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A FRACTIONAL INTEGRAL OPERATOR INVOLVING THE MITTAG-LEFFLER TYPE FUNCTION WITH FOUR PARAMETERS*

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Abstract. In this paper our main aim is establishing a fractional integration formula (of pathway type) involving the Mittag-Leffler type function with four parameters ${}_{\zeta,\eta}E_{\mu,\nu}[z]$ which was recently introduced by Garg, Sharma and Manohar [Thai J. Math. (2015)]. Some interesting special cases of the main result are also considered and shown to be connected with certain known ones.
Keywords: Pathway fractional Integral operators, Mittag-Leffler type function, Generalized Wright function.

1. Introduction and Preliminaries

The Mittag-Leffler function and its various generalizations have been investigated by many researchers in both mathematics and engineering. Yet, during the twentieth century, they were practically unknown to the majority of scientists since they were ignored in the common books on special functions. Nowadays the Mittag-Leffler function and its numerous generalizations have gotten real and new lives. An extremely growing interest in the study of their diverse properties is due mainly to the close connection of the Mittag-Leffler function to fractional calculus, its application to the study of differential and integral equations of, in particular, fractional orders (see [1], [2], [5]–[7], [10], [11], [14], [20]–[23], [27], [28]).

In recent years, the fractional calculus has become one of the most rapidly growing parts of science as well as mathematics. The present far-reaching development of the fractional calculus has been shown by the remarkably large number of contributions (cf. [8], [9], [12], [13], [17], [19], [29]–[32] and the related references therein).

Very recently, Garg, Sharma and Manohar [3] introduced the generalized Mittag-Leffler type function with four parameters and study of its various properties, which

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mainly motivated our present investigation. Throughout this paper, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}_0^+$, $\mathbb{N}$ be sets of complex numbers, real and positive real numbers, nonpositive integers, and positive integers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The following definition is given in [3]:

Definition 1.1. The Mittag-Leffler type function with four parameters is defined in the following manner:

$$(1.1) \qquad {}_{\zeta,\eta}E_{\mu,\nu}[z] = \sum_{n=0}^{+\infty} \frac{(\zeta)_{\eta n}}{\Gamma(\mu n + \nu)}\, z^n \qquad (\mu, \nu, \zeta, \eta,\, z \in \mathbb{C},\ \mathrm{Re}(\mu) > \mathrm{Re}(\nu) > 0),$$

where $(\zeta)_n$ is the Pochhammer symbol defined, for $\zeta \in \mathbb{C}$, as follows (cf. [25, p. 2 and p. 5]):

$$(\zeta)_n = \begin{cases} 1, & n = 0, \\ \zeta(\zeta+1)\cdots(\zeta+n-1), & n \in \mathbb{N}, \end{cases}$$

i.e., $(\zeta)_n = \Gamma(\zeta+n)/\Gamma(\zeta)$ $\left(\zeta \in \mathbb{C} \setminus \mathbb{Z}_0^-\right)$ and Γ being the familiar Gamma function (cf. [24, Section 1.1] and [25, Section 1.1]).

A detail account of several results which include integral representations, recurrence relations, differential formula, fractional derivative and integral, Mellin Barnes integral representation and fractional calculus integral operator involving (1.1) can be found in the article [3]. Some important special cases of this function are listed below:

(i) When $\eta = 1$, with $\mu, \nu, \zeta, z \in \mathbb{C}$, $\mathrm{Re}(\mu) > \mathrm{Re}(\nu) > 0$, the function (1.1) reduces to the one that has been considered by Garg *et al.*:

$$ {}_{\zeta,1}E_{\mu,\nu}[z] = \sum_{n=0}^{+\infty} \frac{(\zeta)_n}{\Gamma(\mu n + \nu)} z^n.$$

(ii) If we set $\eta = 0$ with $\min\{\mathrm{Re}(\mu), \mathrm{Re}(\nu)\} > 0$, then (1.1) reduces to the generalized Mittag-Leffler function considered by Wiman [33]. The case when $\eta = 0$ and $\nu = 1$ can be found in [16].

In recent years, the Mittag-Leffler function and its various generalizations have become a very popular subject of mathematics and its applications. Among the large number of works regarding the Mittag-Leffler function, for a remarkably clear, insightful, and systematic exposition of the investigations carried out by various authors in the field of mathematical analysis and its applications, we refer the interested reader to a survey-cum-expository book by Gorenflo, Kilbas, Mainardi, and Rogosin[4], which contains a fairly comprehensive bibliography on this subject.

Definition 1.2. The H-function is defined in terms of a Mellin-Barnes integral in the following manner (see [15]):

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]=H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\(b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{\mathfrak{L}}\Theta(s)\,z^{-s}\mathrm{d}s,$$

where

$$\Theta(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{i=1}^{n}\Gamma(1-a_i-\alpha_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i+\alpha_i s)\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)},$$

and m, n, p, q are integers such that $0\le m\le q$, $0\le n\le p$, and for parameters a_i, $b_i\in\mathbb{C}$ and for parameters α_i, $\beta_j\in\mathbb{R}^+$ $(i=1,\ldots,p;\ j=1,\ldots,q)$ with the contour $\mathfrak{L}$ suitably chosen, and an empty product, if it occurs, is taken to be unity.

The theory of the H-function is well explained in the book of Srivastava, Gupta and Goyal ([26, Chap. 1]).

Definition 1.3. The generalized Wright's function is defined as follows (see, e.g., [34]):

$$(1.2)\qquad {}_p\Psi_q\left[\begin{matrix}(\alpha_1,\,A_1),\,\ldots,\,(\alpha_p,\,A_p);\\(\beta_1,\,B_1),\,\ldots,\,(\beta_q,\,B_q);\end{matrix}\ z\right]=\sum_{k=0}^{+\infty}\frac{\prod_{j=1}^{p}\Gamma(\alpha_j+A_j\,k)}{\prod_{j=1}^{q}\Gamma(\beta_j+B_j\,k)}\frac{z^k}{k!},$$

where the coefficients $A_1,\,\ldots,\,A_p\in\mathbb{R}^+$ and $B_1,\,\ldots,\,B_q\in\mathbb{R}^+$ with

$$1+\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j\geqq 0.$$

Here, in this paper, our main aim is to establish a (presumably) new fractional integration formula (of pathway type) involving the Mittag–Leffler type function ${}_{\zeta,\eta}E_{\mu,\nu}[z]$. Some interesting special cases of our main result are also considered. Our main result is obtained by applying the ${}_{\zeta,\eta}E_{\mu,\nu}[z]$ to the pathway type fractional integral operator given in (1.3). So we continue to recall the following definition.

Definition 1.4. Let $f(x)\in L(a,b)$, $\rho\in\mathbb{C}$, $\mathrm{Re}(\rho)>0$, $a>0$ and let us take a pathway parameter $\alpha<1$. Then the pathway fractional integration operator is defined and represented as follows (see [18, p. 239]):

$$(1.3)\qquad \left(P_{0^+}^{(\rho,\alpha,a)}f\right)(t)=t^{\rho}\int_0^{\frac{t}{a(1-\alpha)}}\left[1-\frac{a(1-\alpha)\,\tau}{t}\right]^{\frac{\rho}{1-\alpha}}f(\tau)\,\mathrm{d}\tau,$$

where $L(a,b)$ is the set of Lebesgue measurable functions defined on $(a,\,b)$.

Let $[a,\,b]$ $(-\infty<a<b<+\infty)$ be a finite interval on the real line $\mathbb{R}$. The left-sided and right-sided Riemann-Liouville fractional integrals $I_{a+}^{\rho}\,f$ and $I_{b-}^{\rho}\,f$ of order $\rho\in\mathbb{C}$ $(\mathrm{Re}(\rho)>0)$ are defined, respectively, by

$$(1.4)\qquad \left(I_{a+}^{\rho}\,f\right)(x):=\frac{1}{\Gamma(\rho)}\int_a^x\frac{f(t)\,\mathrm{d}t}{(x-t)^{1-\rho}}\qquad (x>a;\ \mathrm{Re}(\rho)>0)$$

and

$$\left(I_{b-}^{\rho} f\right)(x) := \frac{1}{\Gamma(\rho)} \int_x^b \frac{f(t)\,\mathrm{d}t}{(t-x)^{1-\rho}} \qquad (x < b;\ \mathrm{Re}(\rho) > 0),$$

where $f \in C_\xi$ $(\xi \geq -1)$ (cf. [8, p. 69]).

Remark 1.1. It is easy to see that the pathway fractional integration operator (1.3) with several parameters is essentially a special case of the left-sided Riemann-Liouville fractional integral (1.4). Indeed, by setting $t = a(1-\alpha)x$ in (1.3), after a little simplification, we find the following relationship between the two integral operators:

$$\big(P_{0+}^{(\rho,\alpha,a)} f\big)\big(a(1-\alpha)x\big) = a^{\rho}\,\rho\,(1-\alpha)^{\rho-1}\,\Gamma\left(\frac{\rho}{1-\alpha}\right) x^{\frac{\alpha\rho}{\alpha-1}} \left(I_{0+}^{1+\frac{\rho}{1-\alpha}} f\right)(x),$$

where $x > 0,\ a > 0,\ \alpha < 1,\ \mathrm{Re}(\rho) > 0$.

2. Pathway Fractional Integration of Generalized Multiindex Mittag-Leffler Functions

In this section we consider composition of the pathway fractional integral $P_{0+}^{(\eta,\alpha)}$ given by (1.3) with the Mittag-Leffler type function ${}_{\zeta,\eta}E_{\mu,\nu}[z]$ given by (1.1). We begin by stating the following theorem.

Theorem 2.1. *Let the parameters $\beta, \mu, \nu, \zeta, \eta, \rho \in \mathbb{C}$, $a > 0$, $c \in \mathbb{R}$, $\rho > 0$, $\mathrm{Re}(\mu) > \mathrm{Re}(\nu) > 0$, $\beta > 0$, and $\alpha < 1$. Then we have the following relation:*

$$P_{0+}^{(\rho,\alpha,a)} \left\{x^{\lambda-1}\,{}_{\zeta,\eta}E_{\mu,\nu}[c\,x^{\beta}]\right\} = \frac{x^{\rho+\lambda}\,\Gamma\left(1+\dfrac{\rho}{1-\alpha}\right)}{\Gamma(\zeta)\,(a(1-\alpha))^{\lambda}} \times {}_3\Psi_2\left[\begin{matrix}(\zeta,\eta),\ (\lambda,\beta),(1,1);\\ \Big(\nu,\mu\Big),\Big(1+\lambda+\dfrac{\rho}{1-\alpha},\ \beta\Big);\end{matrix}\ c\left(\frac{x}{a(1-\alpha)}\right)^{\beta}\right]. \tag{2.1}$$

Proof. Let the left-hand side of the formula (2.1) be denoted by $\mathcal{J}$. Applying (1.1) and using the definition (1.3) to (2.1), we get

$$\begin{aligned}\mathcal{J} &= P_{0+}^{(\rho,\alpha,a)}\left\{x^{\lambda-1}\sum_{n=0}^{+\infty}\frac{(\zeta)_{\eta n}}{\Gamma(\mu n+\nu)}\,[cx^{\beta}]^n\right\}\\ &= \sum_{n=0}^{+\infty}\frac{(\zeta)_{\eta n}}{\Gamma(\mu n+\nu)}\,c^n P_{0+}^{(\rho,\alpha,a)}\left\{x^{\lambda+\beta n-1}\right\}.\end{aligned}$$

By using the well-known relationship between the Beta function $B(\alpha,\beta)$ and the Gamma function (cf. [24, pp. 9–11] and [25, pp. 7–10]), it is easy to find the following formula (see also [18, Eq. (12)]):

$$P_{0^+}^{(\rho,\alpha,a)}\left\{t^{\beta-1}\right\} = \frac{t^{\rho+\beta}}{[a(1-\alpha)]^{\beta}}\,\frac{\Gamma(\beta)\,\Gamma\left(1+\dfrac{\rho}{1-\alpha}\right)}{\Gamma\left(\dfrac{\rho}{1-\alpha}+\beta+1\right)}, \tag{2.2}$$

where $\alpha < 1$; $\mathrm{Re}(\rho) > 0$; $\mathrm{Re}(\beta) > 0$. Here, applying (2.2) with β replaced by $(\lambda + \beta\, n)$ to the pathway integral, after a little simplification, we obtain the following expression

$$\mathcal{J} = \sum_{n=0}^{+\infty} \frac{\Gamma(\zeta+\eta\, n)}{\Gamma(\zeta)\Gamma(\mu\, n+\nu)}\,\frac{\Gamma\left(\lambda+\beta\, n\right)\Gamma\left(1+\dfrac{\rho}{1-\alpha}\right)}{\Gamma\left(1+\frac{\rho}{1-\alpha}+\lambda+\beta\, n\right)}\,\frac{c^n x^{\rho+\beta n+\lambda}}{[a(1-\alpha)]^{\lambda+n\beta}}$$

$$= \frac{x^{\rho+\lambda}\,\Gamma(1+\dfrac{\rho}{1-\alpha})}{\Gamma(\zeta)\,(a(1-\alpha))^{\lambda}} \sum_{n=0}^{+\infty} \frac{\Gamma\left(\zeta+\eta\, n\right)\Gamma\left(\lambda+\beta\, n\right)\Gamma\left(1+n\right)}{\Gamma\left(1+\dfrac{\rho}{1-\alpha}+\lambda+\beta\, n\right)\Gamma(\mu\, n+\nu) n!}\,\frac{c^n x^{\beta\, n}}{[a(1-\alpha)]^{\beta\, n}},$$

whose last summation, in view of (1.2), is easily seen to arrive at the expression in (2.1). This completes the proof. □

Indeed, by suitably specializing the values of the parameters $\beta, \mu, \nu, \zeta, \eta, \rho \in \mathbb{C}$, one can deduce numerous fractional calculus results involving the various types of Mittag-Leffler functions as the corollary of our main result. Further we can present a large number of special cases of our main result (2.1). Here we give only two examples of this type.

Setting $\eta = 1$ in the result of Theorem 2.1 yields the following result.

Corollary 2.1. *Let the parameters $\beta, \mu, \nu, \zeta, \rho \in \mathbb{C}$, $a > 0$, $c \in \mathbb{R}$, $\rho > 0$,* $\mathrm{Re}(\mu) >$ $\mathrm{Re}(\nu) > 0$*, $\beta > 0$, and $\alpha < 1$, then we have the following relation:*

$$P_{0^+}^{(\rho,\alpha,a)}\left\{x^{\lambda-1}\,{}_{\zeta,1}E_{\mu,\nu}[c\, x^{\beta}]\right\} = \frac{x^{\rho+\lambda}\,\Gamma(1+\frac{\rho}{1-\alpha})}{\Gamma(\zeta)\,(a(1-\alpha))^{\lambda}} \times {}_3\Psi_2\left[\begin{matrix}(\zeta,1),\ (\lambda,\beta),(1,1);\\ \left(\nu,\mu\right),\left(1+\lambda+\dfrac{\rho}{1-\alpha},\ \beta\right);\end{matrix}\ c\left(\frac{x}{a(1-\alpha)}\right)^{\beta}\right]. \tag{2.3}$$

If we replace μ, ν, β by λ and taking $\eta = 0$, $\zeta = 1$ in (2.1), Theorem 2.1 reduces to the following corollary.

Corollary 2.2. *Let $x, \rho \in \mathbb{C}$, $a > 0$, $c \in \mathbb{R}$,* $\mathrm{Re}(\lambda) > 0$*, and $\alpha < 1$ then we have the following relation:*

$$P_{0^+}^{(\rho,\alpha,a)}\left\{e_{\lambda}^{c\,x}\right\} = \frac{x^{\rho+\lambda}\Gamma(1+\frac{\rho}{1-\alpha})}{(a(1-\alpha))^{\lambda}} E_{\lambda,1+\lambda+\frac{\rho}{1-\alpha}}\left[\frac{cx^{\lambda}}{(a(1-\alpha))^{\lambda}}\right], \tag{2.4}$$

where $e_{\lambda}^{c\,x}$ *is called the* λ*-Exponential function defined by* (see [8, p. 50, Eq. (1.10.12)])*:*

$$e_{\lambda}^{c\,x} = x^{\lambda-1} \sum_{n=0}^{+\infty} c^n \frac{x^{\lambda n}}{\Gamma\left[(n+1)\lambda\right]} \qquad (\mathrm{Re}(\lambda) > 0).$$

Remark 2.1. Setting $\eta = 0$ in the main result (2.1), the resulting formula is seen to become the known result given by Nair [18, p. 245, Eq. (26)] and if we set $\eta = 0$ and $\nu = 1$ in (2.1), it becomes in to the known result given by Nair [18, p. 245, Eq. (27)].

3. Further Special Cases and Concluding Remarks

By setting $\alpha = 0$, $a = 1$ and $\rho \to \rho - 1$ in (2.1), (2.3) and (2.4), respectively, and applying the following easily-derivable relation:

$$\left(P_{0+}^{(\rho-1,0,1)} f\right)(t) = \int_0^t (t-\tau)^{\rho-1}\, f(\tau)\,\mathrm{d}\tau = \Gamma(\eta)\, \left(I_{0+}^{\eta}\, f\right)(t) \quad (\mathrm{Re}(\rho) > 0),$$

we obtain three fractional integral formulas involving left-sided Riemann–Liouville fractional integral operators stated in the next corollaries below.

Corollary 3.1. *Let the parameters* $\beta, \mu, \nu, \zeta, \eta, \rho \in \mathbb{C}$*,* $a > 0$*,* $c \in \mathbb{R}$*,* $\rho > 0$*,* $\mathrm{Re}(\mu) > \mathrm{Re}(\nu) > 0$*,* $\beta > 0$*, and* $\alpha < 1$*. Then the following relation*

$$I_{0+}^{\rho}\left\{x^{\lambda-1}\,{}_{\zeta,\eta}E_{\mu,\nu}[c\,x^{\beta}]\right\} = \frac{x^{\rho+\lambda-1}\,\Gamma(\rho)}{\Gamma(\zeta)}\;{}_3\Psi_2\left[\begin{matrix}(\zeta,\eta),\,(\lambda,\beta),(1,1);\\ \Big(\nu,\mu\Big),\Big(\lambda+\alpha,\,\beta\Big);\end{matrix}\; c\,(x)^{\beta}\right]$$

holds.

Corollary 3.2. *Let the parameters* $\beta, \mu, \nu, \zeta, \rho \in \mathbb{C}$*,* $a > 0$*,* $c \in \mathbb{R}$*,* $\rho > 1$*,* $\mathrm{Re}(\mu) > \mathrm{Re}(\nu) > 0$*,* $\beta > 0$*, and* $\alpha < 1$*. Then we have the following relation:*

$$I_{0+}^{\rho}\left\{x^{\lambda-1}\,{}_{\zeta,1}E_{\mu,\nu}[c\,x^{\beta}]\right\} = \frac{x^{\rho+\lambda-1}\,\Gamma(\rho)}{\Gamma(\zeta)}\;{}_3\Psi_2\left[\begin{matrix}(\zeta,1),\,(\lambda,\beta),(1,1);\\ \Big(\nu,\mu\Big),\Big(\lambda+\rho,\,\beta\Big);\end{matrix}\; c\left(\frac{x}{a(1-\alpha)}\right)^{\beta}\right].$$

Corollary 3.3. *Let* $x, \rho \in \mathbb{C}$*,* $a > 0$*,* $c \in \mathbb{R}$*,* $\mathrm{Re}(\lambda) > 0$*, and* $\alpha < 1$*. Then we have the following relation:*

$$I_{0+}^{\rho}\left\{e_{\lambda}^{c\,x}\right\} = x^{\rho+\lambda-1}\Gamma(\rho)E_{\lambda,1+\lambda+\rho}\big[cx^{\lambda}\big].$$

It is noted that if we set $\alpha = 0$, $a = 1$, and $f(t)$ is replaced by

$${}_2F_1\left(\rho+\beta, -\gamma; \rho; 1-\frac{t}{x}\right) f(t),$$

(1.3) yields the Saigo fractional integral operator. Thus we can obtain the generalizations of left-sided fractional integrals, like Saigo, Erdélyi-Kober (see [27]; see also [8]), and so on, by suitable substitutions. Therefore, the results presented here are easily shown to be converted to those corresponding to the above well known fractional operators.

Several further consequences of Theorem 2.1, as well as Corollaries 2.1, 2.1, and 3.1 – 3.3, can easily be derived by using some known and new relationship between Mittag-Leffler type function ${}_{\zeta,\eta}E_{\mu,\nu}[z]$, which is an elegant unification of various special functions (see [3]), and Fox H-function as given in Definition 1.2, after some suitable parametric replacements. These relatively simpler fractional integral formulas for pathway fractional integral operator (1.3) can be deduced from Theorem 2.1, and the previous corollaries by appropriately applying the following relationships:

$$ {}_{\zeta,\eta}E_{\mu,\nu}[z] = \frac{1}{\Gamma(\gamma)} H^{1,2}_{2,2}\left[z \,\middle|\, \begin{matrix} (1-\zeta,\eta),(0,1) \\ (0,1),(1-\nu,\mu) \end{matrix} \right]. $$

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