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WELL-POSEDNESS AND ASYMPTOTIC STABILITY OF SOLUTIONS TO A BRESSE SYSTEM WITH TIME VARYING DELAY TERMS AND INFINITE MEMORIES *

Mohamed Ferhat and Ali Hakem

Abstract. We consider the Bresse system in bounded domain with delay terms in the internal feedbacks and infinite memories acting in the three equations of the system. First, we prove the global existence of its solutions in Sobolev spaces by means of semigroup theory. Furthermore, the asymptotic stability is given by using an appropriate Lyapunov functional.
Keywords: Energy decay; infinite memories; time-varying delay terms; Bresse system.

1. Introduction

In this paper, we investigate the existence and decay properties of solutions for the initial boundary value problem of the linear Bresse system of the type

$$
(1.1)\qquad \begin{cases}
\rho_1\varphi_{tt} - k_1(\varphi_x+\psi+l\omega)_x - lk_3(\omega_x - l\varphi) + \mu_1\varphi_t + \mu_2\varphi_t(x,t-\tau_1(t)) \\
+ \displaystyle\int_0^\infty g_1(s)\varphi_{xx}(x,t-s)ds = 0, \\
\\
\rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x+\psi+l\omega) + \widetilde{\mu_1}\psi_t + \widetilde{\mu_2}\psi_t(x,t-\tau_2(t)) \\
+ \displaystyle\int_0^\infty g_2(s)\psi_{xx}(x,t-s)ds = 0, \\
\\
\rho_1\omega_{tt} - k_3(\omega_x - l\varphi)_x + lk_1(\varphi_x+\psi+l\omega) + \widetilde{\widetilde{\mu_1}}\omega_t + \widetilde{\widetilde{\mu_2}}\omega_t(x,t-\tau_3(t)) \\
+ \displaystyle\int_0^\infty g_3(s)\omega_{xx}(x,t-s)ds = 0,
\end{cases}
$$

where $(x,t)\in(0,L)\times(0,+\infty)$, $\tau_i(t)>0$ $(i=1,2,3)$ is a time delay, $\mu_1,\mu_2,\widetilde{\mu_1},\widetilde{\mu_2}$, $\widetilde{\widetilde{\mu_1}},\widetilde{\widetilde{\mu_2}}$ are positive real numbers. This system is subject to the Dirichlet boundary

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conditions

$$\varphi(0,t)=\varphi(L,t)=\psi(0,t)=\psi(L,t)=\omega(0,t)=\omega(L,t)=0, \quad t>0$$

and to the initial conditions

$$\begin{cases} \varphi(x,-t)=\varphi_0(x), \quad \varphi_t(x,0)=\varphi_1(x), \quad \psi(x,-t)=\psi_0(x), \quad x\in(0,L), \\ \psi_t(x,0)=\psi_1(x), \quad \omega(x,-t)=\omega_0(x), \quad \omega_t(x,0)=\omega_1(x), \quad x\in(0,L), \\ \varphi_t(x,t-\tau_1(t))=f_0(x,t-\tau_1(t)), \quad \text{in } (0,L)\times[0,\tau_1(0)], \\ \psi_t(x,t-\tau_2(t))=\tilde{f_0}(x,t-\tau_2(t)), \quad \text{in } (0,L)\times[0,\tau_2(0)], \\ \omega_t(x,t-\tau_3(t))=\widetilde{\tilde{f}}_0(x,t-\tau_3(t)), \quad \text{in } (0,L)\times[0,\tau_3(0)]. \end{cases}$$

The initial data $(\varphi_0,\varphi_1,\psi_0,\psi_1,\omega_0,\omega_1,f_0,\tilde{f_0},\widetilde{\tilde{f}}_0)$ belong to a suitable Sobolev space. By ω,ψ and φ we are denoting the longitudinal, vertical and shear angle displacements. The original Bresse system is given by the following equations (see [1])

$$\begin{cases} \rho_1\varphi_{tt}=Q_x+lN+F_1, \\ \rho_2\psi_{tt}=M_x-Q+F_2, \\ \rho_1\omega_{tt}=N_x-lQ+F_3, \end{cases}$$

where we use N,Q and M to denote the axial force, the shear force and the bending moment respectively. These forces are stress-strain relations for elastic behavior and given by

$$N=Eh(\omega_x-l\varphi), \quad Q=Gh(\varphi_x+\psi+l\omega), \quad \text{and } M=EI\psi_x,$$

where G,E,I and h are positive constants. Finally, by the terms F_i we are denoting external forces.
The Bresse system without delay (i.e $\mu_2=\widetilde{\mu_2}=\widetilde{\widetilde{\mu_2}}=0$), is more general than the well-known Timoshenko system where the longitudinal displacement ω is not considered $l=0$. There are a number of publications concerning the stabilization of Timoshenko system with different kinds of damping (see [2], [3], [4] and [5]). For the Timoshenko system, along with the new theory og Green and Naghdi [20], Messaoudi and Said-Houari [21] considered a Timoshenko system of thermoelasticity of type III of the form

$$\begin{cases} \rho_1\varphi_{tt}-k(\varphi_x+\psi)_x=0, &]0,L[\times R_+, \\ \rho_2\psi_{tt}-b\psi_{xx}+k(\varphi_x+\psi)+\beta\theta_x=0, &]0,L[\times R_+, \\ \rho_3\theta_{tt}-\delta\theta_{xx}+\gamma\psi_{ttx}-k\theta_{txx}=0, &]0,L[\times R_+, \end{cases}$$

where φ,ψ and θ are function of (x,t), which model the transverse displacement of the the beam, the rotation angle of the filament and the difference temperature, respectively. They proved an exponential decay in the case of equal speeds $\left(\frac{k}{\rho_1}=\frac{b}{\rho_2}\right)$. This result was later established by Messaoudi and Said-Houari [22] for above system in the presence of a viscoelastic damping of the form

$$\int_0^\infty g(s)\psi_{xx}(x,t-s)ds$$

acting in the second equation. Moreover, the case of nonequal speeds $\left(\frac{k}{\rho_1} \neq \frac{b}{\rho_2}\right)$ was studied and a polynomial decay result was proved for solutions with smooth initial data. A more general decay result, from which the exponential and polynomial rates of decay are only special cases, was also established by Kafini **[23]**. Raposo et al **[6]** proved the exponential decay of the solution for the following linear system of Timoshenko-type beam equations with linear frictional dissipative terms

$$\begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x + \psi + l\omega)_x - lEh(\omega_x - l\varphi) + \mu_1\varphi_t = 0 \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + \psi + l\omega) + \widetilde{\mu_1}\psi_t = 0. \end{cases}$$

Messaoudi and Mustafa **[3]** (see also **[11]**, **[5]**) considered the stabilization for the following Timoshenko system with nonlinear internal feedbacks

$$\begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x + \psi + l\omega)_x - lEh(\omega_x - l\varphi) + g_1(\psi_t) = 0 \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + \psi + l\omega) + g_2(\psi_t) = 0. \end{cases}$$

Time delay is the property of a physical system by which the response to an applied force is delayed in its effect (see **[8]**). Whenever material, information or energy is physically transmitted from one place to another, there is a delay associated with the transmission. In recent years, the PDEs with time delay effects have become an active area of research and arise in many practical problems (see for example **[9]**, **[10]**). The presence of delay may be a source of instability. For example, it was proved in **[11]** that an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay. To stabilize a hyperbolic system involving input delay terms, additional control terms will be necessary (see **[12]** and **[13]**). For instance, in **[12]** the authors studied the wave equation with a linear internal damping term with constant delay and determined suitable relations between μ_1 and μ_2, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they found a sequence of delays for which the solution will be instable if $\mu_2 \geq \mu_1$. The main approach used in **[12]**, is an observability inequality obtained with a Carleman estimate. The same results were showed if both the damping and the delay acting in the boundary domain. We also recall the result by Xu, Yung and Li **[13]**, where the authors proved the same result as in **[12]** for the one space dimension by adopting the spectral analysis approach.
Motivated by the previous works it is interesting to give more general decay result to (1.1), by combining the idea of (**[17]**,**[18]**). Our purpose in this paper is to give a global solvability in Sobolev spaces and energy decay estimates of the solutions to the problem (1.1) for linear damping, time varying delay terms and infinite memories. To obtain global solutions to the problem (1.1), we use the argument combining the semigroup theory (see **[12]** and **[14]**) with the energy estimate method. For the decay estimates, we use a Lyapunov functional's method.

2. Preliminary Results

First assume the following hypotheses:
(H1) τ_i is a function such that

$$\tau_i \in W^{2.\infty}([0,T]), \quad \forall\, T>0, \quad i=1,2,3 \tag{2.1}$$

$$\begin{aligned} 0<\tau_{01}\le \tau_1(t)\le \tau_{11},\ \forall t>0,\\ 0<\tau_{02}\le \tau_2(t)\le \tau_{22},\ \forall t>0,\\ 0<\tau_{03}\le \tau_3(t)\le \tau_{33},\ \forall t>0, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \tau_1'(t)\le d_1<1,\\ \tau_2'(t)\le d_2<1,\\ \tau_3'(t)\le d_3<1, \end{aligned} \tag{2.3}$$

where $\tau_{01},\ \tau_{02},\ \tau_{03}$ and $\tau_{11},\ \tau_{22},\ \tau_{33}$ are two positive constants.
(H2)

$$\begin{aligned} \mu_2 &< \sqrt{1-d_1}\mu_1,\\ \widetilde{\mu_2} &< \sqrt{1-d_2}\widetilde{\mu_1},\\ \widetilde{\widetilde{\mu_2}} &< \sqrt{1-d_3}\widetilde{\widetilde{\mu_1}}, \end{aligned} \tag{2.4}$$

(H3) $g_i:\ R_+\to\ R_+$ are differentiable non-increasing function and integrable on R_+ such that there exists a non-increasing differentiable function $\zeta:R^+\to R^+$ satisfying

$$g_i'(t)\le -\zeta(t)g_i(t),$$

and there exists a positive constant k_0 satisfying, for any $(\varphi,\psi,\omega)\in(H_0^1(]0,L[))^3$

$$\begin{aligned} k_0\int_0^L\left(\varphi_x^2+\psi_x^2+\omega_x^2\right)dx &\le \int_0^L(k_2\psi_x^2+k_1\left(\varphi_x+\psi+l\omega)^2+k_3(\omega_x-l\omega)^2\right)dx\\ &-\int_0^L\left(\int_0^{+\infty} g_1(s)ds\right)\varphi_x^2dx+\int_0^L\left(\int_0^{+\infty} g_2(s)ds\right)\psi_x^2dx\\ &+\int_0^L\left(\int_0^{+\infty} g_3(s)ds\omega_x^2\right)dx. \end{aligned}$$

By contradiction arguments, it is easy to see that there exists a positive constant $\widetilde{k_0}$ such that, for $(\varphi,\psi,\omega)\in(H_0^1(]0,L[))^3$,

$$\begin{aligned} &\widetilde{k_0}\int_0^L\left(\varphi_x^2+\psi_x^2+\omega_x^2\right)dx\le\\ &\int_0^L\left(k_2\psi_x^2+k_1(\varphi_x+\psi+l\omega)^2+k_3(w_x-l\varphi)^2\right)dx. \end{aligned} \tag{2.5}$$

The above inequality will be proved later in lemma 4.1. Also, if

$$g_i^0 := \int_0^{+\infty} g_i(s)ds < \widetilde{k_0}, \quad i = 1, 2, 3, \tag{2.6}$$

then (2.2) is satisfied with

$$k_0 = \widetilde{k_0} - \max\left\{g_1^0, g_2^0, g_3^0\right\}.$$

On the other hand, thanks to Poincare's inequality, there exists a positive constant $\widetilde{k_0}$ such that, for $(\varphi, \psi, \omega) \in (H_0^1(]0, L[))^3$,

$$\begin{aligned}&\int_0^L (k_2\psi_x^2 + k_1\left(\varphi_x + \psi + l\omega)^2 + k_3(\omega_x - l\varphi)^2\right) dx \leq \\ &\widetilde{k_0}\int_0^L \left(\varphi_x^2 + \psi_x^2 + \omega_x^2\right) dx.\end{aligned} \tag{2.7}$$

Lemma 2.1. *(Sobolev-Poincare's inequality). Let q be a number with $2 \leq q < +\infty$. Then there is a constant $c_* = c_*((0,1), q)$ such that*

$$\|\psi\|_q \leq c_*\|\psi_x\|_2 \quad for \quad \psi \in H_0^1((0,1)).$$

3. Well-posedness

In order to prove the well-posedness result, we have to make the following operations: we introduce, as in **[12]**, the new variables

$$\begin{aligned} z_1(x,\rho,t) &= \phi_t(x, t - \tau_1(t)\rho), \quad x \in (0, L),\ \rho \in (0,1), \quad t > 0,\\ z_2(x,\rho,t) &= \psi_t(x, t - \tau_2(t)\rho), \quad x \in (0, L),\ \rho \in (0,1), \quad t > 0,\\ z_3(x,\rho,t) &= \omega_t(x, t - \tau_3(t)\rho), \quad x \in (0, L),\ \rho \in (0,1), \quad t > 0.\end{aligned} \tag{3.1}$$

Also as in [17], the new variables

$$\begin{cases} \eta_1(x,t,s) = \varphi(x,t) - \varphi(x, t-s), & in \quad]0, L[\times R_+ \times R_+,\\ \eta_2(x,t,s) = \psi(x,t) - \psi(x, t-s), & in \quad]0, L[\times R_+ \times R_+,\\ \eta_3(x,t,s) = \omega(x,t) - \omega(x, t-s), & in \quad]0, L[\times R_+ \times R_+.\end{cases}$$

These functionals satisfy

$$\begin{cases} \partial_t\eta_1 + \partial_s\eta_1 - \varphi_t = 0, & in \quad]0, L[\times R_+ \times R_+,\\ \partial_t\eta_2 + \partial_s\eta_2 - \psi_t = 0, & in \quad]0, L[\times R_+ \times R_+,\\ \partial_t\eta_3 + \partial_s\eta_3 - \omega_t = 0, & in \quad]0, L[\times R_+ \times R_+,\\ \eta_i(0,t,s) = \eta_i(L,t,s) = 0, & in \quad R_+ \times R_+,\\ \eta_i(x,t,0) = 0, & in \quad]0, L[\times R_+, i = 1,2,3.\end{cases}$$

In order to convert our problem to a system of first-order ordinary differential equations, we note the following:

$$\eta_i^0(x,s)=\eta_i(x,0,s),\quad i=1,2,3. \tag{3.2}$$

Then, we have for $i=1,2,3$

$$\tau_i(t)z_{it}(x,\rho,t)+(1-\tau_i'(t)\rho)z_{i\rho}(x,\rho,t)=0,\ \text{ in } (0,L)\times(0,1)\times(0,+\infty). \tag{3.3}$$

Therefore, problem (1.1) takes the form:

$$\begin{cases}\rho_1\varphi_{tt}(x,t)-k_1(\varphi_x+\psi+l\omega)_x(x,t)-lk_3(\omega_x-l\varphi)(x,t)+\mu_1\varphi_t(x,t)\\ +\mu_2 z_1(x,1,t)+\displaystyle\int_0^\infty g_1(s)\partial_{xx}\eta_1 ds=0,\\ \tau_1(t)z_{1t}(x,\rho,t)+(1-\tau_1'(t)\rho)z_{1\rho}(x,\rho,t)=0,\\ \rho_2\psi_{tt}(x,t)-k_2\psi_{xx}(x,t)+k_1(\varphi_x+\psi+l\omega)(x,t)+\widetilde{\mu_1}\psi_t(x,t)\\ +\widetilde{\mu_2}z_2(x,1,t)+\displaystyle\int_0^\infty g_2(s)\partial_{xx}\eta_2 ds=0,\\ \tau_2(t)z_{2t}(x,\rho,t)+(1-\tau_2'(t)\rho)z_{2\rho}(x,\rho,t)=0,\\ \rho_1\omega_{tt}(x,t)-k_3(\omega_x-l\varphi)_x(x,t)+lk_1(\varphi_x+\psi+l\omega)(x,t)+\widetilde{\widetilde{\mu_1}}\omega_t(x,t)\\ +\widetilde{\widetilde{\mu_2}}z_3(x,1,t)+\displaystyle\int_0^\infty g_3(s)\partial_{xx}\eta_3 ds=0,\\ \tau_3(t)z_{3t}(x,\rho,t)+(1-\tau_3'(t)\rho)z_{3\rho}(x,\rho,t)=0.\end{cases} \tag{3.4}$$

The above system subjected to the following initial and boundary conditions

$$\begin{cases}\varphi(0,t)=\varphi(L,t)=\psi(0,t)=\psi(L,t)=\omega(0,t)=\omega(L,t), & t>0,\\ z_1(x,0,t)=\varphi_t(x,t), z_2(x,0,t)=\psi_t(x,t), & x\in(0,L), t>0,\\ z_3(x,0,t)=\omega_t(x,t), x\in(0,L), & x\in(0,L), t>0,\\ \varphi(x,0)=\varphi_0, \varphi_t(x,0)=\varphi_1, \psi(x,0)=\psi_0, \psi_t(x,0)=\psi_1, & \\ \omega(x,0)=\omega_0,\quad \omega_t(x,0)=\omega_1, & x\in(0,L),\\ z_1(x,1,t)=f_1(x,t-\tau_1(t)), & \text{in } (0,L)\times(0,\tau_1(0)),\\ z_2(x,1,t)=f_2(x,t-\tau_2(t)), & \text{in } (0,L)\times(0,\tau_2(0)),\\ z_3(x,1,t)=f_3(x,t-\tau_3(t)), & \text{in } (0,L)\times(0,\tau_3(0)),\\ \eta_1(x,t,s)=\eta_1(L,t,s)=0, & x\in(0,L), t>0, in\ R_+\times R_+,\\ \eta_2(x,t,s)=\eta_2(L,t,s)=0, & x\in(0,L), t>0, in\ R_+\times R_+,\\ \eta_3(x,t,s)=\eta_3(L,t,s)=0, & x\in(0,L), t>0, in\ R_+\times R_+,\\ \eta_1(x,t,0)=0, & x\in(0,L), t>0,\ in\ R_+\times R_+,\\ \eta_2(x,t,0)=0, & x\in(0,L), t>0,\ in\ R_+\times R_+,\\ \eta_3(x,t,0)=0, & x\in(0,L), t>0,\ in\ R_+\times R_+.\end{cases} \tag{3.5}$$

Let ξ_1,ξ_2 and ξ_3 be positive constants such that

$$\begin{cases}\dfrac{\mu_2}{\sqrt{1-d_1}}\le\xi_1\le 2\mu_2-\dfrac{\mu_2}{\sqrt{1-d_1}},\\ \dfrac{\widetilde{\mu_2}}{\sqrt{1-d_2}}\le\xi_2\le 2\widetilde{\mu_2}-\dfrac{\widetilde{\mu_2}}{\sqrt{1-d_2}},\\ \dfrac{\widetilde{\widetilde{\mu_2}}}{\sqrt{1-d_3}}\le\xi_3\le 2\widetilde{\widetilde{\mu_2}}-\dfrac{\widetilde{\widetilde{\mu_2}}}{\sqrt{1-d_3}}.\end{cases} \tag{3.6}$$

We define the energy associated to the solution of the problem (3.4)-(3.5) by the following formula

$$
\begin{aligned}
E(t) &= \tfrac{\rho_1}{2}\|\varphi_t\|_2^2 + \tfrac{\rho_2}{2}\|\psi_t\|_2^2 + \tfrac{\rho_1}{2}\|\omega_t\|_2^2 + \tfrac{k_3}{2}\|\psi_x\|_2^2 \\
&+\tfrac{k_3}{2}\|\varphi_x + \psi + l\omega\|_2^2 + \tfrac{k_3}{2}\|\omega_x - l\varphi\|_2^2 + \textstyle\sum_{i=1}^{3} \tfrac{\xi_i(t)\tau_i(t)}{2} \displaystyle\int_0^1 \|z_i(x,\rho,t)\|_2^2 d\rho \\
&- \int_0^L (g_1^0\varphi_x^2 + g_2^0\psi_x^2 + g_3^0\omega_x^2)dx + \|\eta_1\|_{H_1^*}^2 + \|\eta_2\|_{H_2^*}^2 + \|\eta_3\|_{H_3^*}^2
\end{aligned}
\tag{3.7}
$$

where

$$
H_i^* = \left\{ v : R_+ \to H_0^1(]0,L[), \int_0^L \int_0^\infty g_i(s) v_x^2(s) ds dx < +\infty \right\}.
$$

We have the following theorem.

Theorem 3.1. *Assume that the hypotheses* $(H1)-(H3)$ *hold.*
Let $(\varphi_0, \varphi_1, f_1(.,-.\tau_1(0)), \psi_0, \psi_1, f_2(.,-.\tau_2(0)), \omega_0, \omega_1, f_3(.,-.\tau_3(0)), \eta_0^1, \eta_0^2, \eta_0^3) \in (H_0^1(0,L)\times L^2(0,L)\times L^2((0,L)\times(0,1)))^3$. *Then problem* $(3.4)-(3.5)$ *admits a unique solution*

$$
\begin{cases}
\varphi \in C([0,+\infty); H_0^1(0,L)) \cap C^1([0,+\infty); L^2(0,L)), \\
\psi \in C([0,+\infty); H_0^1(0,L)) \cap C^1([0,+\infty); L^2(0,L)), \\
\omega \in C([0,+\infty); H_0^1(0,L)) \cap C^1([0,+\infty); L^2(0,L)), \\
z_1, z_2, z_3 \in C([0,+\infty); L^2((0,L)\times(0,1))), \\
\eta_1, \eta_2, \eta_3 \in C([0,+\infty); H_0^1(0,L)) \cap C^1([0,+\infty); L^2(0,L)).
\end{cases}
$$

We finish this section by giving an explicit upper bound for the derivative of the energy.

Lemma 3.1. *Let* $(\varphi, \psi, \omega, z_1, z_2, z_3, \eta_1, \eta_2, \eta_3)$ *be a solution of the problem (3.4)-(3.5). Then, the energy functional defined by (3.7) satisfies*

$$
\begin{aligned}
E'(t) &\le -\left(\mu_1 - \tfrac{\xi_1}{2} - \tfrac{\mu_1}{2\sqrt{1-d_1}}\right)\|\varphi_t\|_2^2 - \left(\widetilde{\mu_1} - \tfrac{\xi_1}{2} - \tfrac{\widetilde{\mu_1}}{2\sqrt{1-d_2}}\right)\|\psi_t\|_2^2 \\
&- \left(\widetilde{\mu_1} - \tfrac{\xi_1}{2} - \tfrac{\widetilde{\mu_1}}{2\sqrt{1-d_3}}\right)\|\omega_t\|_2^2 \\
&- \left(\tfrac{\xi_1(1-\tau_1'(t))}{2} - \tfrac{\mu_2\sqrt{1-d_1}}{2}\right)\|z_1(x,1,t)\|_2^2 \\
&- \left(\tfrac{\xi_1(1-\tau_2'(t))}{2} - \tfrac{(\widetilde{\mu_2}\sqrt{1-d_2}}{2}\right)\|z_2(x,1,t)\|_2^2 \\
&- \left(\tfrac{\xi_1(1-\tau_3'(t))}{2} - \tfrac{\widetilde{\widetilde{\mu_2}}\sqrt{1-d_3}}{2}\right)\|z_3(x,1,t)\|_2^2 \\
&+\tfrac{1}{2}\int_0^L\int_0^\infty g_1'(s)(\partial_x\eta_1)^2 ds dx + \frac{1}{2}\int_0^L\int_0^\infty g_2'(s)(\partial_x\eta_2)^2 ds dx \\
&+\tfrac{1}{2}\int_0^L\int_0^\infty g_3'(s)(\partial_x\eta_3)^2 ds dx.
\end{aligned}
\tag{3.8}
$$

Proof. Multiplying the first equation in (3.4) by φ_t, the third equation by ψ_t, the five equation by ω_t, integrating over $(0, L)$ and using integration by parts, we get

$$\begin{aligned}
&\tfrac{1}{2}\rho_1 \tfrac{d}{dt}\|\varphi_t\|_2^2 - k_1 \int_0^L (\varphi_x + \psi + l\omega)_x \varphi_t dx - lk_3 \int_0^L (\omega_x - l\varphi)\varphi_t dx + \mu_1 \|\varphi_t\|_2^2 \\
&+\mu_2 \int_0^L z_1(x,1,t)\varphi_t dx + \int_0^\infty g_1(s)\partial_{xx}\eta_1 \varphi_t ds dx = 0 \\
&\tfrac{1}{2}\rho_2 \tfrac{d}{dt}\|\psi_t\|_2^2 + \tfrac{k_2}{2}\|\psi_x\|_2^2 + k_1 \int_0^L (\varphi_x + \psi + l\omega)\psi_t dx + \widetilde{\mu_1}\|\psi_t\|_2^2 \\
&+\widetilde{\mu_2} \int_0^L z_2(x,1,t)\psi_t dx + \int_0^\infty g_2(s)\partial_{xx}\eta_2 \psi_t ds dx = 0 \\
&\tfrac{1}{2}\rho_1 \tfrac{d}{dt}\|\omega_t\|_2^2 - k_3 \int_0^L (\omega_x - l\varphi)_x \omega_t dx + lk_1 \int_0^L (\varphi_x + \psi + l\omega)\omega_t dx + \widetilde{\widetilde{\mu_1}}\|\omega_t\|_2^2 \\
&+\widetilde{\widetilde{\mu_2}} \int_0^L z_3(x,1,t)\omega_t dx + \int_0^\infty g_3(s)\partial_{xx}\eta_3 \omega_t ds dx = 0.
\end{aligned}$$

Then, if we put

$$F(t) = \left(\frac{\rho_1}{2}\|\varphi_t\|_2^2 + \frac{\rho_2}{2}\|\psi_t\|_2^2 + \frac{\rho_1}{2}\|\omega_t\|_2^2 + \frac{k_1}{2}\|\psi_x\|_2^2 + \frac{k_2}{2}\|\varphi_x + \psi + l\omega\|_2^2 + \frac{k_3}{2}\|\omega_x - l\varphi\|_2^2 \right)$$

we get

$$\begin{aligned}
&\tfrac{d}{dt}F(t) + \mu_1\|\varphi_t\|_2^2 + \widetilde{\mu_1}\|\psi_t\|_2^2 + \widetilde{\widetilde{\mu_1}}\|\omega_t\|_2^2 \\
&+\widetilde{\mu_2} \int_0^L z_1(x,1,t)\psi_t dx + \mu_2 \int_0^L z_2(x,1,t)\varphi_t dx \\
&+\widetilde{\widetilde{\mu_2}} \int_0^L z_3(x,1,t)\omega_t dx + \int_0^\infty g_1(s)(\partial_x\eta_1)^2 ds + \int_0^\infty g_2(s)(\partial_x\eta_2)^2 ds \\
&+ \int_0^\infty g_3(s)(\partial_x\eta_3)^2 ds + \int_0^\infty g_1(s)\partial_s(\partial_x\eta_1)^2 ds + \int_0^\infty g_2(s)\partial_s(\partial_x\eta_2)^2 ds \\
&+ \int_0^\infty g_3(s)\partial_s(\partial_x\eta_3)^2 ds = 0.
\end{aligned} \tag{3.9}$$

Multiplying the second equation in (3.4) by $\xi_i z_i$ and integrating over $(0, L) \times (0, 1)$, to obtain

$$\begin{aligned}
&\xi_i(t)e^{-\rho\tau_i(t)} \int_0^L \int_0^1 z_{it} z_i(x,\rho,t) d\rho dx \\
&= -\tfrac{\xi_i(t)e^{\rho\tau_i(t)}}{2\tau_{0i}} \int_0^L \int_0^1 (1 - \tau_i'(t)\rho) \tfrac{\partial}{\partial\rho}(z_i(x,\rho,t))^2 d\rho dx.
\end{aligned} \tag{3.10}$$

Consequently,

$$
\begin{aligned}
&\tfrac{d}{dt}\left(\tfrac{\xi_i(t)e^{-\rho\tau_i(t)}}{2}\int_0^L\int_0^1 z_i^2(x,\rho,t)d\rho dx\right)\\
&= -\tfrac{\xi_i(t)}{2}\int_0^1\int_0^L \frac{\partial}{\partial\rho}((1-\tau_i'(t)\rho)e^{-\rho\tau_i(t)}z_i^2(x,\rho,t))d\rho dx\\
&+\tfrac{\xi_i'(t)e^{-\rho\tau_i(t)}}{2}\int_0^1\int_0^L z_i^2(x,\rho,t)d\rho dx\\
&= \tfrac{\xi_i(t)}{2}\int_0^L\left[z_i^2(x,0,t)-z_i^2(x,1,t)\right]e^{-\tau_i(t)}dx\\
&+\tfrac{\xi_i(t)\tau_i'(t)e^{-\tau_i(t)}}{2}\int_0^L z_i^2(x,1,t)dx\\
&+\tfrac{\xi_i'(t)e^{-\rho\tau_i(t)}}{2}\int_0^1\int_0^L z_i^2(x,\rho,t)d\rho dx,
\end{aligned}
\tag{3.11}
$$

where $z_1(x,0,t)=\varphi_t(x,t)$, $z_2(x,0,t)=\psi_t(x,t)$ and $z_3(x,0,t)=\omega_t(x,t)$. From (3.9), (3.11), integrating by parts and using Young's inequality, we get

$$
\begin{aligned}
&E'(t) = -\left(\mu_1-\tfrac{\xi_1}{2}\right)\|\varphi_t\|_2^2-\left(\widetilde{\mu_1}-\tfrac{\xi_2}{2}\right)\|\psi_t\|_2^2-\left(\widetilde{\widetilde{\mu_1}}-\tfrac{\xi_3}{2}\right)\|\omega_t\|_2^2\\
&-\sum_{i=1}^{3}\frac{\xi_i(1-\tau_i'(t))}{2}\|z_i(x,1,t)\|_2^2-\mu_2\int_0^L z_1(x,1,t)\varphi_t\,dx\\
&-\widetilde{\mu_2}\int_0^L z_2(x,1,t)\psi_t\,dx-\widetilde{\widetilde{\mu_2}}\int_0^L z_3(x,1,t)\omega_t\,dx\\
&+\tfrac{1}{2}\int_0^L\int_0^\infty g_1'(s)(\partial_x\eta_1)^2dsdx+\frac{1}{2}\int_0^L\int_0^\infty g_2'(s)(\partial_x\eta_2)^2dsdx\\
&+\tfrac{1}{2}\int_0^L\int_0^\infty g_3'(s)(\partial_x\eta_3)^2dsdx.
\end{aligned}
\tag{3.12}
$$

Due to Young's inequality, we have

$$
\begin{aligned}
&\mu_2\int_0^L z_1(x,1,t)\varphi_t(x,t)dx \le \tfrac{\mu_2}{2\sqrt{1-d_1}}\|\varphi_t(t)\|_2^2+\tfrac{\mu_2\sqrt{1-d_1}}{2}\|z_1(x,1,t)\|_2^2,\\
&\widetilde{\mu_2}\int_0^L z_2(x,1,t)\psi_t(x,t)dx \le \tfrac{\widetilde{\mu_2}}{2\sqrt{1-d_2}}\|\psi_t(t)\|_2^2+\tfrac{\widetilde{\mu_2}\sqrt{1-d_2}}{2}\|z_2(x,1,t)\|_2^2,\\
&\widetilde{\widetilde{\mu_2}}\int_0^L z_3(x,1,t)\omega_t(x,t)dx \le \tfrac{\widetilde{\widetilde{\mu_2}}}{2\sqrt{1-d_3}}\|\omega_t(t)\|_2^2+\tfrac{\widetilde{\widetilde{\mu_2}}\sqrt{1-d_3}}{2}\|z_3(x,1,t)\|_2^2.
\end{aligned}
\tag{3.13}
$$

Inserting (3.13) into (3.12), we obtain

$$\begin{aligned}
E'(t) \le & -\left(\mu_1 - \tfrac{\xi_1}{2} - \tfrac{\mu_1}{2\sqrt{1-d_1}}\right)\|\varphi_t\|_2^2 - \left(\widetilde{\mu_1} - \tfrac{\xi_2}{2} - \tfrac{\widetilde{\mu_1}}{2\sqrt{1-d_2}}\right)\|\psi_t\|_2^2 \\
& -\left(\widetilde{\mu_1} - \tfrac{\xi_3}{2} - \tfrac{\widetilde{\widetilde{\mu_1}}}{2\sqrt{1-d_3}}\right)\|\omega_t\|_2^2 - \left(\tfrac{\xi_1(1-\tau_1'(t))}{2} - \tfrac{\mu_2\sqrt{1-d_1}}{2}\right)\|z_1(x,1,t)\|_2^2 \\
& -\left(\tfrac{\xi_2(1-\tau_2'(t))}{2} - \tfrac{(\widetilde{\mu_2}\sqrt{1-d_2}}{2}\right)\|z_2(x,1,t)\|_2^2 \\
& -\left(\tfrac{\xi_3(1-\tau_3'(t))}{2} - \tfrac{\widetilde{\widetilde{\mu_2}}\sqrt{1-d_3}}{2}\right)\|z_3(x,1,t)\|_2^2 \\
& +\tfrac{1}{2}\int_0^L \int_0^\infty g_1'(s)(\partial_x\eta_1)^2 ds\, dx + \frac{1}{2}\int_0^L \int_0^\infty g_2'(s)(\partial_x\eta_2)^2 ds\, dx \\
& +\tfrac{1}{2}\int_0^L \int_0^\infty g_3'(s)(\partial_x\eta_3)^2 ds\, dx.
\end{aligned}$$

This completes the proof of the lemma. □

Now, we will give well-posedness results for problem (3.4)-(3.5) by using semigroup theory. Let us introduce the semigroup representation of the Bresse system (3.4)-(3.5). Let $U = (\varphi, \psi, \omega, \varphi_t, \psi_t, \omega_t, z_1, z_2, z_3, \eta_1, \eta_2, \eta_3)^T$ and rewrite (3.4)-(3.5) as

$$\begin{cases} U' = AU, \\ U(x,0) = U^0(x). \end{cases} \tag{3.14}$$

$$U^0(x) = (\varphi_0, \psi_0, \omega_0, \varphi_1, \psi_1, \omega_1, f_1(.,-.\tau_1(0)), f_2(.,-.\tau_2(0)), f_3(.,-.\tau_3(0)), \eta_1^0, \eta_2^0, \eta_3^0),$$

where the operator A is defined by

$$A\begin{pmatrix} \varphi \\ \psi \\ \omega \\ \varphi_t \\ \psi_t \\ \omega_t \\ z_1 \\ z_2 \\ z_3 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} \varphi_t \\ \psi_t \\ \omega_t \\ \frac{1}{\rho_1}(k_1 - \int_0^\infty g_1(s)ds)\varphi_{xx} - \frac{l^2k_3}{\rho_1}\varphi + \frac{k_1}{\rho_1}\psi_x + \frac{l}{\rho_1}(k_1+k_3)\omega_x + \frac{1}{\rho_1}\int_0^\infty g_1(s)\partial_{xx}\eta_1 ds \\ -\frac{\mu_1}{\rho_1}\varphi_t - \frac{\mu_2}{\rho_1}z_1(.,1) \\ \frac{-k_1}{\rho_2}\varphi_x + \frac{1}{\rho_2}(k_2 - \int_0^\infty g_2(s)ds)\psi_{xx} - \frac{k_1}{\rho_2}\psi - \frac{-lk_1}{\rho_2}\omega + \frac{1}{\rho_2}\int_0^\infty g_2(s)ds)\partial_{xx}\eta_2 ds \\ -\frac{\widetilde{\mu_1}}{\rho_1}\psi_t - \frac{\widetilde{\mu_2}}{\rho_1}z_2(.,1) \\ \frac{-l}{\rho_1}(k_1+k_3)\varphi_x - \frac{lk_1}{\rho_1}\psi + \frac{1}{\rho_1}(k_3 - \int_0^\infty g_3(s)ds)\omega_{xx} - \frac{l^2k_1}{\rho_1}\omega + \frac{1}{\rho_1}\int_0^\infty g_3(s)\partial_{xx}\eta_3 ds \\ -\frac{\widetilde{\widetilde{\mu_1}}}{\rho_1}\omega_t - \frac{\widetilde{\widetilde{\mu_2}}}{\rho_1}z_3(.,1) \\ -\frac{(1-\tau_2'(t))}{\tau_2(t)}z_{1\rho} \\ -\frac{(1-\tau_2'(t))}{\tau_2(t)}z_{2\rho} \\ -\frac{(1-\tau_2'(t))}{\tau_2(t)}z_{3\rho} \\ \varphi_t - \partial_s\eta_1 \\ \psi_t - \partial_s\eta_2 \\ \omega_t - \partial_s\eta_3 \end{pmatrix}$$

with the domain H given by, $(H^* = (L^2(0,L;H^1(0,1))^3 \times H_1^* \times H_2^* \times H_3^*)$

$$H = (H^2(]0,L[) \cap (H_0^1(]0,L[)))^3 \times (H_0^1(]0,L[))^3 \times H^*. \tag{3.15}$$

The domain $D(A)$ of A is defined by

$$D(A) = \{U \in H; AU \in H, \eta_i(x,t,0) = 0, i = 1,2,3\}. \tag{3.16}$$

Now, under hypothesis (H1), the sets H_i^* and H are Hilbert spaces equipped, respectively, with the inner products that generate the norms

$$\|\eta_i\|_{H_i^*}^2 = \int_0^L \int_0^{+\infty} g_i(s)(\partial_x \eta_i)^2 ds\, dx,$$

$$\begin{aligned}\|U\|_H^2 = &\int_0^L (\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1\omega_t^2 + k_2\psi_x^2 + k_1(\varphi_x + \psi + l\omega)^2 + k_3(\omega_x - l\varphi)^2)dx \\ &+ \int_0^L \sum_{i=1}^{3} \xi_i(t)\tau_i(t) \int_0^1 z_i^2\, d\rho - \int_0^L (g_1^0\varphi_x^2 + g_2^0\psi_x^2 + g_3^0\omega_x^2)\, dx \\ &+ \|\eta_1\|_{H_1^*}^2 + \|\eta_2\|_{H_2^*}^2 + \|\eta_3\|_{H_3^*}^2.\end{aligned}$$

We show that the operator A generates a C_0- semigroup in H. In this step, we prove that the operator A is dissipative. Let $U = (\varphi, \psi, \omega, u, v, \tilde{\omega}, z_1, z_2, z_3, \eta_1, \eta_2, \eta_3)^T$. Using (3.8) and the fact that

$$E(t) = \frac{1}{2}\|U\|_H^2, \tag{3.17}$$

we get

$$\begin{aligned}\langle AU, U\rangle_H = &-\mu_1 \int_0^L u^2\, dx - \widetilde{\mu}_1 \int_0^L v^2\, dx - \widetilde{\widetilde{\mu}}_1 \int_0^L \widetilde{\omega}^2\, dx \\ &-\mu_2 \int_0^L z_1(x,1)u\, dx - \widetilde{\mu}_2 \int_0^L z_2(x,1)v\, dx - \widetilde{\widetilde{\mu}}_2 \int_0^L z_3(x,1)\widetilde{\omega}\, dx \\ &- \sum_{i=1}^{3} \xi_i(t)\tau_i(t) \int_0^L \int_0^1 z_i(x,\rho) z_{i\rho}(x,\rho)\, d\rho\, dx \\ &-\tfrac{1}{2} \int_0^L g_1(s) \int_0^\infty \partial_s(\partial_x\eta_1)^2 ds dx - \frac{1}{2} \int_0^L g_2(s) \int_0^\infty \partial_s(\partial_x\eta_2)^2 ds dx \\ &-\tfrac{1}{2} \int_0^L g_3(s) \int_0^\infty \partial_s(\partial_x\eta_3)^2 ds dx \\ &\le 0,\end{aligned} \tag{3.18}$$

by using the integration by parts and the boundary conditions in (3.5), yields

$$
\begin{aligned}
\langle AU, U\rangle_H = &-\mu_1 \int_0^L u^2\,dx - \widetilde{\mu}_1 \int_0^L v^2\,dx - \widetilde{\widetilde{\mu}}_1 \int_0^L \widetilde{\omega}^2\,dx \\
&-\mu_2 \int_0^L z_1(x,1)u\,dx - \widetilde{\mu}_2 \int_0^L z_2(x,1)v\,dx - \widetilde{\widetilde{\mu}}_2 \int_0^L z_3(x,1)\widetilde{\omega}\,dx \\
&-\sum_{i=1}^{3} \xi_i(t)\tau_i(t) \int_0^L \int_0^1 z_i(x,\rho) z_{i\rho}(x,\rho)\,d\rho\,dx + \frac{1}{2}\int_0^L \int_0^\infty (g_1'(s)\partial_x\eta_1)^2 ds dx \\
&+\tfrac{1}{2}\int_0^L \int_0^\infty (g_2'(s)\partial_x\eta_2)^2 ds dx + \frac{1}{2}\int_0^L \int_0^\infty (g_3'(s)\partial_x\eta_3)^2 ds dx \le 0
\end{aligned}
\tag{3.19}
$$

and then, because for any $i = 1,2,3$, the kernel g_i is non-increasing,

$$
\langle AU, U\rangle \le 0. \tag{3.20}
$$

Consequently, the operator A is dissipative. Now, we will prove that the operator $\lambda I - A$ is surjective for $\lambda > 0$. For this purpose, let

$$(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}, f_{12})^T \in H,$$

we seek

$$U = (v_1, v_2, v_3, v_4, v_5, v_6, z_1, z_2, z_3, v_7, v_8, v_9)^T \in D(A),$$

solution of the following system of equations

$$
\begin{cases}
\lambda v_4 + \frac{\mu_1}{\rho_1} v_4 + \frac{\mu_2}{\rho_1} z_1(.,1) + \frac{1}{\rho_1}(k_1 - g_0^1)\partial_{xx} v_1 - \frac{l}{\rho_1}(k_1 \partial_x v_3 - l k_3 v_1) \\
-\frac{l}{\rho_1}\left\{k_1 \partial_x v_3 + k_2 \partial_x v_3 - g_0^1 v_7\right\} = f_4, \\[2mm]
-\frac{1}{\rho_2}\left\{(k_2 - g_0^2)\partial_{xx} v_2 - k_1 \partial_x v_1 + \frac{g_0^3 \rho_2}{\rho_1}\partial_{xx} v_9\right\} + \frac{l k_1}{\rho_2} v_3 + \lambda v_5 \\
+\frac{\widetilde{\mu}_1}{\rho_1} v_5 + \frac{\widetilde{\mu}_2}{\rho_1} z_2(.,1) = f_6, \\[2mm]
\frac{1}{\rho_1}\left\{(k_1 + k_2)\partial_x v_1 - (k_3 - g_0^3)\partial_{xx} v_3 - g_0^3 \partial_{xx} v_9\right\} + \frac{l k_1}{\rho_1} v_2 + \frac{l^2 k_1}{l_1} v_3 \\
+\frac{\widetilde{\widetilde{\mu}}_1}{\rho_1} v_6 + \lambda v_8 + \frac{\widetilde{\widetilde{\mu}}_2}{\rho_1} z_3(.,1) = f_5, \\[2mm]
\lambda z_1 + \frac{(1-\tau_2'(t))}{\tau_2(t)} z_{1\rho} = f_7, \\
\lambda z_2 + \frac{(1-\tau_2'(t))}{\tau_2(t)} z_{2\rho} = f_8, \\
\lambda z_3 + \frac{(1-\tau_2'(t))}{\tau_2(t)} z_{3\rho} = f_9, \\
\lambda v_1 - v_5 = f_1, \\
\lambda v_2 - v_6 = f_2, \\
\lambda v_3 - v_4 = f_3, \\[2mm]
-v_4 + \lambda v_7 + \partial_s v_7 = f_{10}, \\
-v_5 + \lambda v_8 + \partial_s v_8 = f_{11}, \\
-v_6 + \lambda v_9 + \partial_s v_9 = f_{12}.
\end{cases}
\tag{3.21}
$$

Suppose that we have found v_1, v_2 and v_3. Therefore, the seventh, the eighth and the ninth equation in (3.21) give

$$\begin{cases} v_5 = \lambda v_1 - f_1, \\ v_6 = \lambda v_2 - f_2, \\ v_4 = \lambda v_3 - f_3. \end{cases} \tag{3.22}$$

Then it is clear that $v_1 \in H_0^1(0,L), v_2 \in H_0^1(0,L)$ and $v_3 \in H_0^1(0,L)$. Furthermore, by (3.21) we can find $z_i(i = 1,2,3)$ as

$$z_1(x,0) = v_5(x), z_2(x,0) = v_6(x), z_3(x,0) = v_7(x), \quad \text{for } x \in (0,L). \tag{3.23}$$

Following the same approach as in [12], we obtain by using equations for z_i in (3.21),

$$\begin{cases} z_1(x,\rho) = v_5(x)e^{-\lambda\tau_1(t)\rho} + \tau_1(t)e^{-\lambda\tau_1(t)\rho} \displaystyle\int_0^\rho f_7(x,s)e^{\lambda\tau_1(t)s}\,ds, \\ z_2(x,\rho) = v_6(x)e^{-\lambda\tau_2(t)\rho} + \tau_2(t)e^{-\lambda\tau_2(t)\rho} \displaystyle\int_0^\rho f_8(x,s)e^{\lambda\tau_2(t)s}\,ds, \\ z_3(x,\rho) = v_7(x)e^{-\lambda\tau_3(t)\rho} + \tau_3(t)e^{-\lambda\tau_3(t)\rho} \displaystyle\int_0^\rho f_9(x,s)e^{\lambda\tau_3(t)s}\,ds, \end{cases} \tag{3.24}$$

From (3.22), we obtain

$$\begin{cases} z_1(x,\rho) = \lambda v_1(x)e^{-\lambda\tau_1(t)\rho} - f_1 e^{-\lambda\tau_1(t)\rho} \\ +\tau_1(t)e^{-\lambda\tau_1(t)\rho} \displaystyle\int_0^\rho f_7(x,s)e^{\lambda\tau_1(t)s}\,ds, \\ z_2(x,\rho) = \lambda v_2(x)e^{-\lambda\tau_2(t)\rho} - f_2 e^{-\lambda\tau_2(t)\rho} \\ +\tau_2(t)e^{-\lambda\tau_2(t)\rho} \displaystyle\int_0^\rho f_8(x,s)e^{\lambda\tau_2(t)s}\,ds, \\ z_3(x,\rho) = \lambda v_3(x)e^{-\lambda\tau_3(t)\rho} - f_3 e^{-\lambda\tau_3(t)\rho} \\ +\tau_3(t)e^{-\lambda\tau_3(t)\rho} \displaystyle\int_0^\rho f_9(x,s)e^{\lambda\tau_3(t)s}\,ds, \end{cases} \tag{3.25}$$

$$\begin{cases} v_7 = \left(\int_0^s (v_4 + f_{10}) e^{\tau} d\tau\right) e^{-s}, \\ v_8 = \left(\int_0^s (v_5 + f_{11}) e^{\tau} d\tau\right) e^{-s}, \\ v_9 = \left(\int_0^s (v_6 + f_{12}) e^{\tau} d\tau\right) e^{-s}. \end{cases} \tag{3.26}$$

By using (3.21) and (3.26), the functions v_1, v_2 and v_3 satisfying the following system

$$\begin{cases} \lambda^2 v_3 + \frac{1}{\rho_1}\left(k_1 + \frac{\mu_1\lambda}{\rho_1} v_3 - g_0^1\right)\partial_{xx} v_1 - \frac{l}{\rho_1}\left(k_1 \partial_x v_3 - l k_3 v_1\right) \\ -\frac{l}{\rho_1}\left(k_1 \partial_x v_3 + k_2 \partial_x v_3\right) + \frac{\mu_2}{\rho_1} z_1(.,1) \\ = \left(\frac{\mu_1 \lambda}{\rho_1} + 1\right) f_3 + \left(\int_0^s (\lambda v_3 - f_3 + f_{10}) e^{\tau} d\tau\right) e^{-s}, \\ \lambda^2 \left(\frac{l k_1}{\rho_1} + \frac{\widetilde{\widetilde{\mu_1}}\lambda}{\rho_2}\right) v_2 + \frac{l^2 k_1}{\rho_1} v_3 \\ + \frac{1}{\rho_1}\left(k_1 + k_2 \partial_x v_1 - \frac{1}{\rho_1}(k_3 - g_0^3\right)\partial_x v_3 \\ -\frac{\lambda g_0^3}{\rho_1}\left(\int_0^s (\lambda v_1 - f_1 + f_{12}) e^{\tau} d\tau\right) e^{-s} \\ +\lambda\left(\int_0^s (\lambda v_2 - f_2 + f_{11}) e^{\tau} d\tau\right) e^{-s}\lambda + \frac{\widetilde{\widetilde{\mu_2}}}{\rho_1} z_3(.,1) = f_5 + \frac{\widetilde{\widetilde{\mu_1}}}{\rho_1} f_2, \\ \left(\lambda^2 + \frac{\widetilde{\mu_1}\lambda}{\rho_1}\right) v_1 + \frac{l k_1}{\rho_1} v_3 + \frac{k_1}{\rho_1}\partial_x v_1 + \frac{l k_1}{\rho_2} v_3 \\ -\frac{1}{\rho_1}\left(k_2 - g_0^2\right)\partial_{xx} v_2 + \frac{\widetilde{\mu_2}}{\rho_1} z_2(.,1) \\ -\frac{g_0^3}{\rho_1}\partial_{xx}\left(\int_0^s (\lambda v_2 - f_2 + f_{12}) e^{\tau} d\tau\right) e^{-s} = \left(\lambda + \frac{\widetilde{\mu_1}}{\rho_1}\right) f_1 + f_6. \end{cases} \tag{3.27}$$

Solving system (3.27) is equivalent to finding $(v_1, v_2, v_3) \in (H^2 \cap H_0^1(0, L))^3$ such that

$$
(3.28)\qquad
\begin{cases}
\int_0^L \left\{ \lambda^2 v_3 + \frac{1}{\rho_1}\left(k_1 + \frac{\mu_1\lambda}{\rho_1} v_3 - g_0^1\right)\partial_{xx} v_1 \right\} \phi_1 dx \\
- \int_0^L \left\{ \frac{l}{\rho_1}(k_1\partial_x v_3 - lk_3 v_1) - \frac{l}{\rho_1}(k_1\partial_x v_3 + k_2\partial_x v_3) \right\} \phi_1 dx \\
+ \int_0^L \frac{\mu_2}{\rho_1} z_1(., 1)\phi_1 dx \\
= \int_0^L \left\{ \left(\frac{\mu_1\lambda}{\rho_1} + 1 \right) f_3 + \left(\int_0^s (\lambda v_3 - f_3 + f_{10}) e^\tau d\tau \right) e^{-s} \right\} \phi_1 dx, \\[2ex]
\int_0^L \left\{ \lambda^2 \left(\frac{lk_1}{\rho_1} + \frac{\widetilde{\widetilde{\mu}}\lambda}{\rho_2} \right) v_2 \right\} \phi_2 dx \\
+ \int_0^L \left\{ \frac{1}{\rho_1}\left(k_1 + k_2\partial_x v_1 - \frac{1}{\rho_1}(k_3 - g_0^3\right)\partial_x v_3 \right\} \phi_2 dx \\
+ \int_0^L \left\{ \frac{l^2 k_1}{\rho_1} v_3 + \frac{\widetilde{\widetilde{\mu_2}}}{\rho_1} z_3(., 1) \right\} \phi_2 dx \\
= \int_0^L \left\{ \frac{\lambda g_0^3}{\rho_1} \left(\int_0^s (\lambda v_1 - f_1 + f_{12}) e^\tau d\tau \right) e^{-s} + f_5 + \frac{\widetilde{\widetilde{\mu_1}}}{\rho_1} f_2 \right\} \phi_2 dx \\
+ \int_0^L \left\{ \lambda \left(\int_0^s (\lambda v_2 - f_2 + f_{11}) e^\tau d\tau \right) e^{-s}\lambda \right\} \phi_2 dx, \\[2ex]
\int_0^L \left\{ \left(\lambda^2 + \frac{\widetilde{\mu_1}\lambda}{\rho_1} \right) v_1 + \frac{lk_1}{\rho_1} v_3 + \frac{k_1}{\rho_1}\partial_x v_1 \right\} \phi_3 dx \\
- \int_0^L \left\{ \frac{1}{\rho_1}\left(k_2 - g_0^2\right)\partial_{xx} v_2 \right\} \phi_3 dx + \int_0^L \left\{ \frac{\widetilde{\mu_2}}{\rho_1} z_2(., 1) + \frac{lk_1}{\rho_2} v_3 \right\} \phi_3 dx \\
= \int_0^L \left\{ \frac{g_0^3}{\rho_1}\partial_{xx} \left(\int_0^s (\lambda v_2 - f_2 + f_{12}) e^\tau d\tau \right) e^{-s} \right\} \phi_3 dx \\
+ \int_0^L \left\{ \left(\lambda + \frac{\widetilde{\mu_1}}{\rho_1} \right) f_1 + f_6 \right\} \phi_3 dx.
\end{cases}
$$

Consequently, problem (3.28) is equivalent to the problem

$$
(3.29)\qquad a((v_1, v_2, v_3), (\phi_1, \phi_2, \phi_3)) = L(\phi_1, \phi_2, \phi_3),
$$

where the bilinear form $a : [H_0^1(0,L)\times H_0^1(0,L)\times H_0^1(0,L)]^2 \to R$ and the linear form $L : H_0^1(0,L)\times H_0^1(0,L)\times H_0^1(0,L) \to R$ are defined by

$$
\begin{aligned}
&a((v_1,v_2,v_3,\phi_1,\phi_2,\phi_3)\\
&= \int_0^L \left\{ \lambda^2 v_3 + \frac{1}{\rho_1}\left(k_1 + \frac{\mu_1\lambda}{\rho_1}v_3 - g_0^1\right)\partial_{xx}v_1 \right\}\phi_1 dx\\
&- \int_0^L \left\{ \frac{l}{\rho_1}\left(k_1\partial_x v_3 - lk_3 v_1\right)\right\}\phi_1 dx\\
&- \int_0^L \left\{ \frac{l}{\rho_1}\left(k_1\partial_x v_3 + k_2\partial_x v_3\right) + \frac{\mu_2}{\rho_1}z_1(.,1)\right\}\phi_1 dx\\
&+ \int_0^L \left\{ \lambda^2\left(\frac{lk_1}{\rho_1} + \frac{\widetilde{\widetilde{\mu}}\lambda}{\rho_2}\right)v_2 + \frac{l^2k_1}{\rho_1}v_3\right\}\phi_2 dx\\
&+ \int_0^L \left\{ \frac{1}{\rho_1}\left(k_1 + k_2\partial_x v_1 - \frac{1}{\rho_1}(k_3 - g_0^3)\right)\partial_x v_3 + \frac{\widetilde{\widetilde{\mu_2}}}{\rho_1}z_3(.,1)\right\}\phi_2 dx\\
&+ \int_0^L \left\{ \left(\lambda^2 + \frac{\widetilde{\mu_1}\lambda}{\rho_1}\right)v_1 + \frac{lk_1}{\rho_1}v_3 + \frac{k_1}{\rho_1}\partial_x v_1 + \frac{lk_1}{\rho_2}v_3\right\}\phi_3 dx\\
&- \int_0^L \left\{ \frac{1}{\rho_1}\left(k_2 - g_0^2\right)\partial_{xx}v_2 + \frac{\widetilde{\mu_2}}{\rho_1}z_2(.,1)\right\}\phi_3 dx,
\end{aligned}
\tag{3.30}
$$

$$
\begin{aligned}
&L(\phi_1,\phi_2,\phi_3)\\
&= \int_0^L \left\{ \left(\frac{\mu_1\lambda}{\rho_1} + 1\right)f_3 + \left(\int_0^s (\lambda v_3 - f_3 + f_{10})e^{\tau}d\tau\right)e^{-s}\right\}\phi_1 dx\\
&+ \int_0^L \left\{ \frac{\lambda g_0^3}{\rho_1}\left(\int_0^s (\lambda v_1 - f_1 + f_{12})e^{\tau}d\tau\right)e^{-s} + f_5 + \frac{\widetilde{\widetilde{\mu_1}}}{\rho_1}f_2\right\}\phi_2 dx\\
&+ \int_0^L \left\{ \lambda\left(\int_0^s (\lambda v_2 - f_2 + f_{11})e^{\tau}d\tau\right)e^{-s}\lambda\right\}\phi_2 dx\\
&+ \int_0^L \left\{ \frac{g_0^3}{\rho_1}\partial_{xx}\left(\int_0^s (\lambda v_2 - f_2 + f_{12})e^{\tau}d\tau\right)e^{-s}\right\}\phi_3 dx,\\
&+ \int_0^L \left\{ \left(\lambda + \frac{\widetilde{\mu_1}}{\rho_1}\right)f_1 + f_6\right\}\phi_3 dx.
\end{aligned}
\tag{3.31}
$$

It is easy to verify that a is continuous, coercive and L is continuous. So applying the Lax-Milgram theorem, we deduce that for all $(\phi_1,\phi_2,\phi_3) \in H_0^1(0,L)\times H_0^1(0,L)\times H_0^1(0,L)$ problem (3.4)-(3.5) admits a unique solution $(v_1,v_2,v_3) \in H_0^1(0,L)\times H_0^1(0,L)\times H_0^1(0,L)$. Applying the classical elliptic regularity, it follows from (3.29) that $(v_1,v_1,v_3) \in H^2(0,L)\times H^2(0,L)\times H^2(0,L)$. Therefore, the operator $\lambda I - A$ is surjective for any $\lambda > 0$. Consequently, the existence result of Theorem 3.1 follows from the Hille-Yosida theorem.

4. Asymptotic Stability

In this section, we prove the asymptotic stability result by constructing a suitable Lyapunov functional. Now, let us introduce the following functionals

$$I_1(t) = -\rho_1 \int_0^L \varphi_t \int_0^{+\infty} g_1(s)\eta_1 ds dx, \tag{4.1}$$

$$I_2(t) = -\rho_2 \int_0^L \psi_t \int_0^{+\infty} g_2(s)\eta_2 ds dx, \tag{4.2}$$

$$I_3(t) = -\rho_1 \int_0^L \omega_t \int_0^{+\infty} g_3(s)\eta_3 ds dx, \tag{4.3}$$

$$I_4(t) = \int_0^L (\rho_1 \varphi\varphi_t + \rho_2 \psi\psi_t + \rho_1 \omega\omega_t)\, dx, \tag{4.4}$$

$$I_5(t) = \int_0^L \int_0^1 \sum_{i=1}^{3} \xi_i(t) e^{-2\tau_i(t)\rho} z_i^2(x,t,\rho)\, d\rho\, dx, \tag{4.5}$$

where

$$I_5(t) = I_6(t) + I_7(t) + I_8(t), \tag{4.6}$$

such that

$$I_6(t) = \int_0^L \int_0^1 \xi_1(t) e^{-2\tau_1(t)\rho} z_1^2(x,t,\rho)\, d\rho\, dx, \tag{4.7}$$

$$I_7(t) = \int_0^L \int_0^1 \xi_2(t) e^{-2\tau_2(t)\rho} z_2^2(x,t,\rho)\, d\rho\, dx, \tag{4.8}$$

$$I_8(t) = \int_0^L \int_0^1 \xi_3(t) e^{-2\tau_3(t)\rho} z_3^2(x,t,\rho)\, d\rho\, dx, \tag{4.9}$$

$$I_0(t) = I_1(t) + I_2(t) + I_3(t). \tag{4.10}$$

Then the following result holds.

Lemma 4.1. *(Compactness-Uniqueness). There exists a positive constant C such that the following inequality holds for every* $(\varphi, \psi, \omega) \in (H_0^1(0,L))^3$

$$\int_0^L (|\varphi_x|^2 + |\psi_x|^2 + |\omega_x|^2)\, dx \le C \int_0^L (k_2|\psi_x|^2 + k_1|\varphi_x + \psi + l\omega|^2)dx + k_3|\omega_x - l\varphi|^2\, dx. \tag{4.11}$$

Proof. We will argue by contradiction. Indeed, let us suppose that is not true. So, we can find a sequence $\{(\varphi_\nu, \psi_\nu, \omega_\nu)\}_{\nu\in N}$ in $(H_0^1(0,L))^3$ satisfying

$$\int_0^L (k_2|\psi_{\nu x}|^2 + k_1|\varphi_{\nu x} + \psi + l\omega_\nu|^2 + k_3|\omega_{\nu x} - l\varphi_\nu|^2)\, dx \le \frac{1}{\nu} \tag{4.12}$$

and

$$\int_0^L (|\varphi_{\nu x}|^2 + |\psi_{\nu x}|^2 + |\omega_{\nu x}|^2)\, dx = 1. \tag{4.13}$$

From (4.13), the sequence $\{(\varphi_\nu, \psi_\nu, \omega_\nu)\}_{\nu\in N}$ is bounded in $(H_0^1(0,L))^3$. Since the embedding $H_0^1(0,L) \hookrightarrow L^2(0,L)$ is compact, then the sequence $\{(\varphi_\nu, \psi_\nu, \omega_\nu)\}_{\nu\in N}$ converge strongly in $(L^2(0,L))^3$. From (4.13)

$$\psi_{\nu x} \to 0 \text{ strongly in } L^2(0,L). \tag{4.14}$$

Using Poincaré's inequality we can conclude that

$$\psi_\nu \to 0 \text{ strongly in } L^2(0,L). \tag{4.15}$$

Now, setting $\varphi_\nu \to \varphi$ and $\omega_\nu \to \omega$ strongly in $L^2(0,L)$. From (4.14), we have

$$\varphi_{\nu x} + \psi_\nu + l\omega_\nu \to 0 \text{ strongly in } L^2(0,L). \tag{4.16}$$

Then

$$\varphi_{\nu x} + \psi_\nu + l\omega_\nu = \varphi_{\nu x} + \psi_\nu + l(\omega_\nu - \omega) + l\omega \to 0 \text{ strongly in } L^2(0,L). \tag{4.17}$$

which implies that

$$\varphi_{\nu x} \to -l\omega \text{ strongly in } L^2(0,L). \tag{4.18}$$

Then, $\{\varphi_\nu\}_n$ is a Cauchy sequence in $H^1(0,L)$. Therefore $\{\varphi_\nu\}_n$ converge to a function φ_1 in $H^1(0,L)$. Consequently $\{\varphi_\nu\}_n$ converge to φ_1 in $L^2(0,L)$. Thus by the uniqueness of the limit $\varphi_1 = \varphi$. Moreover $\varphi \in H_0^1(0,L)$, then from (4.18) we deduce that

$$\varphi_x + l\omega = 0 \text{ a.e } x \in (0,L). \tag{4.19}$$

Similarly, we have

$$\omega_x - l\varphi = 0 \text{ a.e } x \in (0,L), \tag{4.20}$$

and $\omega \in H_0^1(0,L)$. Using (4.16) and (4.18), we deduce that $\varphi = \omega = 0$. This contradicts (4.8). Hence the proof is completed. □

Lemma 4.2. *The functional defined in (4.10) satisfies for any* $\delta > 0$

$$
\begin{aligned}
&I_0'(t) \leq -\rho_1(g_1^0 - \delta(1+\mu_2)) \int_0^L \varphi_t^2 dx + \widetilde{\mu_1}\delta \int_0^L \psi_t^2 dx \\
&+\widetilde{\widetilde{\mu_1}}\delta \int_0^L \omega_t^2 dx + c_\delta \int_0^L \left\{\psi_x + (\varphi_x + \psi + l\omega)^2 + (\omega_x - l\varphi)^2\right\} dx \\
&+c_\delta \int_0^L \int_0^\infty g_1(s)(\partial_x \eta_1)^2 ds dx - c_\delta \int_0^L \int_0^\infty g_1'(s)(\partial_x \eta_1)^2 ds dx \\
&+c_\delta \int_0^L \int_0^\infty g_2(s)(\partial_x \eta_2)^2 ds dx - c_\delta \int_0^L \int_0^\infty g_2'(s)(\partial_x \eta_2)^2 ds dx \\
&+c_\delta \int_0^L \int_0^\infty g_3(s)(\partial_x \eta_1)^2 ds dx - c_\delta \int_0^L \int_0^\infty g_3'(s)(\partial_x \eta_3)^2 ds dx \\
&+c_\delta \int_0^L \left\{z_1^2(x,1,t) + z_2^2(x,1,t) + z_3^2(x,1,t)\right\} dx.
\end{aligned}
\tag{4.21}
$$

Proof. Differentiating (4.10) with respect to t and using the third equation in (3.4)-(3.5), integrating by parts and using the fact that

$$
\begin{aligned}
&\tfrac{d}{dt} \int_0^\infty g_1(s)\eta_1 ds = \frac{d}{dt} \int_0^\infty g_1(t-s)\left(\varphi(t) - \varphi(s)\right) ds \\
&= \int_0^\infty g_1'(t-s)\left(\varphi(t) - \varphi(s)\right) ds + \left(\int_0^\infty g_1(t-s) ds\right) \varphi_t \\
&= \int_0^\infty g_1'(s)\eta_1 ds + g_1^0 \varphi_t,
\end{aligned}
\tag{4.22}
$$

in the same way for

$$
\frac{d}{dt} \int_0^\infty g_2(s)\eta_2 ds = \int_0^\infty g_2'(s)\eta_2 ds + g_2^0 \psi_t,
\tag{4.23}
$$

and

$$
\frac{d}{dt} \int_0^\infty g_1(s)\eta_3 ds = \int_0^\infty g_3'(s)\eta_3 ds + g_3^0 \omega_t.
\tag{4.24}
$$

we conclude that

$$
\begin{aligned}
I_0'(t) &= -\rho_1 g_1^0 \int_0^L \varphi_t^2 dx - \rho_1 \int_0^L \varphi_t \int_0^\infty g_1'(s)\eta_1 dsdx \\
&+k_1 \int_0^L (\varphi_x + \psi + l\omega) \int_0^\infty g_1(s)\partial_x \eta_1 dsdx \\
&-k_3 \int_0^L (\omega_x - l\varphi) \int_0^\infty g_1(s)\eta_1 dsdx \\
&- \int_0^L \varphi_x \left(\int_0^\infty g_1(s)\partial_x \eta_1 ds \right) \\
&+ \int_0^L \left(\int_0^\infty g_1(s)\partial_x \eta_1 ds \right)^2 dx + \mu_1 \int_0^L \varphi_t(x,t) \int_0^\infty g_1(s)\eta_1 dsdx \\
&+\mu_2 \int_0^L z_1(x,1,t) \int_0^\infty g_1(s)\eta_1 dsdx + \widetilde{\mu_1} \int_0^L \varphi_t(x,t) \int_0^\infty g_2(s)\eta_2 dsdx \\
&+\widetilde{\mu_2} \int_0^L z_2(x,1,t) \int_0^\infty g_2(s)\eta_2 dsdx + \widetilde{\widetilde{\mu_1}} \int_0^L \varphi_t(x,t) \int_0^\infty g_3(s)\eta_3 dsdx \\
&+\widetilde{\widetilde{\mu_2}} \int_0^L z_3(x,1,t) \int_0^\infty g_3(s)\eta_3 dsdx.
\end{aligned} \tag{4.25}
$$

Using Young's, Poincaré's and Holder's inequalities for the last six terms of the above equality, using the second and third equations of (3.5), we find

$$
\begin{aligned}
I_0'(t) &\leq -\rho_1(g_1^0 - \delta(1+\mu_2)) \int_0^L \varphi_t^2 dx + \widetilde{\mu_1}\delta \int_0^L \psi_t^2 dx \\
&+\widetilde{\widetilde{\mu_1}}\delta \int_0^L \omega_t^2 dx + c_\delta \int_0^L \left\{ \psi_x + (\varphi_x + \psi + l\omega)^2 + (\omega_x - l\varphi)^2 \right\} dx \\
&+c_\delta \int_0^L \int_0^\infty g_1(s)(\partial_x \eta_1)^2 dsdx - c_\delta \int_0^L \int_0^\infty g_1'(s)(\partial_x \eta_1)^2 dsdx \\
&+c_\delta \int_0^L \int_0^\infty g_2(s)(\partial_x \eta_2)^2 dsdx - c_\delta \int_0^L \int_0^\infty g_2'(s)(\partial_x \eta_2)^2 dsdx \\
&cr + c_\delta \int_0^L \int_0^\infty g_3(s)(\partial_x \eta_1)^2 dsdx - c_\delta \int_0^L \int_0^\infty g_3'(s)(\partial_x \eta_3)^2 dsdx \\
&+c_\delta \int_0^L \left\{ z_1^2(x,1,t) + z_2^2(x,1,t) + z_3^2(x,1,t) \right\} dx.
\end{aligned} \tag{4.26}
$$

The proof is hence complete. □

Lemma 4.3. *The functional defined in (4.4) satisfies for any $\epsilon > 0$*

$$
\begin{aligned}
I_4'(t) \leq & \int_0^L \left\{(\rho_1+\epsilon)\varphi_t^2 + (\rho_2+\epsilon)\psi_t^2 + (\rho_1+\epsilon)\omega_t^2\right\} dx \\
& -c_1 \int_0^L \left\{\psi_x^2 + (\varphi_x+\psi+l\omega)^2 + (\omega_x - l\varphi)^2\right\} dx \\
& +c_\epsilon \int_0^L \int_0^\infty \left\{g_1(s)(\partial_x\eta_1)^2 + g_2(s)(\partial_x\eta_2)^2 + g_3(s)(\partial_x\eta_3)^2\right\} dx \\
& +c_\epsilon \int_0^L \left\{z_1^2(x,1,t) + z_2^2(x,1,t) + z_3^2(x,1,t)\right\} dx.
\end{aligned}
\tag{4.27}
$$

Proof. Differentiating $I_4(t)$ with respect to t, we see that

$$
\begin{aligned}
I_4'(t) = & \int_0^L (\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1\omega_t^2)dx - k_1 \int_0^L (\varphi_x+\psi+l\omega)^2 dx \\
& +g_1^0 \int_0^L \varphi_x^2 dx - (k_2 - g_2^0) \int_0^L \psi_x^2 dx + g_3^0 \int_0^L \omega_x^2 dx \\
& - \int_0^L \varphi_x \int_0^{+\infty} g_1(s)\partial_x\eta_1 ds dx - \int_0^L \psi_x \int_0^{+\infty} g_2(s)\partial_x\eta_2 ds dx \\
& - \int_0^L \omega_x \int_0^{+\infty} g_3(s)\partial_x\eta_3 ds dx - \mu_1 \int_0^L \varphi_t\varphi dx - \widetilde{\mu_1} \int_0^L \psi_t\psi dx \\
& -\widetilde{\widetilde{\mu_1}} \int_0^L \omega_t\omega dx - \mu_2 \int_0^L z_1(x,1,t)\varphi dx - \widetilde{\mu_2} \int_0^L z_2(x,1,t)\psi dx \\
& -\widetilde{\widetilde{\mu_2}} \int_0^L z_3(x,1,t)\omega dx - k_3 \int_0^L (\omega_x - l\varphi)^2 dx.
\end{aligned}
\tag{4.28}
$$

Using Young's and Poincare's inequalities, we get for any $\epsilon > 0$

$$
\begin{aligned}
& - \int_0^L \varphi_x \int_0^\infty g_1(s)\partial_x\eta_1 ds dx - \int_0^L \psi_x \int_0^\infty g_2(s)\partial_x\eta_2 ds dx \\
& - \int_0^L \omega_x \int_0^\infty g_3(s)\partial_x\eta_3 ds dx dx \\
& \leq \epsilon \int_0^L (\varphi_x^2 + \psi_x^2 + \omega_x^2)dx + c \int_0^L \int_0^\infty \sum_{i=1}^3 (g_i(s)(\partial_x\eta_i)^2 ds dx,
\end{aligned}
\tag{4.29}
$$

$$
\begin{aligned}
& -\mu_1 \int_0^L \varphi_t\varphi dx - \widetilde{\mu_1} \int_0^L \psi_t\psi dx - \widetilde{\widetilde{\mu_1}} \int_0^L \omega_t\omega dx \\
& \leq \epsilon \int_0^L (\varphi_t^2 + \psi_t^2 + \omega_t^2)dx + c_\epsilon \int_0^L (\varphi_x^2 + \psi_x^2 + \omega_x^2)dx,
\end{aligned}
\tag{4.30}
$$

$$
\begin{aligned}
& -\mu_2 \int_0^L z_1(x,1,t)\varphi dx - \widetilde{\mu_2} \int_0^L z_2(x,1,t)\psi dx - \widetilde{\widetilde{\mu_2}} \int_0^L z_3(x,1,t)\omega dx \\
& \leq \int_0^L c_\epsilon \left\{z_1^2(x,1,t) + z_2^2(x,1,t) + z_3^2(x,1,t)\right\} dx + \epsilon \int_0^L (\varphi_x^2 + \psi_x^2 + \omega_x^2)dx.
\end{aligned}
\tag{4.31}
$$

Inserting (4.29)-(4.31) into (4.28), we find

$$
\begin{aligned}
I_4'(t) \leq & \int_0^L \left\{(\rho_1+\epsilon)\varphi_t^2 + (\rho_2+\epsilon)\psi_t^2 + (\rho_1+\epsilon)\omega_t^2\right\} dx \\
& -(k_0-2\epsilon)\int_0^L (\varphi_x^2+\psi_x^2+\omega_x^2)dx \\
& +c_\epsilon \int_0^L\int_0^\infty \left\{g_1(s)(\partial_x\eta_1)^2 + g_2(s)(\partial_x\eta_2)^2 + g_3(s)(\partial_x\eta_3)^2\right\} dx \\
& +c_\epsilon \int_0^L \left\{z_1^2(x,1,t)+z_2^2(x,1,t)+z_3^2(x,1,t)\right\} dx.
\end{aligned} \tag{4.32}
$$

Then (4.27) is proved. □

Lemma 4.4. *Then the functional defined in (4.5) satisfies*

$$
\begin{aligned}
\tfrac{d}{dt}I_5(t) \leq & -2c\xi_1(t)I_6(t) - \tfrac{c\xi_1(t)}{2\tau_{11}}\int_0^L z_1^2(x,1,t))dx + \frac{\xi_1(t)}{\tau_{01}}\int_0^L \varphi_t^2(x,t)\,dx \\
& -2c\xi_2(t)I_7(t) - \tfrac{c\xi_2(t)}{2\tau_{22}}\int_0^L z_2^2(x,1,t))dx + \frac{\xi_2(t)}{\tau_{02}}\int_0^L \psi_t^2(x,t)\,dx \\
& -2c\xi_3(t)I_8(t) - \tfrac{c\xi_3(t)}{2\tau_{33}}\int_0^L z_3^2(x,1,t))dx + \frac{\xi_3(t)}{\tau_{03}}\int_0^L \omega_t^2(x,t)\,dx.
\end{aligned} \tag{4.33}
$$

Where τ_{01}, τ_{02} τ_{03} , τ_{11}, τ_{22} and τ_{33} are a positive constants.

Proof. Differentiating (4.5) with respect to t and using the third equation in (3.4), we have

$$
\begin{aligned}
\tfrac{I_6(t)}{dt} &\leq \tfrac{d}{dt}\left[\xi_1(t)e^{-\rho\tau_1(t)}\int_0^L\int_0^1 z_1^2(x,\rho,t))d\rho dx\right]\\
&= \xi_1'(t)e^{-\tau_1(t)\rho}\int_0^L\int_0^1 z_1^2(x,\rho,t))dkdx\\
&-\xi_1(t)\rho e^{-\tau_1(t)\rho}\tau_1'(t)\int_0^L\int_0^1 z_1^2(x,\rho,t))d\rho dx\\
&+\tfrac{1}{\tau_1(t)}e^{-\tau_1(t)\rho}\tau_1(t)\xi_1(t)\int_0^L\int_0^1 \frac{d}{dt}z_1^2(x,\rho,t))d\rho dx\\
&= \xi_1'(t)e^{-\tau_1(t)\rho}\int_0^L\int_0^1 z_1^2(x,\rho,t))d\rho dx\\
&-\xi_1(t)\rho e^{-\tau_1(t)\rho}\tau_1'(t)\int_0^L\int_0^1 z_1^2(x,\rho,t))d\rho dx\\
&+\tfrac{1}{\tau_1(t)}e^{-\tau_1(t)\rho}\xi_1(t)\int_0^L\int_0^1 \frac{\partial}{\partial\rho}(1-\tau_1'(t)\rho)z_1^2(x,\rho,t))d\rho dx\\
&\leq -\xi_1(t)\rho e^{-\tau_1(t)\rho}\tau_1'(t)\int_0^L\int_0^1 z_1^2(x,\rho,t))d\rho dx\\
&+\xi_1(t)\tfrac{\beta}{\tau_1(t)}\int_0^L z_1^2(x,1,t)dx\\
&+\tfrac{1}{\tau_1(t)}\left[\xi_1(t)\int_0^L[z_1^2(x,0,t))dx - z_1^2(x,1,t)]dx\right]\\
&\leq -2c\xi_1(t)I_6(t) - \tfrac{c\xi_1(t)}{2\tau_{11}}\int_0^L z_1^2(x,1,t))dx + \frac{\xi_1(t)}{\tau_{01}}\int_0^L \varphi_t^2(x,t)\,dx,
\end{aligned}
\tag{4.34}
$$

in the same way for $I_7(t)$ and $I_8(t)$

$$
\frac{I_7(t)}{dt} \leq -2c\xi_2(t)I_7(t) - \frac{c\xi_2(t)}{2\tau_{22}}\int_0^L z_2^2(x,1,t))dx + \frac{\xi_2(t)}{\tau_{02}}\int_0^L \psi_t^2(x,t)\,dx,
\tag{4.35}
$$

$$
\frac{I_8(t)}{dt} \leq -2c\xi_3(t)I_8(t) - \frac{c\xi_3(t)}{2\tau_{33}}\int_0^L z_3^2(x,1,t))dx + \frac{\xi_3(t)}{\tau_{03}}\int_0^L \omega_t^2(x,t)\,dx.
\tag{4.36}
$$

Summing (4.34), (4.35) and (4.36), we get the desired result. So the proof of Lemma 4.4 is completed. □

Now, let $N_1, N_2 > 0$ and

$$
L(t) = N_1E(t) + N_2(I_1 + I_2 + I_3) + I_4 + I_5,
\tag{4.37}
$$

where E is the energy functional associated to (3.4) and defined in (3.7). Note that E is non-increasing according to (3.8),

$$E'(t) \leq -\left(\mu_1 - \frac{\xi_1}{2} - \frac{\mu_1}{2\sqrt{1-d_1}}\right)\|\varphi_t\|_2^2 - \left(\widetilde{\mu_1} - \frac{\xi_2}{2} - \frac{\widetilde{\mu_1}}{2\sqrt{1-d_2}}\right)\|\psi_t\|_2^2$$

$$-\left(\widetilde{\mu_1} - \frac{\xi_3}{2} - \frac{\widetilde{\mu_1}}{2\sqrt{1-d_3}}\right)\|\omega_t\|_2^2 - \left(\frac{\xi_1(1-\tau_1'(t))}{2} - \frac{\mu_2\sqrt{1-d_1}}{2}\right)\|z_1(x,1,t)\|_2^2$$

$$-\left(\frac{\xi_2(1-\tau_2'(t))}{2} - \frac{\widetilde{\mu_2}\sqrt{1-d_2}}{2}\right)\|z_2(x,1,t)\|_2^2$$

$$-\left(\frac{\xi_3(1-\tau_3'(t))}{2} - \frac{\widetilde{\widetilde{\mu_2}}\sqrt{1-d_3}}{2}\right)\|z_3(x,1,t)\|_2^2$$

$$+\frac{1}{2}\int_0^L\int_0^\infty g_1'(s)(\partial_x\eta_1)^2dsdx + \frac{1}{2}\int_0^L\int_0^\infty g_2'(s)(\partial_x\eta_2)^2dsdx$$

$$+\frac{1}{2}\int_0^L\int_0^\infty g_3'(s)(\partial_x\eta_3)^2dsdx.$$

Using (3.8),(4.21) and (4.27) with (4.33), we get

$$
\begin{aligned}
L'(t) \le & -N_2(c_1 - c_\delta)\int_0^L \left\{\psi_x + (\varphi_x + \psi + l\omega)^2 + (\omega_x - l\varphi)^2\right\} dx \\
& - \left\{N_2\rho_1(g_1^0 - \delta(1+\mu_1)) - N_2 c - \tfrac{\xi_1(t)}{\tau_{01}}\right\} \int_0^L \varphi_t dx \\
& + N_1\left(\mu_1 - \tfrac{\xi_1(t)}{2} - \tfrac{\mu_1}{2\sqrt{1-d_1}}\right)\int_0^L \varphi_t dx \\
& - \left\{N_1\left(\widetilde{\mu_1} - \tfrac{\xi_2(t)}{2} - \tfrac{\widetilde{\mu_2}}{2\sqrt{1-d_2}}\right) - \tfrac{\xi_1(t)}{\tau_{02}} - N_2\widetilde{\widetilde{\mu_2}}\right\}\int_0^L \psi_t dx \\
& - \left\{N_1\left(\widetilde{\widetilde{\mu_1}} - \tfrac{\xi_3(t)}{2} - \tfrac{\widetilde{\widetilde{\mu_2}}}{2\sqrt{1-d_3}}\right) - \tfrac{\xi_3(t)}{\tau_{03}} - N_2\widetilde{\widetilde{\mu_2}}\right\}\int_0^L \omega_t dx \\
& - \left\{N_1\left(\tfrac{\xi_1(1-\tau_1'(t))}{2} - \tfrac{\mu_2\sqrt{1-d_1}}{2}\right) - (c_\delta + c_\epsilon)N_2 + \tfrac{c\xi_1(t)}{2\tau_{11}}\right\}\|z_1(x,1,t)\|_2^2 \qquad (4.38)\\
& - \left\{N_1\left(\tfrac{\xi_2(1-\tau_1'(t))}{2} - \tfrac{\widetilde{\mu_2}\sqrt{1-d_2}}{2}\right) - (c_\delta + c_\epsilon)N_2 + \tfrac{c\xi_2(t)}{2\tau_{22}}\right\}\|z_2(x,1,t)\|_2^2 \\
& - \left\{N_1\left(\tfrac{\xi_3(1-\tau_1'(t))}{2} - \tfrac{\widetilde{\widetilde{\mu_2}}\sqrt{1-d_3}}{2}\right) - (c_\delta + c_\epsilon)N_2 + \tfrac{c\xi_3(t)}{2\tau_{33}}\right\}\|z_3(x,1,t)\|_2^2 \\
& + N_2(c_\epsilon + c_\delta)\int_0^L \sum_{i=1}^3 \int_0^\infty g_i(s)(\partial_x\eta_i)^2 ds dx \\
& + \left(\tfrac{N_1}{2} - c_\delta N_2\right)\int_0^L\int_0^\infty \sum_{i=1}^3 (g_i'(s)\partial_x\eta_i)^2 ds dx \\
& - 2\int_0^L\int_0^1 \sum_{i=1}^3 \xi_i(t) z_i^2(x,\rho,t) d\rho dx.
\end{aligned}
$$

We choose N_1 large enough so that

$$
\beta_1 = -\left\{N_2\rho_1(g_1^0 - \delta(1+\mu_1)) - N_2 c - \frac{\xi_1(t)}{\tau_{01}} + N_1\left(\mu_1 - \frac{\xi_1(t)}{2} - \frac{\mu_1}{2\sqrt{1-d_1}}\right)\right\} > 0
$$

$$
\beta_2 = -\left\{N_1\left(\widetilde{\mu_1} - \frac{\xi_2(t)}{2} - \frac{\widetilde{\mu_2}}{2\sqrt{1-d_2}}\right) - \frac{\xi_1(t)}{\tau_{02}} - N_2\widetilde{\widetilde{\mu_2}}\right\} > 0
$$

$$
\beta_3 = \left\{N_1\left(\widetilde{\widetilde{\mu_1}} - \frac{\xi_3(t)}{2} - \frac{\widetilde{\widetilde{\mu_2}}}{2\sqrt{1-d_3}}\right) - \frac{\xi_3(t)}{\tau_{03}} - N_2\widetilde{\widetilde{\mu_2}}\right\} > 0
$$

$$\beta_4 = \left\{ N_1 \left(\frac{\xi_1(1-\tau_1'(t))}{2} - \frac{\mu_2\sqrt{1-d_1}}{2} \right) - (c_\delta + c_\epsilon)N_2 + \frac{c\xi_1(t)}{2\tau_{11}} \right\} > 0$$

$$\beta_5 = \left\{ N_1 \left(\frac{\xi_2(1-\tau_2'(t))}{2} - \frac{\widetilde{\mu_2}\sqrt{1-d_2}}{2} \right) - (c_\delta + c_\epsilon)N_2 + \frac{c\xi_2(t)}{2\tau_{22}} \right\} > 0$$

$$\beta_6 = \left\{ N_1 \left(\frac{\xi_3(1-\tau_3'(t))}{2} - \frac{\widetilde{\widetilde{\mu_2}}\sqrt{1-d_3}}{2} \right) - (c_\delta + c_\epsilon)N_2 + \frac{c\xi_3(t)}{2\tau_{33}} \right\} > 0$$

such that $\min\{\beta_1, \beta_2, \beta_3\} > 0$. (Note that $g_i^0 > 0$ because g_i is continuous non-negative and $g_i(0) > 0$) and we find, for some positive constants c_4

$$\begin{aligned} L'(t) \leq & -c_4 E(t) + N_2(c_\epsilon + c_\delta) \int_0^L \sum_{i=1}^{3} \int_0^\infty g_i(s)(\partial_x \eta_i)^2 ds dx \\ & + \left(\tfrac{N_1}{2} - c_\delta N_2 \right) \int_0^L \int_0^\infty \sum_{i=1}^{3} (g_i'(s)\partial_x \eta_i)^2 ds dx. \end{aligned} \tag{4.39}$$

On the other hand, by (4.37) and definition of $E(t)$ and I_i, there exists a positive constant N_4(not depending on N_1) such that

$$(N_1 - N_4)E(t) \leq L(t) \leq ((N_1 + N_4)E(t). \tag{4.40}$$

Thus, choosing $N_1 > N_3$ and using the fact that,$g_i' \leq 0$, we conclude

$$L'(t) \leq -c_4 E(t) + N_2(c_\epsilon + +c_\delta) \int_0^L \sum_{i=1}^{3} \int_0^\infty g_i(s)(\partial_x \eta_i)^2 ds dx. \tag{4.41}$$

Lemma 4.5. ([17]) *For any $i = 1, 2, 3$, there exist positive constants α_i such that the following inequalities hold*

$$\int_0^L \int_0^\infty g_i(s)(\partial_x \eta_i)^2 \leq -\alpha_i E'(t) \; if \; (H2) \; holds \tag{4.42}$$

$$\zeta(t)L'(t) \leq -\eta_1 \zeta(t)E(t) - 2\eta_2 E'(t), \quad \forall \, t \geq t_0. \tag{4.43}$$

Proof. Define $\chi(t) = \zeta(t)L(t) + 2\eta_2 E(t)$, which is equivalent to $E(t)$ and $\zeta'(t) \leq 0 \, \forall t \geq 0$, we obtain

$$\begin{aligned} \chi'(t) & \leq \zeta'(t)L(t) - \eta_{14}\zeta(t)E(t) \\ & \leq -\alpha\zeta(t)E(t), \quad \forall \, t \geq t_0. \end{aligned} \tag{4.44}$$

Integrating the last inequality over (t_0, t), we conclude that

$$\chi(t) \leq \chi(0) \, e^{-\alpha \int_{t_0}^{t} \zeta(s) ds}. \tag{4.45}$$

Then, the equivalent relation between $\chi(t)$ and $E(t)$ yields

$$E(t) \le K e^{-\alpha \int_{t_0}^{t} \zeta(s) ds}. \tag{4.46}$$

This completes the proof. □

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Mohamed FERHAT
University of Oran
Department of Mathematics
P. O. Box 31000
ALGERIA
ferhat22@hotmail.fr

Ali HAKEM
Laboratory ACEDP
Djillali Liabes University
22000 Sidi Bel Abbes
ALGERIA
hakemali@yahoo.com