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COMMON FIXED POINT THEOREMS IN FUZZY METRIC SPACES EMPLOYING CLR_S AND $JCLR_{ST}$ PROPERTIES

Saurabh Manro and Calogero Vetro

Abstract. In this paper, we utilize the CLR_S and $JCLR_{ST}$ properties to prove some existence theorems of common fixed point for contractive mappings in fuzzy metric spaces. Our results generalize and extend many known results from the literature. An example and some applications are given to show the usability of the presented results.

1. Introduction

In 1975, Kramosil and Michalek [22] introduced the notion of fuzzy metric space which could be considered as generalization of probabilistic metric space due to Menger [26], see also [21]. Fixed point theory in fuzzy metric spaces has been developed starting with the work of Heilpern [16]. In [12, 13], George and Veeramani modified the notion given by Kramosil and Michalek, in order to introduce a Hausdorff topology on fuzzy metric spaces. Many authors have contributed to the development of this theory and its applications, for instance [6, 9, 10, 11, 14, 15, 18, 17, 19, 24, 27, 28, 29, 31, 32, 36, 35, 34]. In 2002, Aamri and El Moutawakil [1] defined the property (E.A.) for self-mappings whose class contains the class of noncompatible as well as compatible mappings. It is observed that the property (E.A.) requires the containment and closedness of ranges for the existence of fixed points. In 2009, Abbas et al. [2] introduced the notion of common property (E.A). Later on, Sintunavarat and Kumam [33] coined the idea of "common limit in the range property" which does not require the closedness of the subspaces for the existence of fixed point for a pair of mappings. In 2012, Manro et al. [25] defined the notion of CLR_S property which does not require completeness or closedness of subspaces but only requires containment of any one pair of ranges, see also [3].

Recently, Chauhan et al. [7] defined the notion of $JCLR_{ST}$ property which does not require closedness of subspaces for the existence of fixed points for two pairs of mappings. In this paper, we utilize the CLR_S and $JCLR_{ST}$ properties to prove

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some existence results of common fixed points for contractive mappings in fuzzy metric spaces (in the sense of Kramosil and Michalek or of George and Veeramani). An example and some applications are given to show the usability of the presented results.

2. Preliminaries

The following definitions and results will be needed in the sequel.

Definition 2.1. ([30]) A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous triangular norm (t-norm) if it satisfies the following conditions:

(i) $*$ is associative and commutative;

(ii) $*$ is continuous;

(iii) $a * 1 = a$ for all $a \in [0,1]$;

(iv) $a * b \le c * d$, whenever $a \le c$ and $b \le d$, for all $a, b, c, d \in [0,1]$.

Three basic examples of continuous t-norms are $a *_1 b = \min\{a, b\}, a *_2 b = ab$ and $a *_3 b = \max\{a + b - 1, 0\}$.

Definition 2.2. ([22]) A fuzzy metric space is a triple $(X, M, *)$, where X is a non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times [0, +\infty)$, satisfying the following properties:

(K1) $M(x, y, 0) = 0$ for all $x, y \in X$;

(K2) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$;

(K3) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and for all $t > 0$;

(K4) $M(x, y, \cdot) : [0, +\infty) \to [0, 1]$ is left continuous for all $x, y \in X$;

(K5) $M(x, z, t + s) \ge M(x, y, t) * M(y, z, s)$ for all $x, y, z \in X$ and for all $t, s > 0$.

We denote such space as KM-fuzzy metric space.

Lemma 2.1. ([22]) *In a KM-fuzzy metric space $(X, M, *)$, $M(x, y, \cdot)$ is non-decreasing for all $x, y \in X$.*

If, in the definition of Kramosil and Michalek [22], M is a fuzzy set on $X \times X \times (0, +\infty)$ and (K1), (K2), (K4) are replaced, respectively, with (G1), (G2), (G4) below, then $(X, M, *)$ is called a fuzzy metric space in the sense of George and Veeramani [12].

(G1) $M(x,y,t) > 0$ for all $t > 0$;

(G2) $M(x,x,t) = 1$ for all $t > 0$ and if $M(x,y,t) = 1$ for some $t > 0$, then $x = y$;

(G4) $M(x,y,\cdot) : (0,+\infty) \to [0,1]$ is continuous for all $x, y \in X$.

We denote such space as GV-fuzzy metric space.

Definition 2.3. ([12]) Let $(X, M, *)$ be a fuzzy metric space. Then a sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be convergent to $x \in X$, that is, $\lim_{n\to+\infty} x_n = x$, if, for all $t > 0$, $\lim_{n\to+\infty} M(x_n, x, t) = 1$.

Definition 2.4. ([8]) Two self-mappings f and g of a fuzzy metric space $(X, M, *)$ are said to be compatible if $\lim_{n\to+\infty} M(fgx_n, gfx_n) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = z$ for some $z \in X$.

Definition 2.5. ([8]) Two self-mappings f and g of a fuzzy metric space $(X, M, *)$ are said to be non-compatible if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = z$ for some $z \in X$, but for some $t > 0$, either $\lim_{n\to+\infty} M(fgx_n, gfx_n) \neq 1$ or the limit does not exist.

Definition 2.6. ([20]) A pair (f, g) of self-mappings of a non-empty set X is said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, that is, if $fz = gz$ for some $z \in X$, then $fgz = gfz$.

If two self-mappings A and S of a fuzzy metric space $(X, M, *)$ are compatible then they are weakly compatible but the converse need not be true.

Definition 2.7. ([1]) A pair (f, g) of self-mappings of a fuzzy metric space $(X, M, *)$ is said to satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = z$, for some $z \in X$.

From Definition 2.7, it is easy to see that any two non-compatible self-mappings of a fuzzy metric space $(X, M, *)$ satisfy the property (E.A) but the reverse need not be true.

Definition 2.8. ([2]) Two pairs (A, S) and (B, T) of self-mappings of a fuzzy metric space $(X, M, *)$ are said to satisfy the common property (E.A) if there exist two sequences $\{x_n\}$, $\{y_n\}$ in X such that

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} By_n = \lim_{n\to+\infty} Ty_n = z,$$

for some $z \in X$.

Definition 2.9. ([33]) A pair of self-mappings (f, g) of a fuzzy metric space $(X, M, *)$ is said to satisfy the common limit in the range of g property (CLR_g, for short) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = gz$, for some $z \in X$.

Inspired by Sintunavarat and Kumam [33], Manro et al. [25] introduced the following notion:

Definition 2.10. ([25]) Two pairs (A, S) and (B, T) of self-mappings of a fuzzy metric space $(X, M, *)$ are said to share the common limit in the range of S property if there exist two sequences $\{x_n\}$, $\{y_n\}$ in X such that

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} By_n = \lim_{n\to+\infty} Ty_n = Sz,$$

for some $z \in X$.

Very recently, Chauhan et al. [7] introduced the following property:

Definition 2.11. ([7]) Two pairs (A, S) and (B, T) of self-mappings of a fuzzy metric space $(X, M, *)$ are said to satisfy $JCLR_{ST}$ property if there exist two sequences $\{x_n\}$, $\{y_n\}$ in X such that

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} By_n = \lim_{n\to+\infty} Ty_n = Tz = Sz,$$

for some $z \in X$.

Definition 2.12. ([17]) Two families of self-mappings $\{A_i\}$ and $\{S_j\}$ are said to be pairwise commuting if:

1. $A_iA_j = A_jA_i$, $i, j \in \{1, 2, \ldots, m\}$,
2. $S_iS_j = S_jS_i$, $i, j \in \{1, 2, \ldots, n\}$,
3. $A_iS_j = S_jA_i$, $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, n\}$.

3. Main Results

Our results involve the class Φ of all functions $\phi : [0, 1] \to [0, 1]$ satisfying the following properties:

(A) ϕ is continuous and non-decreasing on $[0, 1]$;

(B) $\phi(x) > x$ for all $x \in (0, 1)$.

Clearly by using properties (A) and (B), we also have:

(C) $\phi(1) = 1$;

(D) $\phi(x) \geq x$ for all $x \in [0, 1]$.

The basic example of function $\phi \in \Phi$ is $\phi(x) = \sqrt{x}$, for all $x \in [0, 1]$.

We begin with the following theorem:

Theorem 3.1. *Let A, B, S and T be four self-mappings of a KM-fuzzy metric space $(X, M, *)$ satisfying the following conditions:*

(i) for all $x, y \in X$ and $t > 0$ with $0 < M(Ax, By, t) < 1$, there exists $\phi \in \Phi$ such that

$$\begin{aligned} M(Ax, By, t) \quad \geq \quad & \phi(\min\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), \\ & M(By, Sx, t), M(Ax, Ty, t)\}); \end{aligned}$$

(ii) (A, S) and (B, T) share the CLR_S property (or CLR_T property);

(iii) $A(X) \subset T(X)$ (or $B(X) \subset S(X)$).

Then, the pairs (A, S) and (B, T) have a coincidence point. Further if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.

Proof. Since the pairs (A, S) and (B, T) share the common limit in the range of S property, then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n \to +\infty} Ax_n = \lim_{n \to +\infty} Sx_n = \lim_{n \to +\infty} By_n = \lim_{n \to +\infty} Ty_n = Sz,$$

for some $z \in X$. Firstly, we assert that $Az = Sz$, or equivalently, $M(Az, Sz, t) = 1$. Suppose not, that is $0 < M(Az, Sz, t) < 1$ for all $t > 0$. Then by using (i), we have

$$\begin{aligned} M(Az, By_n, t) \quad \geq \quad & \phi(\min\{M(Sz, Ty_n, t), M(Az, Sz, t), M(By_n, Ty_n, t), \\ & M(By_n, Sz, t), M(Az, Ty_n, t)\}), \end{aligned}$$

and taking the limit as $n \to \infty$, we get

$$\begin{aligned} (3.1) \qquad M(Az, Sz, t) \quad & \geq \quad \phi(\min\{M(Sz, Sz, t), M(Az, Sz, t), M(Sz, Sz, t), \\ & \qquad M(Sz, Sz, t), M(Az, Sz, t)\}) \\ & = \quad \phi(\min\{1, M(Az, Sz, t), 1, 1, M(Az, Sz, t)\}) \\ & = \quad \phi(M(Az, Sz, t)). \end{aligned}$$

As we know, by definition of KM-fuzzy metric space, $M(Az, Sz, \cdot)$ is left-continuous and by Lemma 2.1, $M(Az, Sz, \cdot)$ is non-decreasing. Thus, it has at most countable points of discontinuity. Since $0 < M(Az, Sz, t) < 1$ for all $t > 0$, then

$0 < M(Az, Sz, t_0) < 1$ for some $t_0 > 0$. Let t_0 be a point where $M(Az, Sz, \cdot)$ is continuous and thus by using the definition of ϕ, from (3.1), we get

$$M(Az, Sz, t_0) \geq \phi(M(Az, Sz, t_0)) > M(Az, Sz, t_0),$$

which is a contradiction. Therefore, $Az = Sz$ and hence z is a coincidence point of the pair (A, S). Since, $A(X) \subset T(X)$, there exists $v \in X$ such that $Az = Tv$.

Secondly, we assert that $Bv = Tv$. If not, that is $0 < M(Bv, Tv, t) < 1$ for all $t > 0$, then by (i), we get

$$\begin{aligned} M(Tv, Bv, t) &= M(Az, Bv, t) \\ &\geq \phi(\min\{M(Sz, Tv, t), M(Az, Sz, t), M(Bv, Tv, t), \\ &\qquad M(Bv, Sz, t), M(Az, Tv, t)\}) \\ &= \phi(\min\{M(Tv, Tv, t), M(Tv, Tv, t), M(Bv, Tv, t), \\ &\qquad M(Bv, Tv, t), M(Tv, Tv, t)\}) \\ &= \phi(M(Tv, Bv, t)). \end{aligned} \tag{3.2}$$

By definition of KM-fuzzy metric space, $M(Tv, Bv, \cdot)$ is left-continuous and by Lemma 2.1, $M(Tv, Bv, \cdot)$ is non-decreasing. Thus, it has at most countable points of discontinuity. Since $0 < M(Tv, Bv, t) < 1$ for all $t > 0$, then $0 < M(Tv, Bv, t_0) < 1$ for some $t_0 > 0$. Let t_0 be a point where $M(Tv, Bv, \cdot)$ is continuous and thus by using the definition of ϕ in (3.2), we get

$$M(Tv, Bv, t_0) \geq \phi(M(Tv, Bv, t_0)) > M(Tv, Bv, t_0),$$

which is a contradiction. Therefore, $Bv = Tv$ and hence v is a coincidence point of the pair (B, T). Thus, we have $u = Tv = Bv = Az = Sz$. Since the pairs (A, S) and (B, T) are weakly compatible, this gives $Au = ASz = SAz = Su$ and $Bu = BTv = TBv = Tu$. Finally, we assert that $Au = u$. Again suppose not, that is $0 < M(Au, u, t) < 1$ for all $t > 0$. Then by (i), we get

$$\begin{aligned} M(Au, u, t) &= M(Au, Bv, t) \\ &\geq \phi(\min\{M(Su, Tv, t), M(Au, Su, t), M(Bv, Tv, t), \\ &\qquad M(Bv, Su, t), M(Au, Tv, t)\}) \\ &= \phi(\min\{M(Au, u, t), M(Au, Au, t), M(u, u, t), \\ &\qquad M(u, Au, t), M(Au, u, t)\}) \\ &= \phi(M(Au, u, t)). \end{aligned} \tag{3.3}$$

Again, by definition of KM-fuzzy metric space, $M(Au, u, \cdot)$ is left-continuous and by Lemma 2.1, $M(Au, u, \cdot)$ is non-decreasing. Thus, it has at most countable points of discontinuity. Since $0 < M(Au, u, t) < 1$ for all $t > 0$, then $0 < M(Au, u, t_0) < 1$ for some $t_0 > 0$. Let t_0 be a point where $M(Au, u, \cdot)$ is continuous and thus by using the definition of ϕ in (3.3), we get

$$M(Au, u, t_0) \geq \phi(M(Au, u, t_0)) > M(Au, u, t_0),$$

which is a contradiction. Therefore $Au = u = Su$, which gives u is a common fixed point of A and S. Similarly, one can easily prove that $Bu = u = Tu$, that is u is a common fixed point of B and T. Therefore u is a common fixed point of A, S, B and T. Uniqueness of the common fixed point is an easy consequence of condition (i) and hence we omit details. □

Now we attempt to drop containment of subspaces by using weaker condition $JCLR_{ST}$ in Theorem 3.1.

Theorem 3.2. *Let A, B, S and T be four self-mappings of a KM-fuzzy metric space $(X, M, *)$ satisfying the following conditions:*

(i) for all $x, y \in X$ and $t > 0$ with $0 < M(Ax, By, t) < 1$, there exists $\phi \in \Phi$ such that

$$\begin{aligned} M(Ax, By, t) \quad \geq \quad & \phi(\min\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), \\ & M(By, Sx, t), M(Ax, Ty, t)\}); \end{aligned}$$

(ii) (A, S) and (B, T) share the $JCLR_{ST}$ property.

Then, the pairs (A, S) and (B, T) have a coincidence point. Further if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.

Proof. Since the pairs (A, S) and (B, T) satisfy the $JCLR_{ST}$ property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} By_n = \lim_{n\to+\infty} Ty_n = Tz = Sz,$$

for some $z \in X$. Firstly, by using the same arguments in Theorem 3.1, we can easily show that $Az = Sz$ and hence z is a coincidence point of the pair (A, S).

Now, we assert that $Bz = Tz$. Suppose not, that is $0 < M(Bz, Tz, t) < 1$ for all $t > 0$. Then by (i), we get

$$\begin{aligned} (3.4) \quad M(Tz, Bz, t) &= M(Az, Bz, t) \\ &\geq \phi(\min\{M(Sz, Tz, t), M(Az, Sz, t), M(Bz, Tz, t), \\ &\qquad M(Bz, Sz, t), M(Az, Tz, t)\}) \\ &= \phi(\min\{M(Tz, Tz, t), M(Tz, Tz, t), M(Bz, Tz, t), \\ &\qquad M(Bz, Tz, t), M(Tz, Tz, t)\}) \\ &= \phi(M(Tz, Bz, t)). \end{aligned}$$

As we know, by definition of KM-fuzzy metric space, $M(Tz, Bz, \cdot)$ is left-continuous and by Lemma 2.1, $M(Tz, Bz, \cdot)$ is non-decreasing. Thus, it has at most countable points of discontinuity. Since $0 < M(Tz, Bz, t) < 1$ for all $t > 0$, then

$0 < M(Tz, Bz, t_0) < 1$ for some $t_0 > 0$. Let t_0 be a point where $M(Tz, Bz, \cdot)$ is continuous and thus by using the definition of ϕ in (3.4), we get

$$M(Tz, Bz, t_0) \geq \phi(M(Tz, Bz, t_0)) > M(Tz, Bz, t_0),$$

which is a contradiction. Therefore, $Bz = Tz$ and hence z is a coincidence point of the pair (B, T). Thus, we have $Tz = Bz = Az = Sz$. The rest of the proof is the same of Theorem 3.1 and hence we omit details. □

Remark 3.1. The conclusions of Theorems 3.1 and 3.2 remain true if $(X, M, *)$ is a GV-fuzzy metric space instead of a KM-fuzzy metric space. Precisely, in the proofs of analogous of Theorems 3.1 and 3.2 in a GV-fuzzy metric space, we have only to consider the fact that the fuzzy set M is a continuos function instead of a left continuous function.

The following example illustrates some hypotheses of Theorem 3.1.

Example 3.1. Let $(X, M, *)$ be a KM-fuzzy metric space, where $X = [1, 15)$ with the t-norm defined by $a * b = \min\{a, b\}$ and the fuzzy set M given by

$$M(x, y, t) = \begin{cases} 1 & \text{if } x = y \text{ and } t > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Also define $A, B, S, T : X \to X$ by

$$Ax = \begin{cases} 1 & \text{if } x \in \{1\} \cup (3, 15), \\ x + 6 & \text{if } x \in (1, 3], \end{cases} \qquad Bx = \begin{cases} 1 & \text{if } x \in \{1\} \cup (3, 15), \\ x + 5 & \text{if } x \in (1, 3], \end{cases}$$

$$Sx = \begin{cases} 1 & \text{if } x = 1, \\ 6 & \text{if } x \in (1, 3], \\ \frac{x+1}{4} & \text{if } x \in (3, 15), \end{cases} \qquad Tx = \begin{cases} 1 & \text{if } x = 1, \\ 11 & \text{if } x \in (1, 3], \\ x - 2 & \text{if } x \in (3, 15). \end{cases}$$

If we choose two sequences in X as $\{x_n\} = \{y_n\} = \left\{3 + \frac{1}{n}\right\}$, then the pairs (A, S) and (B, T) satisfy the CLR_S property since

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} By_n = \lim_{n\to+\infty} Ty_n = S(1) = 1 \in X.$$

We note that $A(X) = \{1\} \cup (7, 9]$, $B(X) = \{1\} \cup (6, 8]$, $S(X) = [1, 4) \cup \{6\}$ and $T(X) = [1, 13)$ and so $A(X) \subset T(X)$ but $B(X) \not\subset S(X)$. Finally, in view of the definition of M, the contractive condition of Theorem 3.1 need not to be checked in our case. Thus, we conclude that $u = 1$ is the unique common fixed point of the pairs (A, S) and (B, T), which also remains a point of coincidence as well. Moreover, it should be noted that $A(X), B(X), S(X)$ and $T(X)$ are not closed subspaces of X.

Further, by putting $A = B$ and $S = T$ in Theorem 3.1, we deduce the following result for two self-mappings.

Corollary 3.1. *Let A and S be two self-mappings of a KM-fuzzy metric space $(X, M, *)$ satisfying the following conditions:*

(i) for all $x, y \in X$ and $t > 0$ with $0 < M(Ax, Ay, t) < 1$, there exists $\phi \in \Phi$ such that

$$M(Ax, Ay, t) \geq \phi(\min\{M(Sx, Sy, t), M(Ax, Sx, t), M(Ay, Sy, t), M(Ay, Sx, t), M(Ax, Sy, t)\});$$

(ii) (A, S) has the CLR_S property.

Then, the pair (A, S) has a coincidence point. Further if (A, S) is weakly compatible, then A and S have a unique common fixed point in X.

4. Applications

In this Section we apply the results obtained in Section 3. to solve two special problems.

4.1. Finite families of mappings

As an application of Theorems 3.1 and 3.2, we prove a common fixed point theorem for four finite families of mappings on fuzzy metric spaces. While proving our result, we utilize Definition 2.12 which is a natural extension of commutativity condition to two finite families.

Theorem 4.1. *Let $\{A_1, A_2, \dots, A_m\}$, $\{B_1, B_2, \dots, B_n\}$, $\{S_1, S_2, \dots, S_p\}$ and $\{T_1, T_2, \dots, T_q\}$ be four finite families of self-mappings of a KM-fuzzy metric space $(X, M, *)$ such that $A = A_1A_2 \cdots A_m$, $B = B_1B_2 \cdots B_n$, $S = S_1S_2 \cdots S_p$ and $T = T_1T_2 \cdots T_q$ satisfy the conditions of Theorem 3.1 (or Theorem 3.2). Then*

(a) the pairs (A, S) and (B, T) have a point of coincidence each;

(b) $\{A_i\}$, $\{B_j\}$, $\{S_k\}$ and $\{T_r\}$ have a unique common fixed point provided that the pairs of families $(\{A_i\}, \{S_k\})$ and $(\{B_j\}, \{T_r\})$ commute pairwise, for all $i = 1, \dots, m$, $j = 1, \dots, n$, $k = 1, \dots, p$ and $r = 1, \dots, q$.

Proof. Since the pairs of families $(\{A_i\}, \{S_k\})$ and $(\{B_j\}, \{T_r\})$ commute pairwise, we first show that $AS = SA$. In fact, we have

$$\begin{aligned} AS &= (A_1A_2\cdots A_m)(S_1S_2\cdots S_p) \\ &= (A_1A_2\cdots A_{m-1})(A_mS_1S_2\cdots S_p) \\ &= (A_1A_2\cdots A_{m-1})(S_1S_2\cdots S_pA_m) \\ &= (A_1A_2\cdots A_{m-2})(A_{m-1}S_1S_2\cdots S_pA_m) \\ &= (A_1A_2\cdots A_{m-2})(S_1S_2\cdots S_pA_{m-1}A_m) \\ &= \ldots \\ &= A_1(S_1S_2\cdots S_pA_2\cdots A_m) \\ &= (S_1S_2\cdots S_p)(A_1A_2\cdots A_m) = SA. \end{aligned}$$

Similarly one can prove that $BT = TB$; therefore the pairs (A, S) and (B, T) are weakly compatible. Now, using Theorem 3.1 (or Theorem 3.2), we conclude that A, S, B and T have a unique common fixed point in X, say z. Now, we need to prove that z remains the fixed point of all component mappings. To this aim, consider

$$\begin{aligned} A(A_iz) &= ((A_1A_2\cdots A_m)A_i)z = (A_1A_2\cdots A_{m-1})(A_mA_i)z \\ &= (A_1A_2\cdots A_{m-1})(A_iA_m)z = (A_1A_2\cdots A_{m-2})(A_{m-1}A_iA_m)z \\ &= (A_1A_2\cdots A_{m-2})(A_iA_{m-1}A_m)z = \cdots = A_1(A_iA_2\cdots A_m)z \\ &= (A_1A_i)(A_2\cdots A_m)z = (A_iA_1)(A_2\cdots A_m)z \\ &= A_i(A_1A_2\cdots A_m)z = A_iAz = A_iz. \end{aligned}$$

Similarly, one can prove that $A(S_kz) = S_k(Az) = S_kz, S(S_kz) = S_k(Sz) = S_kz$, $S(A_iz) = A_i(Sz) = A_iz, B(B_jz) = B_j(Bz) = B_jz$, $B(T_rz) = T_r(Bz) = T_rz$, $T(T_rz) = T_r(Tz) = T_rz$ and $T(B_jz) = B_j(Tz) = B_jz$, which show that (for all i, j, k and r) A_iz and S_kz are other fixed points of the pair (A, S) whereas B_jz and T_rz are other fixed points of the pair (B, T). Since A, B, S and T have a unique common fixed point, then we get $z = A_iz = S_kz = B_jz = T_rz$, for all $i = 1, \ldots, m$, $j = 1, \ldots, n$, $k = 1, \ldots, p$ and $r = 1, \ldots, q$. Thus z is the unique common fixed point of $\{A_i\}$, $\{B_j\}$, $\{S_k\}$ and $\{T_r\}$. □

4.2. Product space

As an application of Corollary 3.1, we prove a common fixed point theorem in the product space $X \times X$. In 2006, Bhaskar and Lakshmikantham [5] introduced the notion of coupled fixed point and proved coupled fixed point results with useful applications to the study of the existence and uniqueness of solution for periodic boundary value problems. Further to this, Lakshmikantham and Ćirić in [23] proved coupled coincidence and coupled common fixed point theorems for commuting mappings that extended the results in [5]. Precisely, we have the following notions.

Definition 4.1. ([5]) Let X be a non-empty set and $F : X \times X \to X$ be a given mapping. We say that $(x, y) \in X \times X$ is a coupled fixed point of F iff $F(x, y) = x$ and $F(y, x) = y$.

Definition 4.2. ([23]) An element $(x, y) \in X \times X$ is called a coupled coincidence point of two mappings $F : X \times X \to X$ and $g : X \to X$ if $gx = F(x, y)$ and $gy = F(y, x)$. Moreover, (gx, gy) is called a coupled point of coincidence.

Definition 4.3. ([23]) An element $(x, y) \in X \times X$ is said to be a common coupled fixed point of two mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx = x$ and $F(y, x) = gy = y$.

In 2011, Aydi et al. [4] extended the concepts above as follows:

Definition 4.4. ([4]) An element $(x, y) \in X \times X$ is called a b-coupled coincidence point of two mappings $F, G : X \times X \to X$ if $G(x, y) = F(x, y)$ and $G(y, x) = F(y, x)$. Moreover, $(G(x, y), G(y, x))$ is called a b-coupled point of coincidence.

Definition 4.5. ([4]) An element $(x, y) \in X \times X$ is called a b-common coupled fixed point of two mappings $F, G : X \times X \to X$ if $x = G(x, y) = F(x, y)$ and $y = G(y, x) = F(y, x)$.

Here, we state and prove the following theorem.

Theorem 4.2. *Let $(X, M, *)$ be a KM-fuzzy metric space. Let $F, G : X \times X \to X$ be two mappings satisfying the following conditions:*

(i) for all $(x, y), (u, v) \in X \times X$ and $t > 0$ with $0 < M(F(x, y), F(u, v), t) < 1$, there exists $\phi \in \Phi$ such that

$$\begin{aligned} M(F(x,y),F(u,v),t) \quad \geq \quad & \phi(\min\{M(G(x,y),G(u,v),t), M(F(x,y),G(x,y),t), \\ & M(F(u,v),G(u,v),t), M(F(u,v),G(x,y),t), \\ & M(F(x,y),G(u,v),t)\}) \end{aligned}$$

(ii) for each $y \in X$, the pair $(F(\cdot, y), G(\cdot, y))$ has the CLR_S property and is weakly compatible;

(iii) for each $z : X \to X$, the pair $(F(z(y), y), G(z(y), y))$ has the CLR_S property and is weakly compatible.

Then, there exists a unique point w such that $F(z(w), w) = G(z(w), w) = z(w) = w$.

Proof. Fix $y = v \in X$ and let $A, S : X \to X$ be such that $F(x, y) = Ax$ and $G(u, y) = Su$, for all $x, u \in X$. Then, condition (i) of Theorem 4.2 reduces to condition (i) of Corollary 3.1 and so, applying Corollary 3.1, the pair (A, S) has a unique common fixed point $z(y)$, that is $f(z(y)) = z(y) = g(z(y))$. Again, we can apply Corollary 3.1 to the self-mappings $F(z(y), y)$ and $G(z(y), y)$ on X and therefore we deduce that there exists a unique point w such that $F(z(w), w) = G(z(w), w) = z(w) = w$. $\square$

4.3. Conclusion

In view of their interesting applications, searching for fixed point theorems in fuzzy metric spaces has received considerable attention through the last decades. In particular, researchers are currently focusing on weaker form of contractive conditions. In this connection, the main aim of this paper is to present some fixed point results involving the so-called "common limit in the range property". The new theory leads to further investigations and applications, for instance in the setting of intuitionistic fuzzy metric spaces.

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Saurabh Manro
Thapar University
School of Mathematics and Computer Applications
Patiala, Punjab, Indiaa
sauravmanro@hotmail.com

Calogero Vetro
Università degli Studi di Palermo
Dipartimento di Matematica e Informatica
Via Archirafi, 34
90123 Palermo, Italy
cvetro@math.unipa.it