

FACTA UNIVERSITATIS (NIŠ)
SER. MATH. INFORM. Vol. 31, No 2 (2016), 543–557

SCREEN SLANT RADICAL TRANSVERSAL NULL SUBMANIFOLDS OF PARA-SASAKIAN MANIFOLDS

Bilal Eftal Acet, Selcen Yüksel Perktas

Abstract. In our paper we introduce totally paracontact umbilical radical transversal null submanifolds and screen slant radical transversal null submanifolds of para-Sasakian manifolds. On a screen slant radical transversal null submanifold of a para-Sasakian manifold, we find integrability conditions of distributions.

Keywords: Null submanifold, Para-Sasakian manifold

1. Introduction

Differential geometry of null (lightlike) submanifolds is different from non-degenerate submanifolds because of the fact that the normal vector bundle has non-trivial intersection with the tangent vector bundle. So, one cannot use the classical submanifold theory for null submanifolds. For this problem K. L. Duggal and A. Bejancu introduced a new method and presented a book about null submanifolds [13] (see also [14]). The term of totally contact umbilical null submanifolds was considered by several geometers ([9, 12, 17]). In 2009, B. Şahin studied screen slant null submanifolds [5]. Radical transversal null submanifolds were defined and studied by C. Yıldırım and B. Şahin in 2010. Since then many authors have studied null submanifolds ([3, 6, 7, 8, 15, 16, 20]).

In 1985, on a semi-Riemannian manifold M^{2n+1} , S. Kaneyuki and M. Konzai [19] introduced a structure which is called almost paracontact structure and then they characterized the almost paracontact structure on $M^{2n+1} \times \mathbb{R}$. Recently, S. Zamkovoy [21] studied paracontact metric manifolds and some subclasses which are known para-Sasakian manifolds. The study of paracontact geometry was continued by several papers ([10, 11, 18, 22, 23]) which include role of paracontact geometry about semi-Riemannian geometry, mathematical physics and relationships with the para-Kähler manifolds.

The goal of the present article is to examine some null submanifolds of a para-Sasakian manifold. There are some basic definitions for almost paracontact metric

Received January 25, 2015; accepted March 25, 2016
2010 *Mathematics Subject Classification.* Primary 53C15; Secondary 53C25

manifolds and null submanifolds in section 2. Totally paracontact umbilical radical transversal null submanifolds of a para-Sasakian manifold are introduced in Section 3. Finally, Section 4 is devoted to screen slant radical transversal null submanifolds of a para-Sasakian manifold and integrability conditions of distributions on screen slant radical transversal null submanifolds.

2. Preliminaries

2.1. Null Submanifolds

Let (\bar{M}^{n+m}, \bar{g}) be a semi-Riemannian manifold with index q , such that $m, n \geq 1, 1 \leq q \leq m+n-1$ and (M^m, g) be a submanifold of \bar{M} , where g induced metric from \bar{g} on M . In this case M is called a *null (lightlike) submanifold* of \bar{M} if g is degenerate on M . Now consider a degenerate metric g on M . Thus TM^\perp is a degenerate n -dimensional subspace of $T_x\bar{M}$ and orthogonal subspaces T_xM and T_xM^\perp are degenerate but no longer complementary. So, there exists a subspace $RadT_xM = T_xM \cap T_xM^\perp$ which is called radical space. If the mapping $RadTM : x \in M \rightarrow RadT_xM$, defines a smooth distribution, named *Radical distribution*, on M of rank $r > 0$ then the submanifold M is called an *r -null submanifold* [13].

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $RadTM$ in TM . So we can state

$$(2.1) \quad TM = S(TM) \perp RadTM,$$

and $S(TM^\perp)$ is a complementary vector subbundle to $RadTM$ in TM^\perp . Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}|_M$ and $RadTM$ in $S(TM^\perp)^\perp$, respectively. In this case, we arrive at

$$(2.2) \quad tr(TM) = ltr(TM) \perp S(TM^\perp),$$

$$(2.3) \quad T\bar{M}|_M = TM \oplus tr(TM) = \{RadTM \oplus ltr(TM)\} \perp S(TM) \perp S(TM^\perp).$$

Theorem 2.1. [13] *Let $(M, g, S(TM), S(TM^\perp))$ be a null submanifold of a semi-Riemannian manifold (\bar{M}, \bar{g}) . Then there exist a complementary vector subbundle $ltr(TM)$ of $RadTM$ in $S(TM^\perp)^\perp$ and a basis of $\Gamma(ltr(TM))|_U$ consisting of smooth section $\{N_i\}$ of $S(TM^\perp)^\perp|_U$, where U is a coordinate neighborhood of M , such that*

$$(2.4) \quad \bar{g}(N_i, E_i) = 1, \quad \bar{g}(N_i, N_j) = 0,$$

where $\{E_1, E_2, \dots, E_n\}$ is a null basis of $\Gamma(RadTM)$.

For a null submanifold $(M, g, S(TM), S(TM^\perp))$,

* If $r < \min\{m, n\}$ then M is a *r -null submanifold*,

- * If $r = n < m$, $S(TM^\perp) = \{0\}$ then M is a *coisotropic* null submanifold,
- * If $r = m < n$, $S(TM) = \{0\}$ then M is a *isotropic* null submanifold,
- * If $r = m = n$, $S(TM) = \{0\} = S(TM^\perp)$ then M is a *totally* null submanifold.

In view of (2.3), the Gauss and Weingarten formulas are given by

$$(2.5) \quad \bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM),$$

$$(2.6) \quad \bar{\nabla}_X U = -A_U X + \nabla_X^t U, \quad \forall X \in \Gamma(TM), U \in \Gamma(tr(TM)),$$

where $\{\nabla_X Y, A_U X\}$ belong to $\Gamma(TM)$ and $\{h(X, Y), \nabla_X^t U\}$ belong to $\Gamma(tr(TM))$. $\bar{\nabla}$ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$, respectively. In view of (2.2), we consider the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^\perp)$. Therefore (2.5) and (2.6) become

$$(2.7) \quad \bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad X, Y \in \Gamma(TM),$$

$$(2.8) \quad \bar{\nabla}_X N = -A_N X + \nabla_X^l N + D^s(X, N), \quad X \in \Gamma(TM), N \in \Gamma(ltr(TM)),$$

$$(2.9) \quad \bar{\nabla}_X W = -A_W X + \nabla_X^s W + D^l(X, W), \quad X \in \Gamma(TM), W \in \Gamma(S(TM^\perp)),$$

where $h^l(X, Y) = L(h(X, Y))$, $h^s(X, Y) = S(h(X, Y))$, $\nabla_X^l N, D^l(X, W) \in \Gamma(ltr(TM))$, $\nabla_X^s W, D^s(X, N) \in \Gamma(S(TM^\perp))$ and $\nabla_X Y, A_N X, A_W X \in \Gamma(TM)$.

Let P be a projection of TM on $S(TM)$, from (2.1), we have

$$(2.10) \quad \nabla_X PY = \nabla_X^* PY + h^*(X, PY), \quad X, Y \in \Gamma(TM),$$

$$(2.11) \quad \nabla_X E = -A_E^* X + \nabla_X^{*t} E, \quad X \in \Gamma(TM), E \in \Gamma(RadTM),$$

where $\{\nabla_X^* PY, A_E^* X\}$ belong to $\Gamma(S(TM))$ and $\{h^*(X, PY), \nabla_X^{*t} E\}$ belong to $\Gamma(RadTM)$.

Using (2.10) and (2.11), we have

$$(2.12) \quad \bar{g}(h^*(X, PY), N) = \bar{g}(A_N X, PY),$$

$$(2.13) \quad \bar{g}(h^l(X, PY), E) = \bar{g}(A_E^* X, PY),$$

$$(2.14) \quad A_E^* E = 0, \quad \bar{g}(h^l(X, E), E) = 0.$$

In general, the induced connection ∇ on M is not metric connection. Since $\bar{\nabla}$ is a metric connection, by using (2.7), we have

$$(2.15) \quad (\nabla_X g)(Y, Z) = \bar{g}(h^l(X, Y), Z) + \bar{g}(h^l(X, Z), Y).$$

However, it is important to note that ∇^* is a metric connection on $S(TM)$.

2.2. Almost paracontact metric manifolds

A paracontact manifold \bar{M}^{2n+1} is a smooth manifold equipped with a 1-form η , a characteristic vector field ξ and a tensor field $\bar{\phi}$ of type $(1, 1)$ such that [19]:

$$(2.16) \quad \eta(\xi) = 1,$$

$$(2.17) \quad \bar{\phi}^2 = I - \eta \otimes \xi,$$

$$(2.18) \quad \bar{\phi}\xi = 0,$$

$$(2.19) \quad \eta \circ \bar{\phi} = 0,$$

If we set $D = \ker \eta = \{X \in \Gamma(T\bar{M}) : \eta(X) = 0\}$, then $\bar{\phi}$ induces an almost paracomplex structure on the codimension 1 distribution defined by D [19].

Moreover, if the manifold \bar{M} is equipped with a semi-Riemannian metric \bar{g} of signature $(n+1, n)$ which is called *compatible metric* satisfying [21]

$$(2.20) \quad \bar{g}(\bar{\phi}X, \bar{\phi}Y) = -\bar{g}(X, Y) + \eta(X)\eta(Y), \quad X, Y \in \Gamma(T\bar{M}),$$

then we say that \bar{M} is an *almost paracontact metric manifold* with an *almost paracontact metric structure* $(\bar{\phi}, \xi, \eta, \bar{g})$.

From the definition, one can see that [21],

$$(2.21) \quad \bar{g}(\bar{\phi}X, Y) = -\bar{g}(X, \bar{\phi}Y),$$

$$(2.22) \quad \bar{g}(X, \xi) = \eta(X).$$

If $\bar{g}(X, \bar{\phi}Y) = d\eta(X, Y)$ the almost paracontact metric manifold is said to be a *paracontact metric manifold*.

For an almost paracontact metric manifold $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$, one can always find a local orthonormal basis which is called $\bar{\phi}$ -basis $(X_i, \bar{\phi}X_i, \xi)$ ($i = 1, 2, \dots, n$) [21].

An almost paracontact metric manifold $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ is a *para-Sasakian manifold* if and only if [21]

$$(2.23) \quad (\bar{\nabla}_X \bar{\phi})Y = -\bar{g}(X, Y)\xi + \eta(Y)X, \quad X, Y \in \Gamma(T\bar{M}),$$

where $\bar{\nabla}$ is Levi-Civita connection of \bar{M} .

From (2.23), we also have

$$(2.24) \quad \bar{\nabla}_X \xi = -\bar{\phi}X.$$

Example 2.1. [1] Let $\bar{M} = \mathbb{R}^{2n+1}$ be $(2n + 1)$ -dimensional real number space with $(x_1, y_1, x_2, y_2, \dots, x_n, y_n, z)$ standard coordinate system. Defining

$$(2.25) \quad \begin{aligned} \bar{\phi} \frac{\partial}{\partial x_\alpha} &= \frac{\partial}{\partial y_\alpha}, & \bar{\phi} \frac{\partial}{\partial y_\alpha} &= \frac{\partial}{\partial x_\alpha}, & \bar{\phi} \frac{\partial}{\partial z} &= 0, \\ \xi &= \frac{\partial}{\partial z}, & \bar{\eta} &= dz, \\ \bar{g} &= \eta \otimes \eta + \sum_{\alpha=1}^n (dx_\alpha \otimes dx_\alpha - dy_\alpha \otimes dy_\alpha), \end{aligned}$$

where $\alpha = 1, 2, \dots, n$, then the set $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ is an almost paracontact metric manifold.

3. Totally paracontact umbilical radical transversal null submanifold

The aim of this section is to examine totally paracontact umbilical radical transversal null submanifolds of a para-Sasakian manifold. We state the following definition given [9] for a radical transversal null submanifold of a para-Sasakian manifold.

Definition 3.1. Let $(M, g, S(TM), S(TM^\perp))$ be a null submanifold of a para-Sasakian manifold (\bar{M}, \bar{g}) such that $\xi \in \Gamma(TM)$. If the following conditions given by

$$(3.1) \quad \bar{\phi}(RadTM) = ltr(TM),$$

$$(3.2) \quad \bar{\phi}(D) = D,$$

are provided on M then M is called radical transversal null submanifold, where $S(TM) = D \perp \{\xi\}$ and D is complementary non-degenerate distribution to $\{\xi\}$ in $S(TM)$.

Example 3.1. Let $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ be a 9-dimensional almost paracontact metric manifold given in Example 2.1. Assume that M is a submanifold defined by

$$\begin{aligned} x_1 &= -y_3, & x_2 &= y_4, \\ x_3 &= -y_1, & x_4 &= y_2. \end{aligned}$$

In this case TM of M is spanned by

$$\left\{ \begin{array}{l} \Psi_1 = -\frac{\partial}{\partial x_3} + \frac{\partial}{\partial y_1}, \quad \Psi_2 = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial y_2}, \quad \Psi_3 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_3}, \\ \Psi_4 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_4}, \quad \Psi_5 = \frac{\partial}{\partial z} \end{array} \right\}.$$

Hence the radical distribution $RadTM = Sp\{\Psi_1, \Psi_3\}$ and $ltr(TM)$ is spanned by

$$\Omega_1 = \frac{\partial}{\partial x_3} - \frac{\partial}{\partial y_1}, \quad \Omega_2 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_3}.$$

It follows that $\bar{\phi}\Psi_1 = -\Omega_1$, $\bar{\phi}\Psi_3 = -\Omega_2$, $\bar{\phi}\Psi_2 = \Psi_4$, $\bar{\phi}\Psi_4 = \Psi_2$. Thus

$$\bar{\phi}(RadTM) = ltr(TM)$$

and

$$\bar{\phi}(D) = D,$$

which implies that M is a radical transversal 2-null submanifold.

Proposition 3.1. [2] *There does not exist an isotropic or totally null radical transversal null submanifold of a para-Sasakian manifold.*

Proposition 3.2. [2] *There exists no 1-null radical transversal null submanifold of a para-Sasakian manifold.*

For a radical transversal null submanifold M of a para-Sasakian manifold \bar{M} , assume that ω_1 and ω_2 are the projection morphisms on $S(TM)$ and $RadTM$, respectively. Then, for $X \in \Gamma(TM)$, one can write

$$(3.3) \quad X = \omega_1 X + \omega_2 X,$$

where $\omega_1 X \in \Gamma(S(TM))$ and $\omega_2 X \in \Gamma(RadTM)$.

If we apply $\bar{\phi}$ to (3.3), we get

$$(3.4) \quad \bar{\phi}X = \bar{\phi}\omega_1 X + \bar{\phi}\omega_2 X.$$

Taking $\bar{\phi}\omega_1 X = TX$ and $\bar{\phi}\omega_2 X = QX$ in (3.4), we have

$$(3.5) \quad \bar{\phi}X = TX + QX,$$

where $TX \in \Gamma(S(TM))$ and $QX \in \Gamma(ltr(TM))$.

From (2.23), for a radical transversal null submanifold M , we get

$$\begin{aligned} (\bar{\nabla}_X \bar{\phi})Y &= \bar{\nabla}_X \bar{\phi}Y - \bar{\phi}\bar{\nabla}_X Y \\ &= -\bar{g}(X, Y)\xi + \eta(Y)X. \end{aligned}$$

By use of (2.7), (2.8) with (3.5), we obtain

$$\begin{aligned} -g(X, Y)\xi + \eta(Y)X &= \nabla_X TY + h^l(X, TY) + h^s(X, TY) \\ &\quad - A_{QY}X + \nabla_X^l QY + D^s(X, QY) \\ &\quad - T\nabla_X Y - Q\nabla_X Y - \bar{\phi}h^l(X, Y) \\ &\quad - \bar{\phi}h^s(X, Y). \end{aligned}$$

Considering the tangential, lightlike transversal and screen transversal components of above equation, we get

$$(3.6) \quad (\nabla_X T)Y = \bar{\phi}h^l(X, Y) + A_{QY}X - g(X, Y)\xi + \eta(Y)X,$$

$$(3.7) \quad h^l(X, TY) + \nabla_X^l QY - Q\nabla_X Y = 0,$$

$$(3.8) \quad h^s(X, TY) + D^s(X, QY) - \bar{\phi}h^s(X, Y) = 0.$$

It is well known that the induced connection of a null submanifold is not a metric connection. The following theorem shows the necessary and sufficient condition for the induced connection to be a metric connection.

Theorem 3.1. [2] *Let M be a radical transversal null submanifold of a para-Sasakian manifold \bar{M} . Then ∇ is a metric connection on M if and only if $A_{\bar{\phi}Y}X$ has no component in $S(TM)$, for $X \in \Gamma(TM)$ and $Y \in \Gamma(RadTM)$.*

Lemma 3.1. *Let M be a radical transversal null submanifold of a para-Sasakian manifold \bar{M} . Then for all $X, Y \in (\Gamma(TM) - \{\xi\})$, we have*

$$(3.9) \quad g(\nabla_X Y, \xi) = \bar{g}(Y, \bar{\phi}X).$$

Proof. From (2.7), we obtain

$$(3.10) \quad g(\nabla_X Y, \xi) = g(\bar{\nabla}_X Y, \xi).$$

Since $\bar{\nabla}$ is a metric connection, by use of (3.10), we get

$$g(\nabla_X Y, \xi) = Xg(Y, \xi) - \bar{g}(Y, \bar{\nabla}_X \xi),$$

which implies

$$(3.11) \quad g(\nabla_X Y, \xi) = -\bar{g}(Y, \bar{\nabla}_X \xi).$$

In view of (2.24) and (3.11), we complete the proof. \square

Definition 3.2. For a null submanifold M of a para-Sasakian manifold \bar{M} such that $\xi \in \Gamma(TM)$, if the second fundamental form h of M satisfies;

$$(3.12) \quad h^l(X, Y) = (g(X, Y) - \eta(X)\eta(Y))\alpha_Q + \eta(X)h^l(Y, \xi) + \eta(Y)h^l(X, \xi),$$

$$(3.13) \quad h^s(X, Y) = (g(X, Y) - \eta(X)\eta(Y))\alpha_T + \eta(X)h^s(Y, \xi) + \eta(Y)h^s(X, \xi),$$

then M is said to be totally paracontact umbilical radical transversal null submanifold, where $\alpha_Q \in \Gamma(ltr(TM))$, $\alpha_T \in \Gamma(S(TM^\perp))$ and $X, Y \in \Gamma(TM)$.

Theorem 3.2. *Let M be a totally paracontact umbilical radical transversal null submanifold of a para-Sasakian manifold \bar{M} . Then $\alpha_Q = 0$ if and only if*

$$(3.14) \quad h^*(X, \bar{\phi}Y) = 0,$$

for any $X, Y \in \Gamma(D)$.

Proof. Assume that M is totally paracontact umbilical radical transversal null submanifold of a para-Sasakian manifold \bar{M} . From (3.6), we get

$$(3.15) \quad \begin{aligned} -\bar{g}(X, Y)\xi + \eta(Y)X &= \bar{\nabla}_X \bar{\phi}Y - \bar{\phi}\bar{\nabla}_X Y \\ &= \nabla_X \bar{\phi}Y + h^l(X, \bar{\phi}Y) + h^s(X, \bar{\phi}Y) \\ &\quad - T\nabla_X Y - Q\nabla_X Y - \bar{\phi}h^l(X, Y) - \bar{\phi}h^s(X, Y), \end{aligned}$$

for any $X, Y \in \Gamma(D)$.

Now, using (3.15) for $X, Y \in \Gamma(D)$ and $Z \in \Gamma(\text{Rad}TM)$, we have

$$(3.16) \quad \bar{g}(\nabla_X \bar{\phi}Y, \bar{\phi}Z) - \bar{g}(\bar{\phi}h^l(X, Y), \bar{\phi}Z) = 0.$$

In view of (2.20), (2.10) in (3.16), we obtain

$$(3.17) \quad \bar{g}(h^*(X, \bar{\phi}Y), \bar{\phi}Z) + \bar{g}(h^l(X, Y), Z) = 0.$$

From the definition of totally paracontact umbilical radical transversal null submanifold, we find

$$(3.18) \quad \bar{g}(h^*(X, \bar{\phi}Y), \bar{\phi}Z) + \bar{g}(g(X, Y)\alpha_Q, Z) = 0,$$

completes the proof. \square

Theorem 3.3. *Let M be a totally paracontact umbilical radical transversal null submanifold of a para-Sasakian manifold \bar{M} . Then ∇ is a metric connection if and only if*

$$A_{\bar{\phi}Y}X = \eta(X)Y,$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(\text{Rad}TM)$.

Proof. It is known that the induced connection is a metric connection if and only if [2]

$$(3.19) \quad \nabla_X Y \in \Gamma(\text{Rad}TM), \quad X \in \Gamma(TM), \quad Y \in \Gamma(\text{Rad}TM).$$

By use of (3.6), (3.7) and (3.8) we have

$$(3.20) \quad \begin{aligned} T\nabla_X Y + Q\nabla_X Y \\ + \bar{\phi}h^l(X, Y) + \bar{\phi}h^s(X, Y) &= \nabla_X^l QY - A_{QY}X + D^s(X, QY). \end{aligned}$$

Now, using (3.12) and (3.13) in (3.20), we obtain

$$(3.21) \quad \begin{aligned} T\nabla_X Y + Q\nabla_X Y \\ + \eta(X)\bar{\phi}h^l(Y, \xi) + \eta(X)\bar{\phi}h^s(Y, \xi) &= -A_{LY}X + \nabla_X^l QY + D^s(X, QY). \end{aligned}$$

Taking the tangential part of (3.21), we get

$$(3.22) \quad T\nabla_X Y + \eta(X)\bar{\phi}h^l(Y, \xi) = -A_{\bar{\phi}Y}X.$$

Also, from (2.7) and (2.24), we have

$$(3.23) \quad -\bar{\phi}Y = h^l(Y, \xi), \quad Y \in \Gamma(RadTM).$$

Using (3.23) in (3.22), we find

$$T\nabla_X Y = -A_{\bar{\phi}Y}X + \eta(X).$$

The proof follows from the previous equation. \square

4. Screen slant radical transversal null submanifold

Now, we investigate screen slant radical transversal null submanifolds of a para-Sasakian manifold.

Definition 4.1. Let M be a $2q$ -lightlike submanifold of a para-Sasakian manifold \bar{M} of index $2q$ such that $2q \leq \dim(M)$. We say that M is a screen slant radical transversal null submanifold of \bar{M} if the following conditions are provided:

i) $\bar{\phi}(RadTM) = ltr(TM)$,

ii) For each non-zero vector field X tangent to D at $x \in U \subset M$, the angle $\theta(X)$ between, $\bar{\phi}X$ and the vector space D_x is constant, i.e. it is independent of the choice of $x \in U \subset M$ and $X \in D_x$, where D is complementary non-degenerate distribution to $\{\xi\}$ in $S(TM)$ such that $S(TM) = D \perp \{\xi\}$.

This constant angle $\theta(X)$ is called *slant angle* of distribution D . From the definition, we have the following decomposition:

$$(4.1) \quad TM = RadTM \perp D \perp \{\xi\}.$$

Now, we denote the projection on $RadTM$ and D in TM by f_1 and f_2 , respectively. Thus, we get

$$(4.2) \quad X = f_1X + f_2X + \eta(X)\xi, \quad \forall X \in \Gamma(TM).$$

Applying $\bar{\phi}$ to the both sides of equation (4.2), we get

$$(4.3) \quad \bar{\phi}X = \bar{\phi}f_1X + \bar{\phi}f_2X,$$

which implies

$$(4.4) \quad \bar{\phi}X = \bar{\phi}f_1X + TX + LX, \quad \forall X \in \Gamma(TM),$$

where TX and LX denote the tangential and transversal component of $\bar{\phi}f_2X$, respectively. Therefore we have $\bar{\phi}f_1X \in \Gamma(ltr(TM))$, $TX \in \Gamma(D)$ and $LX \in \Gamma(S(TM^\perp))$. Similarly, we denote the projections on $ltr(TM)$ and $S(TM^\perp)$ in $tr(TM)$ by t_1 and t_2 , respectively. Then, we get

$$(4.5) \quad U = t_1U + t_2U, \quad \forall U \in \Gamma(tr(TM)).$$

On applying $\bar{\phi}$ to (4.5), we have

$$(4.6) \quad \bar{\phi}U = \bar{\phi}t_1U + \bar{\phi}t_2U,$$

which gives

$$(4.7) \quad \bar{\phi}U = \bar{\phi}t_1U + P_1U + P_2U,$$

where P_1U and P_2U denote tangential and transversal component of $\bar{\phi}t_2U$, respectively. Hence, we have $\bar{\phi}t_1U \in \Gamma(\text{Rad}TM)$, $P_1U \in \Gamma(D)$ and $P_2U \in \Gamma(S(TM^\perp))$.

By use of (2.7), (2.8), (2.9), (2.23), (4.4) and (4.7), and comparing tangential, lightlike transversal and screen transversal components, we have

$$(4.8) \quad -\bar{g}(X, Y)\xi + \eta(Y)X = \nabla_X TY - A_{LY}X - A_{\bar{\phi}f_1Y}X \\ - T\nabla_X Y + P_1h^s(X, Y) + \bar{\phi}h^l(X, Y),$$

$$(4.9) \quad h^l(X, TY) + D^l(X, LY) + \nabla_X^l \bar{\phi}f_1Y = \bar{\phi}f_1\nabla_X^l Y,$$

$$(4.10) \quad D^s(X, \bar{\phi}f_1Y) + h^s(X, TY) = P_2h^s(X, Y) - \nabla_X^s LY + L\nabla_X Y.$$

Theorem 4.1. *Let M be a $2q$ -null submanifold of a para-Sasakian manifold \bar{M} with $\xi \in \Gamma(TM)$ such that $\bar{\phi}(\text{Rad}TM) = \text{ltr}(TM)$. Then M is a screen slant radical transversal null submanifold if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$P^2X = \lambda(X - \eta(X)\xi), \forall X \in \Gamma(D).$$

Proof. Assume that there exists a constant λ such that for all $X \in \Gamma(D)$, $P^2X = \lambda(X - \eta(X)\xi) = \lambda\bar{\phi}^2X$. Then we have

$$\begin{aligned} \cos \theta(X) &= \frac{g(\bar{\phi}X, PX)}{\|PX\| \|\bar{\phi}X\|} \\ &= -\frac{g(X, \bar{\phi}PX)}{\|PX\| \|\bar{\phi}X\|} \\ &= -\frac{g(X, P^2X)}{\|PX\| \|\bar{\phi}X\|} \\ &= -\frac{g(X, \bar{\phi}^2X)}{\|PX\| \|\bar{\phi}X\|} \\ &= \lambda \frac{g(\bar{\phi}X, \bar{\phi}X)}{\|PX\| \|\bar{\phi}X\|}. \end{aligned}$$

From above equation, we find

$$(4.11) \quad \cos \theta(X) = \lambda \frac{\|\bar{\phi}X\|}{\|PX\|}.$$

On the other hand, $\|PX\| = \|\bar{\phi}X\| \cos \theta(X)$, implies

$$(4.12) \quad \cos \theta(X) = \frac{\|PX\|}{\|\bar{\phi}X\|}.$$

From (4.11) and (4.12), we get

$$\cos^2 \theta(X) = \lambda \text{ (constant)}.$$

Thus M is a screen slant radical transversal null submanifold.

On the other hand, suppose that M is a screen slant radical transversal null submanifold. Then $\cos^2 \theta(X) = \lambda$ (constant). From (4.12), we have

$$\frac{\|PX\|^2}{\|\bar{\phi}X\|^2} = \lambda.$$

Now $g(PX, PX) = \lambda g(\bar{\phi}X, \bar{\phi}X)$, which gives $g(X, P^2X) = \lambda g(X, \bar{\phi}^2X)$. Therefore $g(X, (P^2 - \lambda\bar{\phi}^2)X) = 0$. Since X is a non-null vector, we obtain

$$\begin{aligned} P^2 - \lambda\bar{\phi}^2X &= 0, \\ P^2 &= \lambda\bar{\phi}^2X = \lambda(X - \eta(X)\xi). \end{aligned}$$

This completes the proof. \square

Theorem 4.2. *Let M be a screen slant radical transversal null submanifold of a para-Sasakian manifold \bar{M} . Then $RadTM$ is integrable if and only if*

$$D^s(Y, \bar{\phi}X) = D^s(X, \bar{\phi}Y) \quad \text{and} \quad A_{\bar{\phi}X}Y = A_{\bar{\phi}Y}X,$$

for all $X, Y \in \Gamma(RadTM)$.

Proof. Suppose that M is a screen slant radical transversal null submanifold of a para-Sasakian manifold \bar{M} . From (4.10), we get

$$(4.13) \quad D^s(X, \bar{\phi}Y) = P_2h^s(X, Y) + L\nabla_XY, \quad X, Y \in \Gamma(RadTM).$$

Interchanging the roles of X and Y in (4.13), we obtain

$$(4.14) \quad D^s(Y, \bar{\phi}X) = P_2h^s(Y, X) + L\nabla_YX.$$

From (4.13) with (4.14), we have

$$(4.15) \quad D^s(X, \bar{\phi}Y) - D^s(Y, \bar{\phi}X) = L(\nabla_XY - \nabla_YX) = L[X, Y].$$

By using (4.8), we get

$$(4.16) \quad A_{\bar{\phi}Y}X + T\nabla_YX = P_1h^s(X, Y) + \bar{\phi}h^l(X, Y).$$

Again interchanging roles of X and Y in (4.16), we get

$$(4.17) \quad A_{\bar{\phi}X}Y + T\nabla_X Y = P_1 h^s(Y, X) + \bar{\phi}h^l(Y, X).$$

From (4.16) and (4.17), we obtain

$$(4.18) \quad A_{\bar{\phi}X}Y - A_{\bar{\phi}Y}X = T[X, Y].$$

In view of (4.15) and (4.18), the proof completes. \square

Theorem 4.3. *Let M be a screen slant radical transversal null submanifold of a para-Sasakian manifold \bar{M} . Then the distribution D is integrable if and only if*

$$h^l(X, TY) + D^l(X, LY) = h^l(Y, TX) + D^l(Y, LX),$$

for all $X, Y \in \Gamma(D)$.

Proof. From (4.9), we get

$$(4.19) \quad h^l(X, TY) + D^l(X, LY) = \bar{\phi}f_1 \nabla_X Y, \quad X, Y \in \Gamma(D).$$

If we change the roles of X and Y in (4.19), we find

$$(4.20) \quad h^l(Y, TX) + D^l(Y, LX) = \bar{\phi}f_1 \nabla_Y X.$$

From (4.19) and (4.20), we have

$$(4.21) \quad h^l(X, TY) - h^l(Y, TX) + D^l(X, LY) - D^l(Y, LX) = \bar{\phi}f_1[X, Y].$$

The proof follows from (4.21). \square

Theorem 4.4. *Let M be a screen slant radical transversal null submanifold of a para-Sasakian manifold \bar{M} . Then $S(TM)$ defines a totally geodesic foliation if and only if*

$$\bar{g}(A_{\bar{\phi}N}^* X, TY) = -\bar{g}(D^l(X, LY), \bar{\phi}N),$$

for all $X, Y \in \Gamma(S(TM))$ and $N \in \Gamma(\text{ltr}(TM))$.

Proof. We know that, $S(TM)$ defines a totally geodesic foliation if and only if

$$\bar{g}(\nabla_X Y, N) = 0, \quad X, Y \in \Gamma(S(TM)), \quad N \in \Gamma(\text{ltr}(TM)).$$

From (2.7), it can be easily seen that

$$(4.22) \quad \bar{g}(\nabla_X Y, N) = \bar{g}(\bar{\nabla}_X Y, N).$$

From (2.20), (2.23) and (4.22), we obtain

$$(4.23) \quad \bar{g}(\nabla_X Y, N) = -\bar{g}(\bar{\nabla}_X \bar{\phi}Y, \bar{\phi}N).$$

By use of (2.7), (2.8), (4.4) and (4.23), we get

$$\bar{g}(\nabla_X Y, N) = -\bar{g}(h^l(X, TY) + D^l(X, LY), \bar{\phi}N).$$

From (2.13), we have

$$\bar{g}(\nabla_X Y, N) = -\bar{g}(A_{\bar{\phi}N}^* X, TY) - \bar{g}(D^l(X, LY), \bar{\phi}N).$$

Thus the proof completes. \square

Theorem 4.5. *Let M be a screen slant radical transversal null submanifold of a para-Sasakian manifold \bar{M} . Then the distribution $\mathring{D} = \text{Rad}TM \perp \{\xi\}$ defines a totally geodesic foliation on M if and only if*

$$A_{LZ} = h^*(X, TV) + \eta(V)X,$$

for all $X \in \Gamma(\mathring{D})$ and $V \in \Gamma(D)$.

Proof. By using the definition of radical transversal null submanifold, \mathring{D} is a totally geodesic foliation if and only if

$$\bar{g}(\nabla_X Y, V) = 0, \quad X, Y \in \Gamma(\mathring{D}), V \in \Gamma(S(TM)).$$

From (2.7), we obtain

$$(4.24) \quad \bar{g}(\nabla_X Y, V) = -\bar{g}(Y, \bar{\nabla}_X V).$$

In view of equations (2.7), (2.20), (2.23) and (4.24), we get

$$\bar{g}(\nabla_X Y, V) = \bar{g}(\bar{\phi}Y, X)\eta(V) + \bar{g}(\bar{\phi}Y, \nabla_X \bar{\phi}V).$$

By using (2.7), (2.8), (2.10), (4.4) and (4.23), we can write

$$\bar{g}(\nabla_X Y, V) = \eta(V)\bar{g}(\bar{\phi}Y, X) + \bar{g}(\bar{\phi}Y, h^*(X, TV)) - \bar{g}(\bar{\phi}Y, A_{LV}),$$

which implies

$$(4.25) \quad \bar{g}(\nabla_X Y, Z) = \bar{g}(\bar{\phi}Y, -A_{LZ} + h^*(X, TZ) + \eta(Z)X).$$

The proof follows from (4.25). \square

REFERENCES

1. B. E. ACET, S. YÜKSEL PERKTAŞ and E. KILIÇ: *On lightlike geometry of para-Sasakian manifolds*. Scientific World J. (2014), Article ID 696231.
2. B. E. ACET, S. YÜKSEL PERKTAŞ and E. KILIÇ: *Lightlike submanifolds of a para-Sasakian manifold*. General Math Notes **22(2)** (2014), 22–45.

3. B. E. ACET and S. YÜKSEL PERKTAŞ: *On geodesic paracontact CR-lightlike submanifolds*. British J. Math. and Computer Sci. **14(6)** (2016), 1–11.
4. B. O'NEILL: *Semi-Riemannian geometry with appl. to relativity*. Academic Press, 1983.
5. B. ŞAHİN: *Screen slant lightlike submanifolds*. Int Elect J of Geom. **2** (2009), 41–54.
6. B. ŞAHİN: *Transversal lightlike submanifolds of indefinite Kaehler manifolds*. An Univ Vest Timis Ser Mat. **44** (2006), 119–145.
7. B. ŞAHİN: *Screen transversal lightlike submanifolds of indefinite Kaehler manifolds*. Chaos, Solutions and Fractals **38** (2008), 1439–1448.
8. B. ŞAHİN, C. YILDIRIM: *Slant lightlike submanifolds of indefinite Sasakian manifolds*. Faculty of Sci and Math Univ of Nis **2** (2012), 71–81.
9. C. YILDIRIM, B. ŞAHİN: *Transversal lightlike submanifolds of indefinite Sasakian manifolds*. Turk J Math. **34** (2010), 561–583.
10. D. V. ALEKSEEVSKI, V. CORTÉS, A. S. GALAEV and T. LEISTNER: *Cones over pseudo-Riemannian manifolds and their holonomy*. J Reine Angew Math. **635** (2009), 23–69.
11. D. V. ALEKSEEVSKI, C. MEDORI and A. TOMASSINI: *Maximally homogeneous para-CR manifolds*. Ann Global Anal Geom. **30** (2006), 1–27.
12. F. MASSAMBA: *Totally contact umbilical lightlike hypersurfaces of indefinite Sasakian manifolds*. Kodai Math J. **31** (2008), 338–358.
13. K. L. DUGGAL and A. BEJANCU: *Lightlike submanifolds of semi-Riemannian manifolds and applications*. Mathematics and Its Applications, Kluwer Publisher, 1996.
14. K. L. DUGGAL and B. ŞAHİN: *Differential geometry of lightlike submanifolds*. Frontiers in Mathematics, 2010.
15. K. L. DUGGAL and B. ŞAHİN: *Lightlike submanifolds of indefinite submanifolds*. Int J of Math and Math Sci. (2007), Article ID 57585.
16. K. L. DUGGAL and B. ŞAHİN: *Screen Cauchy Riemann lightlike submanifolds*. Acta Mathematica Hungarica **106(1-2)** (2005), 125–153.
17. K. L. DUGGAL and D. H. JIN: *Totally umbilical lightlike submanifolds*. Kodai Math J. **26(1)** (2003), 49–63.
18. S. ERDEM: *On almost (para)contact (hyperbolic) metric manifolds and harmonicity of (φ, φ') -holomorphic maps between them*. Houston J Math. **28** (2002), 21–45.
19. S. KANEYUKI and M. KONZAI: *Paracomplex structure and affine symmetric spaces*. Tokyo J Math. **8** (1985), 301–308.
20. S. YÜKSEL PERKTAŞ, E. KILIÇ and B. E. ACET: *Lightlike hypersurfaces of para-Sasakian space form*. Gulf J Math. **2(2)** (2014), 7–18.
21. S. ZAMKOVY: *Canonical connection on paracontact manifolds*. Ann Glob Anal Geo. **36** (2009), 37–60.
22. V. CORTÉS, C. MAYER, T. MOHAUPT and F. SAUERESSING: *Special geometry of Euclidean supersymmetry 1. Vector multiplets*. J High Energy Phys. **0403(028)** (2004), 73pp.

23. V. CORTÉS, M. A. LAWN and L. SCHÄFER: *Affine hyperspheres associated to special para-Kähler manifolds*. Int J Geom Methods Mod Phys. **3** (2006), 995–1009.

Bilal Eftal ACET
Faculty of Arts and Sciences
Department of Mathematics
P. O. Box 02040
Adiyaman, Turkey
eacet@adiyaman.edu.tr

Selcen YÜKSEL PERKTAŞ
Faculty of Arts and Sciences
Department of Mathematics
P.O. Box 02040
Adiyaman, Turkey
sperktas@adiyaman.edu.tr