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DUALITY IN INVEX PROGRAMMING PROBLEM IN HILBERT SPACE

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Abstract. In this paper the concept of duality is introduced for the invex programming problem in infinite dimensional Hilbert spaces. A generalization of the concept of Wolfe-duality is proposed for such class of problems. Some important theorems regarding the characterization of the dual problems are also discussed.
Keywords: Invexity, Wolfe-Duality, KKT Conditions.

1. Introduction

In the last few decades, generalized monotonicity, duality and optimality conditions in the theory of *invex optimization* have been discussed rigorously by several authors. In most of the cases, such problems have been characterized in R^n [1, 2, 3, 4, 5, 6, 7, 12, 14, 16, 18, 19, 22]. Chatterjee and Mukherjee [16] very recently generalized the concept of invex optimization from R^n to an arbitrary Hilbert Space. Such problems are known as the invex programming problem (IPP). This paper generalizes the concept of Wolf-duality for such class of infinite dimensional optimization problems. Some important theorems regarding the characterization of the dual problems, e.g., weak duality, strong duality and strict converse duality, have been proved in this generalized structure. This generalization allows us to study a wider class of infinite dimensional optimization problems and their duals.

2. Prerequisites

Definition 2.1. A subset C of $\mathbb{R}^n$ is *convex* [11] if for every pair of points x_1, x_2 in C, the line segment

$$[x_1, x_2] = \{x : x = \alpha x_1 + \beta x_2,\ \alpha \geq 0,\ \beta \geq 0, \alpha + \beta = 1\}$$

belongs to C.

The set C is said to be *invex* [18] if there is a vector function $\eta : C \times C \to \mathbb{R}^n$ such that

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$$x_1 + \lambda\eta(x_1, x_2) \in C \quad \forall\, x_1, x_2 \in C \; and \; \forall\lambda \in [0, 1]$$

Definition 2.2. Let C be an open convex set in $\mathbb{R}^n$ and let f be real valued and differentiable on C. Then, f is *convex* [11] if

$$f(x) - f(y) \geq \langle \nabla f(y), x - y \rangle\,, \quad \forall\, x, y \in C.$$

The function f is said to be *invex* [18] if there is a vector function $\eta : C \times C \to \mathbb{R}^n$

$$f(x) - f(y) \geq \langle \nabla f(y), \eta(x, y) \rangle\,, \quad \forall\, x, y \in C.$$

Definition 2.4. Let X and Y be two normed vector spaces. A continuous linear transformation A: $X \to Y$ is said to be the *Fréchet* (Strong) derivative [23] of $f : X \to Y$ at x, if for every $\epsilon > 0$, $\exists\, \delta > 0$ such that,

$$\| f(x + h) - f(x) - Ah \|_Y \leq \epsilon \| h \|_X, \text{ for all } h \text{ with } \| h \|_X \leq \delta$$

When the derivative exists it is denoted by $Df(x)$.

Proposition 2.1. [23] Let X be a vector space and Y be a normed space. Let S be a transformation, mapping an open set $D \subset X$ into an open set $E \subset Y$ and let P be a transformation, mapping E into a normed space Z. Let $T = PS$, S is *Fréchet* differentiable at $x \in D$ and P is *Fréchet* differentiable at $y = S(x) \in E$. Then, T is *Fréchet* Differentiable at x and $DT(x) = DP(y)DS(x)$.

Remark. [11] It is to be noted that in $\mathbb{R}^n$, $Df(x) = \nabla f(x)$.

Let H_1 and H_2 be two Hilbert spaces with some archimedean ordering " $\geq$ " and $X \subseteq H_1$ is an open set. Let $f : X \to H_2$ be a differentiable(*Fréchet*) function, $\eta : X \times X \to H_2$ be a vector function and $e \in H_2$ such that $\|e\|_{H_2} = 1$.

Definition 2.5. [16] The function f is said to be $(\eta, e) - invex$ if

$$f(x) - f(y) \geq \langle Df(y)\,,\, \eta(x, y) \rangle e, \quad \forall\, x, y \in X, \tag{2.1}$$

where $Df(y)$ stands for the *Fréchet* derivative of f at y.

The function f is said to be *strictly* (η, e)*-invex* if there exists $e \in H_2$ with $\|e\|_{H_2} = 1$, such that

$$f(x) - f(y) > \langle D(f(y), \eta(x, y) \rangle e, \quad \forall x, y \in X, x \neq y.$$

Remark. It is to be noted that, if H_1=H_2= $\mathbb{R}^n$, $e = (1, 1, 1, \ldots, 1)$ and $\eta(x, y) = (x - y)$, then f becomes a convex function in $\mathbb{R}^n$. The norm in this case can be taken as $(n)^{-\frac{1}{2}}$-multiple of the usual Euclidean norm.

Let H_1 and H_2 be two real Archimedean ordered Hilbert spaces. Let $\phi : H_1 \to H_2$ and $f : H_1 \to H_2$ be $(\eta, e) - invex$ functions (i.e., both the functions are invex with respect to the same η and e) such that

$$f(x) = f(y) \Rightarrow (x - y) \in Ker f.$$

Let us consider the following program:

$$\begin{aligned} & Min \;\; \phi(x) \\ s.t. \quad & f(x) = y \qquad x \in H_1 \,,\; y \in H_2 \\ & x \geq \theta_{H_1}. \end{aligned}$$

Let us denote the program by IP(H_1, H_2, ϕ, f) or simply by IP(if there is no confusion). In our further discussion, we shall refer this problem as *invex primal*.

Example 2.1. Detection Filter Problem (Fortmann, Athans) [20]. Following is an example of an IP:

$$\begin{gathered} Min\{-\langle u, x\rangle : u \in L^2[0, T]\} \\ \text{s.t. } \langle u, s_t\rangle - \epsilon\langle u, s\rangle \leq 0 \quad \delta \leq |t| \leq T \\ -\langle u, s_t\rangle - \epsilon\langle u, s\rangle \leq 0 \quad \delta \leq |t| \leq T \\ \|u\| \leq 1 \end{gathered}$$

Where $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denotes the usual norm and inner product in L^2-space, and s is a signal function with the assumption that the energy of s equals to 1, i.e., $\|s\|^2 = 1$.

KKT Conditions. $x^* \in H_1$ is a solution of IP iff there exist a scalar $\lambda^* \geq 0$ such that
i) $\lambda^* f(x^*) = \theta$
ii)$D(\phi(x^*) + \lambda^* f(x^*)) = \theta$
iii)$\lambda^* \geq 0$

3. Fritz-John Condition

Let H_1 and H_2 be two real Archimedean ordered Hilbert spaces and I be an open invex set in H_1. Let $f, g, h : I \to H_2$ be differentiable (*Fréchet*) $(\eta, e) - invex$ functions with respect to same $\eta(\cdot,\cdot)$ and e. Let us consider the following IPP:

$$\begin{aligned} & Min \;\; f(x) \\ s.t. \;\; & g(x) \leq \theta_{H_2} \\ & h(x) = \theta_{H_2} \\ & x \geq \theta_{H_1}. \end{aligned} \tag{3.1}$$

The following theorem is a generalization of the very well known Fritz-John conditions. Under the assumption of invexity, the conditions are not only necessary but also sufficient. The proof of the necessity of the conditions is motivated by McShane [11]. In our discussion, whenever we consider topology we mean weak topology.

Theorem 3.1. $x^* \in I$ is a solution of IPP (3.1) iff there exist non-zero scalars λ, μ and ν such that
(i) $\lambda g(x^*) = \theta$
(ii)$\lambda Df(x^*) + \mu Dg(x^*) + \nu Dh(x^*) = \theta$.

Proof. Let K be a strictly increasing differentiable real valued function defined on H_2 such that, $K(x) > 0$ whenever $x > \theta$ and $K(x) = 0$ elsewhere. It is to be noted that, $DK(x) > 0$ for $x > \theta$. Since g is continuous and I is open , there exist an $\epsilon_0 > 0$ such that $B(\theta, \epsilon_0) \subset I$ and $g(x) \leq \theta$ for $x \in B(\theta, \epsilon_0)$. Now, let us define a function

$$F(x, p) = \|f(x)\| + \|x\|^2 + p\{K(g(x)) + \|h(x)\|^2\}, \quad x \in I \text{ and } p \in \mathbb{Z}^+. \tag{3.2}$$

We assert that, for each ϵ satisfying $0 < \epsilon < \epsilon_0$, there exist a positive integer $p(\epsilon)$ such that, for x with $\|x\| = \epsilon$, $F(x, p(\epsilon)) > \theta$. If not, then there would exist an ϵ' with $0 < \epsilon' < \epsilon_0$ such that, for each positive integer p, there exist a vector x_p with $\|x_p\| = \epsilon'$ and $F(x_p, p) \leq \theta$. Hence form (3.2),

$$\|f(x)\| \leq -\{\|x\|^2 + p\{Kg(x_p) + \|h(x_p)\|^2\}\} \tag{3.3}$$

Now since $\|x_p\| = \epsilon'$ and since $S(0, \epsilon') = \{y : \ \|y\| = \epsilon'\}$ is weakly compact, then there exist sub-sequences, which we relabel as x_p and p ,and a point x_0 with $\|x_0\| = \epsilon'$, such that, $x_p \mapsto x_0$. Since f, g and h are continuous, $f(x_p) \mapsto f(x_0)$; $g(x_p) \mapsto g(x_0)$; $h(x_p) \mapsto h(x_0)$. Therefore, dividing (3.3) by $-p$ and letting $p \to \infty$ we get, $K(g(x_0)) + \|h(x_0)\|^2 = 0$. Hence by definition of $K(\cdot, \cdot)$, $g(x_0) \leq \theta$ and $h(x_0) = \theta$. Thus x_0 is a feasible vector. Now, by a suitable affine transformation x^* can be assumed as θ and $f(x^*) = f(\theta) = 0$. Therefore, $f(x_0) \geq f(\theta) = 0$. Now, from (3.3), $\|f(x_p)\| \leq -(\epsilon')^2 < 0$, which is a contradiction. Hence the assertion is true.

Again, for each $\epsilon \in (0, \epsilon_0)$, the function $F(\cdot, p(\epsilon))$ is continuous on the closed ball $\overline{B(0, \epsilon)}$. Since $\overline{B(0, \epsilon)}$ is weakly compact, $F(\cdot, p(\epsilon))$ attains its minimum on $\overline{B(0, \epsilon)}$ at an interior point x_ϵ of $\overline{B(0, \epsilon)}$. Hence

$$DF(x_\epsilon, p(\epsilon)) = 0 \tag{3.4}$$

Now let us assume that

$$L(\epsilon) = 1 + (p(\epsilon)DK(g(x_\epsilon)))^2 + (p(\epsilon)D(\|h(x_\epsilon)\|))^2 \tag{3.5}$$

$$\lambda(\epsilon) = \frac{D(\|f(x)\|)}{\sqrt{L(\epsilon)}} \tag{3.6}$$

$$\mu(\epsilon) = \begin{cases} \frac{p(\epsilon)DK(g(x_\epsilon))}{\sqrt{L(\epsilon)}} & if \quad g(\theta) = \theta \\ 0 & else \end{cases} \tag{3.7}$$

$$\nu(\epsilon) = \frac{p(\epsilon)D(\|h(x_\epsilon\|)}{\sqrt{L(\epsilon)}}. \tag{3.8}$$

It is to be noted that $\lambda(\epsilon) \geq 0$,and $\mu(\epsilon), \nu(\epsilon) \geq \theta$.

Now, from (3.2), (3.4) and (3.5), we get

$$\lambda(\epsilon)Df(x_\epsilon) + \frac{D(\|x_\epsilon\|^2)}{\sqrt{L(\epsilon)}} + \mu(\epsilon)Dg(x_\epsilon) + \nu(\epsilon)Dh(x_\epsilon) = 0. \tag{3.9}$$

Let $\epsilon \to 0$ through a sequence of values ϵ_k. Then since $\|x_\epsilon\| < \epsilon$, we have

$$x_{\epsilon_k} \to \theta, \ \lambda(\epsilon_k) \to \lambda, \ \mu(\epsilon_k) \to \mu, \ \nu(\epsilon_k) \to \nu. \tag{3.10}$$

Therefore from (6) and (7) we get $\lambda Df(\theta)+\mu Dg(\theta)+\nu Dh(\theta)=\theta$ and from the definition of μ, $\mu g(\theta)=\theta$. This proves the necessity of the conditions.
Let us now consider on the sufficiency of the conditions. Since f is $(\eta, e)-invex$,

$$\begin{aligned}
f(x)-f(x^*) &\geq \langle Df(x^*), \eta(x,x^*)\rangle e \\
&= -\langle \mu Dg(x^*)+\nu Dh(x^*), \eta(x,x^*\rangle e \\
&= -\{\mu\langle Dg(x^*), \eta(x,x^*\rangle e+\nu\langle Dh(x^*), \eta(x,x^*\rangle e\} \\
&\geq -\{\mu(g(x)-g(x^*))+\nu(h(x)-h(x^*))\} \\
&= -\mu g(x) \\
&\geq \theta
\end{aligned}$$

which proves the sufficiency of the conditions. □

It is quite obvious that using any constraint qualification assuring the positivity of λ, we can obtain the generalization of the very popular Karush-Kuhn-Tucker conditions stated earlier.

4. Duality in IPP

Definition 4.1. The *generalized Wolfe dual* for the invex primal is defined as

$$\begin{gathered}
Max_{u,\lambda}(f(u)+\lambda g(u)) \\
\text{subject to } D(f(u)+\lambda g(u))=\theta \\
\lambda \geq 0 \\
u \in H_1 \text{ and } \lambda \text{ is a scalar.}
\end{gathered}$$

This problem will be referred to as *invex dual* denoted by ID in further discussion.

Theorem 4.1. Strong Duality. Under the condition of a suitable constraint qualification [18] for IP, if x° is minimal for IP, then (x°, λ°) is minimal for ID, where λ° is given by the KKT Conditions and f and g are $(\eta, e)-invex$.

Proof: Let (u,λ) be any vector feasible for ID. Then,

$$\begin{aligned}
&(f(x^\circ)+\lambda^\circ g(x^\circ))-(f(u)+\lambda g(u)) \\
&= f(x^\circ)-(f(u)+\lambda g(u)) \\
&\geq \langle Df(u),\ \eta(x^\circ,u)\rangle e-\lambda g(u) \\
&= -\langle \lambda Dg(u),\ \eta(x^\circ,u)\rangle e-\lambda g(u) \\
&= \lambda\{-\langle Dg(u),\ \eta(x^\circ,u)\rangle e\}-\lambda g(u) \\
&\geq \lambda\{g(u)-g(x^\circ)\}-\lambda g(u) \\
&= -\lambda g(x^\circ) \\
&\geq \theta
\end{aligned}$$

Therefore, (x°, λ°) is maximal in the dual problem and since $\lambda^\circ g(x^0)=0$, the extreme of the two problems are same. □

Theorem 4.2. Weak Duality. Let x be feasible for IP and (u, λ) be feasible for ID, then we have $f(x) \geq f(u)+\lambda g(u)$.

Proof. From the invexity assumption, we have

$$(f(x)-f(u)-\langle Df(u), \eta(x,u)\rangle e)+\lambda(g(x)-g(u)-\langle Dg(u), \eta(x,u)\rangle e) \geq \theta$$

which further implies

$$\begin{aligned}(f(x)-(f(u)+\lambda g(u))) &\geq \langle Df(u),\eta(x,u)\rangle e+\lambda\langle Dg(u),\eta(x,u)\rangle e-\lambda g(u)\\ &= \langle Df(u)+\lambda Dg(u),\eta(x,u)\rangle e-\lambda g(u)\\ &= -\lambda g(u)\\ &\geq \theta. \quad \square\end{aligned}$$

Theorem 4.3. Strict Converse Duality. Let f and g are $(\eta, e)-$ *invex*. Let x^* be optimal for IP and $(\bar{x}, \bar{\lambda})$ be optimal for ID. If a suitable constraint qualification[18] is satisfied for IP and f is strictly invex at $\bar{x}$, then $x^* = \bar{x}$.

Proof. Let $x^* \neq \bar{x}$. By the strong duality theorem, there exists λ^* such that (x^*, λ^*) is optimal for ID.
Hence,

$$f(x^*) = f(x^*) + \lambda^* g(x^*) = f(\bar{x}) + \bar{\lambda} g(\bar{x}) \tag{4.1}$$

Now by strict invexity of f we get,

$$f(x^*) - f(\bar{x}) > \langle Df(\bar{x}, \eta(x^*, \bar{x})\rangle e \tag{4.2}$$

And by invexity of g with $\bar{\lambda} \geq 0$ we get,

$$\bar{\lambda} g(x^*) - \bar{\lambda} g(\bar{x}) \geq \langle \bar{\lambda} Dg(\bar{x}), \eta(x^*, \bar{x})\rangle e. \tag{4.3}$$

Adding 4.2 and 4.3 we get,

$$(f(x^*) - f(\bar{x})) + (\bar{\lambda} g(x^*) - \bar{\lambda} g(\bar{x})) \geq \theta.$$

But, since $\bar{\lambda} g(x^*) \leq \theta$, we have $f(x^*) - f(\bar{x}) - \bar{\lambda} g(\bar{x}) > \theta$, i.e. $f(x^*) - f(\bar{x}) > \bar{\lambda} g(\bar{x})$, which contradicts 3.1. Therefore $x^* = \bar{x}$. $\square$

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