrices with monotone rows. In this direction, Younis Theorem 5.2 [3] characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms, namely we have

Theorem 1.1. *[3] Let $f\in L^2(\mathbb{R})$. Then the following are equivalents*

(i) $\|f(x+h)-f(x)\| = O\left(\frac{h^\delta}{(\log\frac{1}{h})^\gamma}\right)$, *as* $h\to 0, 0<\delta<1, \gamma\geq 0,$

(ii) $\displaystyle\int_{|\lambda|\geq r}|\widehat{f}(\lambda)|^2 d\lambda = O\left(\frac{r^{-2\delta}}{(\log r)^{2\gamma}}\right)$, *as* $r\to\infty,$

where $\widehat{f}$ stands for the Fourier transform of f.

In this paper, we prove the generalization of Theorem 1.1 for the Cherednik-Opdam transform for functions satisfying the $(\psi,\gamma,2)$-Cherednik-Opdam Lipschitz condition in the space $L^2_{\alpha,\beta}(\mathbb{R})$. For this purpose, we use the generalized translation operator. We point out that similar results have been established in the Jacobi

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transform [8].

In this section, we develop some results from harmonic analysis related to the differential-difference operator $T^{(\alpha,\beta)}$. Further details can be found in [1] and [2]. In the following we fix parameters α, β subject to the constraints $\alpha \geq \beta \geq -\frac{1}{2}$ and $\alpha > \frac{-1}{2}$.

Let $\rho = \alpha+\beta+1$ and $\lambda \in \mathbb{C}$. The Opdam hypergeometric functions $G_\lambda^{(\alpha,\beta)}$ on $\mathbb{R}$ are eigenfunctions $T^{(\alpha,\beta)}G_\lambda^{(\alpha,\beta)}(x) = i\lambda G_\lambda^{(\alpha,\beta)}(x)$ of the differential-difference operator

$$T^{(\alpha,\beta)}f(x) = f'(x) + [(2\alpha+1)\coth x + (2\beta+1)\tanh x]\frac{f(x)-f(-x)}{2} - \rho f(-x),$$

that are normalized such that $G_\lambda^{(\alpha,\beta)}(0) = 1$. In the notation of Cherednik one would write $T^{(\alpha,\beta)}$ as

$$T(k_1+k_2)f(x) = f'(x) + \left\{\frac{2k_1}{1+e^{-2x}} + \frac{4k_2}{1-e^{-4x}}\right\}(f(x)-f(-x)) - (k_1+2k_2)f(x),$$

with $\alpha = k_1+k_2-\frac{1}{2}$ and $\beta = k_2-\frac{1}{2}$. Here k_1 is the multiplicity of a simply positive root and k_2 the (possibly vanishing) multiplicity of a multiple of this root. By [1] or [2], the eigenfunction $G_\lambda^{(\alpha,\beta)}$ is given by

$$G_\lambda^{(\alpha,\beta)}(x) = \varphi_\lambda^{\alpha,\beta}(x) - \frac{1}{\rho - i\lambda}\frac{\partial}{\partial x}\varphi_\lambda^{\alpha,\beta}(x) = \varphi_\lambda^{\alpha,\beta}(x) + \frac{\rho}{4(\alpha+1)}\sinh(2x)\varphi_\lambda^{\alpha+1,\beta+1}(x),$$

where $\varphi_\lambda^{\alpha,\beta}(x) = {}_2F_1(\frac{\rho+i\lambda}{2};\frac{\rho-i\lambda}{2};\alpha+1;-\sinh^2 x)$ is the classical Jacobi function.

Lemma 1.1. [4] *The following inequalities are valids for Jacobi functions* $\varphi_\lambda^{\alpha,\beta}(x)$
(i) $|\varphi_\lambda^{\alpha,\beta}(x)| \leq 1$.
(ii) $1-\varphi_\lambda^{\alpha,\beta}(x) \leq x^2(\lambda^2+\rho^2)$.
(iii) there is a constant $c > 0$ *such that*

$$1-\varphi_\lambda^{\alpha,\beta}(x) \geq c,$$

for $\lambda x \geq 1$.

Denote $L^2_{\alpha,\beta}(\mathbb{R})$, the space of measurable functions f on $\mathbb{R}$ such that

$$\|f\|_{2,\alpha,\beta} = \left(\int_{\mathbb{R}} |f(x)|^2 A_{\alpha,\beta}(x)dx\right)^{1/2} < +\infty,$$

where

$$A_{\alpha,\beta}(x) = (\sinh|x|)^{2\alpha+1}(\cosh|x|)^{2\beta+1}.$$

The Cherednik-Opdam transform of $f \in C_c(\mathbb{R})$ is defined by

$$\mathcal{H}f(\lambda) = \int_{\mathbb{R}} f(x)G_\lambda^{(\alpha,\beta)}(-x)A_{\alpha,\beta}(x)dx \quad \text{for all} \quad \lambda \in \mathbb{C}.$$

The inverse transform is given as

$$\mathcal{H}^{-1}g(x) = \int_{\mathbb{R}} g(\lambda) G_\lambda^{(\alpha,\beta)}(x) \left(1 - \frac{\rho}{i\lambda}\right) \frac{d\lambda}{8\pi |c_{\alpha,\beta}(\lambda)|^2},$$

here

$$c_{\alpha,\beta}(\lambda) = \frac{2^{\rho - i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma(\frac{1}{2}(\rho + i\lambda))\Gamma(\frac{1}{2}(\alpha - \beta + 1 + i\lambda))}.$$

The corresponding Plancherel formula was established in [1], to the effect that

$$\int_{\mathbb{R}} |f(x)|^2 A_{\alpha,\beta}(x) dx \quad = \quad \int_0^{+\infty} \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma(\lambda),$$

where $\check{f}(x) := f(-x)$ and $d\sigma$ is the measure given by

$$d\sigma(\lambda) = \frac{d\lambda}{16\pi |c_{\alpha,\beta}(\lambda)|^2}.$$

According to [2] there exists a family of signed measures $\mu_{x,y}^{(\alpha,\beta)}$ such that the product formula

$$G_\lambda^{(\alpha,\beta)}(x) G_\lambda^{(\alpha,\beta)}(y) = \int_{\mathbb{R}} G_\lambda^{(\alpha,\beta)}(z) d\mu_{x,y}^{(\alpha,\beta)}(z)$$

holds for all $x, y \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, where

$$d\mu_{x,y}^{(\alpha,\beta)}(z) = \begin{cases} \mathcal{K}_{\alpha,\beta}(x,y,z) A_{\alpha,\beta}(z) dz, & \text{if } xy \neq 0 \\ \\ d\delta_x(z), & \text{if } y = 0 \\ d\delta_y(z), & \text{if } x = 0 \end{cases}$$

and

$$\mathcal{K}_{\alpha,\beta}(x,y,z) = M_{\alpha,\beta} |\sinh x . \sinh y . \sinh z|^{-2\alpha} \int_0^\pi g(x,y,z,\chi)_+^{\alpha-\beta-1}$$
$$\times \left[1 - \sigma_{x,y,z}^\chi + \sigma_{x,z,y}^\chi + \sigma_{z,y,x}^\chi + \frac{\rho}{\beta + \frac{1}{2}} \coth x . \coth y . \coth z (\sin\chi)^2\right] \times (\sin\chi)^{2\beta} d\chi$$

if $x, y, z \in \mathbb{R}\backslash\{0\}$ satisfy the triangular inequality $||x| - |y|| < |z| < |x| + |y|$, and $\mathcal{K}_{\alpha,\beta}(x,y,z) = 0$ otherwise. Here

$$\forall x, y, z \in \mathbb{R}, \chi \in [0,1], \sigma_{x,y,z}^\chi = \begin{cases} \frac{\cosh x + \cosh y - \cosh z \cos\chi}{\sinh x \sinh y}, & \text{if } xy \neq 0 \\ \\ 0, & \text{if } xy = 0 \end{cases}$$

and $g(x,y,z,\chi) = 1 - \cosh^2 x - \cosh^2 y . \cosh^2 z + 2\cosh x . \cosh y . \cosh z . \cos\chi$.

Lemma 1.2. [2] *For all $x, y \in \mathbb{R}$, we have*
(i) $\mathcal{K}_{\alpha,\beta}(x,y,z) = \mathcal{K}_{\alpha,\beta}(y,x,z)$.
(ii) $\mathcal{K}_{\alpha,\beta}(x,y,z) = \mathcal{K}_{\alpha,\beta}(-x,z,y)$.
(iii) $\mathcal{K}_{\alpha,\beta}(x,y,z) = \mathcal{K}_{\alpha,\beta}(-z,y,-x)$.

The product formula is used to obtain explicit estimates for the generalized translation operators

$$\tau_x^{(\alpha,\beta)} f(y) = \int_{\mathbb{R}} f(z) d\mu_{x,y}^{(\alpha,\beta)}(z).$$

It is known from [2] that

$$\mathcal{H}\tau_x^{(\alpha,\beta)} f(\lambda) = G_\lambda^{(\alpha,\beta)}(x)\mathcal{H}f(\lambda), \tag{1.1}$$

for $f \in C_c(\mathbb{R})$.

2. Main Result

In this section we give the main result of this paper. We need first to define $(\psi,\gamma,2)$-Cherednik-Opdam Lipschitz class.
Denote N_h by

$$N_h = \tau_h^{(\alpha,\beta)} + \tau_{-h}^{(\alpha,\beta)} - 2I,$$

where I is the unit operator in the space $L^2_{\alpha,\beta}(\mathbb{R})$.

Definition 2.1. Let $\gamma \geq 0$. A function $f \in L^2_{\alpha,\beta}(\mathbb{R})$ is said to be in the $(\psi,\gamma,2)$-Cherednik-Opdam Lipschitz class, denoted by $Lip(\psi,\gamma,2)$, if

$$\|N_h f(x)\|_{2,\alpha,\beta} = O\left(\frac{\psi(h)}{(\log \frac{1}{h})^\gamma}\right) \quad \text{as} \quad h \to 0,$$

where
(a) ψ is a continuous increasing function on $[0,\infty)$,
(b) $\psi(0) = 0$, $\psi(ts) = \psi(t)\psi(s)$ for all $t, s \in [0,\infty)$,
(c) and

$$\int_0^{1/h} s\psi(s^{-2})(\log s)^{-2\gamma} ds = O\left(h^{-2}\psi(h^2)\left(\log \frac{1}{h}\right)^{-2\gamma}\right), \quad h \to 0.$$

Lemma 2.1. *If $f \in C_c(\mathbb{R})$, then*

$$\mathcal{H}\check{\tau}_x^{(\alpha,\beta)} f(\lambda) = G_\lambda^{(\alpha,\beta)}(-x)\mathcal{H}\check{f}(\lambda). \tag{2.1}$$

Proof. For $f \in C_c(\mathbb{R})$, we have

$$\begin{aligned}
\mathcal{H}\check{\tau}_x^{(\alpha,\beta)} f(\lambda) &= \int_{\mathbb{R}} \tau_x^{(\alpha,\beta)} f(-y) G_\lambda^{(\alpha,\beta)}(-y) A_{\alpha,\beta}(y) dy \\
&= \int_{\mathbb{R}} \tau_x^{(\alpha,\beta)} f(y) G_\lambda^{(\alpha,\beta)}(y) A_{\alpha,\beta}(y) dy \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(z)\mathcal{K}_{\alpha,\beta}(x,y,z) A_{\alpha,\beta}(z) dz\right] G_\lambda^{(\alpha,\beta)}(y) A_{\alpha,\beta}(y) dy \\
&= \int_{\mathbb{R}} f(z) \left[\int_{\mathbb{R}} G_\lambda^{(\alpha,\beta)}(y)\mathcal{K}_{\alpha,\beta}(x,y,z) A_{\alpha,\beta}(y) dy\right] A_{\alpha,\beta}(z) dz.
\end{aligned}$$

Since $\mathcal{K}_{\alpha,\beta}(x,y,z)=\mathcal{K}_{\alpha,\beta}(-x,z,y)$, it follows from the product formula that

$$\begin{aligned}\mathcal{H}\check{\tau}_x^{(\alpha,\beta)}f(\lambda) &= G_\lambda^{(\alpha,\beta)}(-x)\int_{\mathbb{R}} f(z)G_\lambda^{(\alpha,\beta)}(z)A_{\alpha,\beta}(z)dz\\ &= G_\lambda^{(\alpha,\beta)}(-x)\int_{\mathbb{R}} f(-z)G_\lambda^{(\alpha,\beta)}(-z)A_{\alpha,\beta}(z)dz\\ &= G_\lambda^{(\alpha,\beta)}(-x)\mathcal{H}\check{f}(\lambda).\end{aligned}$$

□

Lemma 2.2. *For $f\in L^2_{\alpha,\beta}(\mathbb{R})$, then*

$$\|N_hf(x)\|^2_{2,\alpha,\beta}=4\int_0^{+\infty}|\varphi_\lambda^{\alpha,\beta}(h)-1|^2\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma(\lambda).$$

Proof. From formulas (1.1) and (2.1), we have

$$\mathcal{H}(N_hf)(\lambda)=(G_\lambda^{(\alpha,\beta)}(h)+G_\lambda^{(\alpha,\beta)}(-h)-2)\mathcal{H}(f)(\lambda),$$

and

$$\mathcal{H}(\check{N}_hf)(\lambda)=(G_\lambda^{(\alpha,\beta)}(-h)+G_\lambda^{(\alpha,\beta)}(h)-2)\mathcal{H}(\check{f})(\lambda).$$

Since

$$G_\lambda^{(\alpha,\beta)}(h)=\varphi_\lambda^{\alpha,\beta}(h)+\frac{\rho}{4(\alpha+1)}\sinh(2h)\varphi_\lambda^{\alpha+1,\beta+1}(h),$$

and $\varphi_\lambda^{\alpha,\beta}$ is even, then

$$\mathcal{H}(N_hf)(\lambda)=2(\varphi_\lambda^{\alpha,\beta}(h)-1)\mathcal{H}(f)(\lambda)$$

and

$$\mathcal{H}(\check{N}_hf)(\lambda)=2(\varphi_\lambda^{\alpha,\beta}(h)-1)\mathcal{H}(\check{f})(\lambda).$$

Now by Plancherel Theorem, we have the result. □

Theorem 2.1. *Let $f\in L^2_{\alpha,\beta}(\mathbb{R})$. Then the following are equivalents*

(a) $f\in Lip(\psi,\gamma,2)$,

(b) $\displaystyle\int_r^{+\infty}\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma(\lambda)=O\left(\frac{\psi(r^{-2})}{(\log r)^{2\gamma}}\right),\quad as\quad r\to\infty.$

Proof. $(a)\Rightarrow(b)$ Let $f\in Lip(\psi,\gamma,2)$. Then we have

$$\|N_hf(x)\|_{2,\alpha,\beta}=O\left(\frac{\psi(h)}{(\log\frac{1}{h})^\gamma}\right)\quad\text{as}\quad h\to 0.$$

From Lemma 2.2, we have

$$\|N_hf(x)\|^2_{2,\alpha,\beta}=4\int_0^{+\infty}|1-\varphi_\lambda^{\alpha,\beta}(h)|^2\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma\lambda.$$

If $\lambda \in [\frac{1}{h}, \frac{2}{h}]$, then $\lambda h \geq 1$ and (iii) of Lemma 1.1 implies that

$$1 \leq \frac{1}{c^2}|1-\varphi_\lambda^{\alpha,\beta}(h)|^2.$$

Then

$$\begin{aligned}
\int_{\frac{1}{h}}^{\frac{2}{h}} \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma(\lambda) &\leq \frac{1}{c^2}\int_{\frac{1}{h}}^{\frac{2}{h}} |1-\varphi_\lambda^{\alpha,\beta}(h)|^2 \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma(\lambda)\\
&\leq \frac{1}{c^2}\int_0^{+\infty} |1-\varphi_\lambda^{\alpha,\beta}(h)|^2 \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma(\lambda)\\
&\leq \frac{1}{4c^2}\|N_h f(x)\|_{2,\alpha,\beta}^2\\
&= O\left(\frac{\psi(h^2)}{(\log\frac{1}{h})^{2\gamma}}\right).
\end{aligned}$$

We obtain

$$\int_r^{2r} \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma(\lambda) \leq C\frac{\psi(r^{-2})}{(\log r)^{2\gamma}}, \quad r\to\infty,$$

where C is a positive constant. Now,

$$\begin{aligned}
\int_r^{+\infty} \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma(\lambda) &= \sum_{i=0}^{\infty}\int_{2^i r}^{2^{i+1}r} \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma(\lambda)\\
&\leq C\left(\frac{\psi(r^{-2})}{(\log r)^{2\gamma}} + \frac{\psi((2r)^{-2})}{(\log 2r)^{2\gamma}} + \frac{\psi((4r)^{-2})}{(\log 4r)^{2\gamma}} + \cdots\right)\\
&\leq C\frac{\psi(r^{-2})}{(\log r)^{2\gamma}}\left(1+\psi(2^{-2}) + (\psi(2^{-2}))^2 + (\psi(2^{-2}))^3 + \cdots\right)\\
&\leq K_\psi \frac{\psi(r^{-2})}{(\log r)^{2\gamma}},
\end{aligned}$$

where $K_\psi = C(1-\psi(2^{-2}))^{-1}$ since $\psi(2^{-2}) < 1$.
Consequently

$$\int_r^{+\infty} \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma(\lambda) = O\left(\frac{\psi(r^{-2})}{(\log r)^{2\gamma}}\right), \quad as \quad r\to\infty.$$

$(b) \Rightarrow (a)$. Suppose now that

$$\int_r^{+\infty} \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma(\lambda) = O\left(\frac{\psi(r^{-2})}{(\log r)^{2\gamma}}\right), \quad as \quad r\to\infty,$$

and write

$$\|N_h f(x)\|_{2,\alpha,\beta}^2 = 4(I_1 + I_2),$$

where

$$I_1 = \int_0^{\frac{1}{h}} |1 - \varphi_\lambda^{\alpha,\beta}(h)|^2 \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma\lambda,$$

and

$$I_2 = \int_{\frac{1}{h}}^{+\infty} |1 - \varphi_\lambda^{\alpha,\beta}(h)|^2 \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma\lambda.$$

Firstly, we use the formula $|\varphi_\lambda^{\alpha,\beta}(h)| \leq 1$ and

$$I_2 \leq 4 \int_{\frac{1}{h}}^{+\infty} \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma(\lambda) = O\left(\frac{\psi(h^2)}{(\log \frac{1}{h})^{2\gamma}}\right), \quad as \quad h \to 0.$$

To estimate I_1, we use the inequalities (i) and (ii) of Lemma 1.1

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{h}} |1 - \varphi_\lambda^{\alpha,\beta}(h)|^2 \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma\lambda \\ &\leq 2\int_0^{\frac{1}{h}} |1 - \varphi_\lambda^{\alpha,\beta}(h)| \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma\lambda \\ &\leq 2h^2 \int_0^{\frac{1}{h}} (\lambda^2 + \rho^2) \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma\lambda. \end{aligned}$$

Now, we apply integration by parts for a function

$$\phi(s) = \int_s^{+\infty} \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma(\lambda)$$

to get

$$\begin{aligned} I_1 &\leq -2h^2 \int_0^{1/h} (s^2 + \rho^2)\phi'(s)ds \\ &\leq -2h^2 \int_0^{1/h} s^2\phi'(s)ds \\ &\leq h^2 \left(-\frac{1}{h^2}\phi(\frac{1}{h}) + 2\int_0^{1/h} s\phi(s)ds\right) \\ &\leq -\phi(\frac{1}{h}) + 2h^2 \int_0^{1/h} s\phi(s)ds \\ &\leq 2h^2 \int_0^{1/h} s\phi(s)ds. \end{aligned}$$

Since $\phi(s) = O\left(\frac{\psi(s^{-2})}{(\log s)^{2\gamma}}\right)$, we have $s\phi(s) = O\left(\frac{s\psi(s^{-2})}{(\log s)^{2\gamma}}\right)$ and

$$\int_0^{1/h} s\phi(s)ds = O\left(\int_0^{1/h} \frac{s\psi(s^{-2})}{(\log s)^{2\gamma}} ds\right) = O\left(\frac{h^{-2}\psi(h^2)}{(\log \frac{1}{h})^{2\gamma}}\right),$$

so that

$$I_1 = O\left(\frac{\psi(h^2)}{(\log\frac{1}{h})^{2\gamma}}\right).$$

Consequently,

$$\|N_h f(x)\|_{2,\alpha,\beta} = O\left(\frac{\psi(h)}{(\log\frac{1}{h})^{\gamma}}\right) \quad \text{as} \quad h \to 0,$$

and this ends the proof of the theorem. □

3. Conclusion

In this work we have succeded to generalise the theorem in [3] for the Cherednik-Opdam transform in the space $L^2_{\alpha,\beta}(\mathbb{R})$. We proved that $f(x)$ belong to $Lip(\psi, \gamma, 2)$. Then

$$\int_r^{+\infty} \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma(\lambda) = O\left(\frac{\psi(r^{-2})}{(\log r)^{2\gamma}}\right), \quad as \quad r \to \infty.$$

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