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$\delta\beta\mathcal{I}$ APPROXIMATION SPACES

A. E. Radwan, Rodyna A. Hosny and A. M. Abd El-latif

Abstract. The concepts of $\delta\beta\mathcal{I}$-lower and $\delta\beta\mathcal{I}$-upper approximation as a generalization of rough set theory via $\delta\beta\mathcal{I}$-open sets are introduced and studied. Some of its basic properties with the aid of examples are proven. Furthermore, the established relationships between the rough approximations reported in [2] and our new approximations are examined.
Keywords: $\delta\beta$-rough sets; Accuracy measure; Rough sets; $\delta\beta$-open sets; $\delta\beta\mathcal{I}$-open sets

1. Introduction

Rough set theory (RST) is an important mathematical approach that was developed by Pawlak in [19] to overcome the difficulties associated with vague and complicated data. However, (RST) has rapidly found a lot of applications in numerous fields [2, 4, 8, 17, 20, 24]. This method has been developed to manage uncertainties from information that presents some inexactitude, incompleteness and noises. When the available information is insufficient to determine the exact value of a given set, lower and upper approximations can be used by rough set for the representation of the concerned set. The approximation synthesis of concepts from the acquired data is the main objective of the rough set analysis. For example, if it is difficult to define a concept in a given knowledge base, rough sets can approximate with respect to that knowledge. In decision making, it has been confirmed that rough set methods have a powerful essence in dealing with uncertainties. The RST has been applied in several fields including image processing, data mining, pattern recognition, medical informatics, knowledge discovery and expert systems. Using the concepts of lower and upper approximations in rough set theory, knowledge hidden in information systems may be unraveled and expressed in the form of decision rules. The basic operators in rough set theory are approximation operators. Many examples of applications of the rough set theory to process control, economics, medical diagnosis, biochemistry, environmental science, biology, chemistry psychology,

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conflict analysis and other fields. Ideals in topological spaces have been considered since 1930, by Kuratowski [15]. This topic has won its importance by the paper of Vaidyanathaswamy [22]. In 2006, Hatir et al. [10] defined the concept of $\delta\beta$-open sets. Then, in terms of $\delta\beta$-open sets, Abu-Donia et al. [2] generalized rough approximation spaces due to [19].
In this note, the concepts of $\delta\beta\mathcal{I}$-lower and $\delta\beta\mathcal{I}$-upper approximation as a generalization of rough set theory via ideals are introduced. Some of its basic properties with the aid of examples are proven. Furthermore, the established relationships between the rough approximations outlined in [2] and our new approximations are examined.

2. Preliminaries

Throughout this paper, $\mathcal{P}(X)$ denote the power set of X. Let A be a subset of a topological space (X,τ). The closure, interior and the complement of a subset A of X are denoted by $cl(A)$, $int(A)$ and $(A)^c$ respectively.

A subset A is said to be regular open (resp. regular closed) [21] if $A = int(cl(A))$ (resp. $A = cl(int(A))$). The δ-interior $int_\delta(A)$of a subset A of X [23] is the union of all regular open sets of X contained in A. A subset A is called δ-open if $A = int_\delta(A)$ i.e., a set is δ-open if it is the union of regular open sets. The family of δ-open sets is denoted by δO. The complement of a δ-open set is called δ-closed, alternatively, a set A of (X,τ) is called δ-closed [23] if $A = cl_\delta(A)$, where $cl_\delta(A) = \{x \in X \mid A \cap int(cl(U)) \neq \emptyset, U \in \tau \text{ and } x \in U\}$, $cl(A) \subseteq cl_\delta(A)$. In Pawlak approximation space a subset $A \subseteq X$ has two possibilities rough or exact. A subset A of topological space (X,τ) is called exact if its boundary is empty i.e., $b(A) = cl(A) - int(A) = \emptyset$, otherwise A is rough. It is clear that A is exact if and only if $cl(A) = int(A)$.
An ideal $\mathcal{I}$ is a nonempty collection of subsets of X closed with respect to finite union and heredity [15].
Given an ideal topological space $(X,\tau,\mathcal{I})$, a set operator $(.)^\star : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$, is called a local function [15] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subseteq X$, $A^\star(\mathcal{I},\tau) = \{x \in X \mid U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau \mid x \in U\}$. When there is no chance for confusion, we will simply write $A^\star$ for $A^\star(\mathcal{I},\tau)$. Additionally, a Kuratowski closure operator $cl^\star(.)$ for a topology $\tau^\star(\mathcal{I},\tau)$, called the $\star$-topology, finer than τ), is defined by $cl^\star(A) = A \cup A^\star(\mathcal{I},\tau)$ [14].

Definition 2.1. A subset A of a topological space (X,τ) is said to be
(i) β-open [1] or (semi-pre open [3]), if $A \subseteq cl(int(cl(A)))$.
(ii) $\delta\beta$-open [11](e^*-open [7]), if $A \subseteq cl(int(cl_\delta(A)))$.
(iii) Semi-open [16], if $A \subseteq cl(int(A))$.

Definition 2.2.[12] A subset A of an ideal topological space $(X,\tau,\mathcal{I})$ is said to be $\beta\mathcal{I}$-open, if $A \subseteq cl(int(cl^\star(A)))$.

Lemma 2.1. In a topological space (X, τ), the following properties hold
(i) [10] β-open set is $\delta\beta$-open.
(ii) $\beta\mathcal{I}$-open set is β-open.

The rough set theory was introduced by Pawlak [20] based on an equivalence relation R on a finite universe X. The equivalence classes R_x of R, is defined as $R_x = \{x \in X \mid xRy\}$. Let A be a subset of the approximation space $K = (X, R)$, he considered two operators, the lower and upper approximations of subsets.
$\underline{R}(A) = \{x \in X \mid R_x \subseteq A\}$.
$\overline{R}(A) = \{x \in X \mid R_x \cap A \neq \emptyset\}$.
Positive, negative and boundary regions respectively are also defined
$POS_R(A) = \underline{R}(A)$.
$NEG_R(A) = X - \overline{R}(A)$.
$BN_R(A) = \overline{R}(A) - \underline{R}(A)$.

The degree of completeness can also be characterized by the accuracy measure, in which $|A|$ represents the cardinality of a subset A⊆X as follows:

$$\alpha_R(A) = \frac{|\underline{R}(A)|}{|\overline{R}(A)|}, \text{ where } A \neq \emptyset.$$

Accuracy measure try to express the degree of completeness of knowledge. $\alpha_R(A)$ is able to capture how large the boundary region of the data sets is. However, we cannot easily capture the structure of the knowledge. A fundamental advantage of rough set theory is the ability to handle a category that cannot be sharply defined. The characteristics of a potential data set can be measured through the rough sets framework. We can measure inexactness and express topological characterization of imprecision as follows:

(i) If $\underline{R}(A) \neq \emptyset$ and $\overline{R}(A) \neq X$, then A is called roughly R-definable,
(ii) If $\underline{R}(A) = \emptyset$ and $\overline{R}(A) \neq X$, then A is called internally R-undefinable,
(iii) If $\underline{R}(A) \neq \emptyset$ and $\overline{R}(A) = X$, then A is called externally R-undefinable,
(iv) If $\underline{R}(A) = \emptyset$ and $\overline{R}(A) = X$, then A is called totally R-undefinable.

Definition 2.3. [2] Let X be a finite nonempty universe. The pair $(X, R_{\delta\beta})$ is called a $\delta\beta$-approximation space where $R_{\delta\beta}$ is a general relation used to get a subbase for a topology τ on X which generates the class $\delta\beta O(X)$ of all $\delta\beta$-open sets.

Definition 2.4. [2] Let $(X, R_{\delta\beta})$ be a $\delta\beta$-approximation space. $\delta\beta$-upper approximations and $\delta\beta$-lower approximation of any nonempty subset A of X are defined as follows:
(i) $\overline{R}_{\delta\beta}(A) = \cap\{V \mid V \text{ is a } \delta\beta\text{-closed and } A \subseteq V\}$;
(ii) $\underline{R}_{\delta\beta}(A) = \cup\{U \mid U \text{ is a } \delta\beta\text{-open and } U \subseteq A\}$.

3. On $\delta\beta\mathcal{I}$-Open Sets

This section is dedicated to define the local function of A with regard to δ-open sets and an ideal $\mathcal{I}$ to introduce $\delta\beta\mathcal{I}$-open sets. Some of its characterizations are studied.

Definition 3.1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Consider the self-map on $\mathcal{P}(X)$ where for every set A: $A^\star_\delta(\mathcal{I}, \delta O)=\{x \in X \mid A \cap int(cl(U)) \notin \mathcal{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau \mid x \in U\}$. Then, $A^\star_\delta(\mathcal{I}, \delta O)$ is called local function of a set A with respect to δO and $\mathcal{I}$. When there is no chance for confusion, we will simply write $A^\star_\delta$ or $A^\star_\delta(\mathcal{I})$ for $A^\star_\delta(\mathcal{I}, \delta O)$. Additionally, $cl^\star_\delta(A) = A \cup A^\star_\delta$.

The next theorem shows the properties of the local function for any nonempty subset with respect to δO and $\mathcal{I}$.

Theorem 3.1. Let $\mathcal{I}$ and $\mathcal{J}$ be any two ideals on a topological space (X, τ). Let A, B be subsets of X. Then,
(i) $\emptyset^\star_\delta = \emptyset$.
(ii) If $A \subseteq B$, then $A^\star_\delta \subseteq B^\star_\delta$.
(iii) If $\mathcal{I} \subseteq \mathcal{J}$, then $\mathrm{A}^\star{}_\delta(\mathcal{J}) \subseteq \mathrm{A}^\star{}_\delta(\mathcal{I})$.
(iv) $A^\star \subseteq A^\star_\delta \subseteq cl_\delta(A)$.
(v) $A^\star_\delta$ is δ-closed set. Moreover, it is τ-closed.
(vi)$(A^\star_\delta)^\star_\delta \subseteq A^\star_\delta$.
(vii) $(A \cup B)^\star_\delta = \mathrm{A}^\star{}_\delta \cup \mathrm{B}^\star{}_\delta$.
(viii) $\cup_j \mathrm{A}^\star{}_{j\delta} = (\cup_j \mathrm{A}_j)^\star{}_\delta$.
(ix) $(A \cap B)^\star_\delta \subseteq A^\star_\delta \cap B^\star_\delta$.
(x) $A^\star_\delta - B^\star_\delta = (A - B)^\star_\delta - B^\star_\delta \subseteq (A - B)^\star_\delta$.
(xi) If $B \in \mathcal{I}$, then $B^\star_\delta = \emptyset$.
(xii) If $B \in \mathcal{I}$, then $(A \cup B)^\star_\delta = A^\star_\delta = (A - B)^\star_\delta$.

Proof. (i) Follows directly from Definition 3.1.
(ii) Let $x \in A^\star_\delta$, then for every $U \in \tau(x)$, $A \cap int(cl(U)) \notin \mathcal{I}$. Since $A \subseteq B$, then $B \cap int(cl(U)) \notin \mathcal{I}$ and so $x \in B^\star_\delta$. Hence, $A^\star_\delta \subseteq B^\star_\delta$.
(iii) Let $x \in A^\star_\delta(\mathcal{J})$, then for every $U \in \tau(x)$, $A \cap int(cl(U)) \notin \mathcal{J}$. Since $\mathcal{I} \subseteq \mathcal{J}$, then $A \cap int(cl(U)) \notin \mathcal{I}$ and so $x \in A^\star_\delta(\mathcal{I})$. Hence, $A^\star_\delta(\mathcal{J}) \subseteq A^\star_\delta(\mathcal{I})$.
(iv) Let $x \in A^\star$, then for every $U \in \tau(x)$, $A \cap U \notin \mathcal{I}$. Since U is open set, then $U \subseteq int(cl(U))$. Hence, for every $U \in \tau(x)$, $A \cap int(cl(U)) \notin \mathcal{I}$ and so $x \in A^\star_\delta$. Therefore, $A^\star \subseteq A^\star_\delta$. It is obvious, $A^\star_\delta \subseteq cl_\delta(A)$.
(v) Let $x \in cl_\delta(A^\star_\delta)$, then for every $U \in \tau(x)$, $A^\star_\delta \cap int(cl(U)) \neq \emptyset$. Hence, there exist $y \in X$ such that $y \in A^\star_\delta \cap int(cl(U))$. Therefore, $y \in A^\star_\delta$ and $y \in int(cl(U))$. Hence, $A \cap int(cl(V)) \notin \mathcal{I}$, for every $V \in \tau(y)$. Since $int(cl(U)) \in \tau(y)$, then

$A \cap int(cl(int(cl(U)))) \notin \mathcal{I}$ and so $x \in A^\star_\delta$. Which implies that $cl_\delta(A^\star_\delta) = A^\star_\delta$ i.e, $A^\star_\delta$ is δ-closed set. Since $cl(A) \subseteq cl_\delta(A)$, for any $A \subseteq X$, then $cl(A^\star_\delta) = A^\star_\delta$ i.e, $A^\star_\delta$ is τ-closed set.

(vi) It is clear that by using (iv)$(A^\star_\delta)^\star_\delta \subseteq cl_\delta(A^\star_\delta)$. So $cl_\delta(A^\star_\delta) = A^\star_\delta$ from (v) and then $(A^\star_\delta)^\star_\delta \subseteq A^\star_\delta$.

(vii) $A^\star_\delta \cup B^\star_\delta \subseteq (A \cup B)^\star_\delta$ follows directly from (ii). Let $x \notin A^\star_\delta \cup B^\star_\delta$, then $x \notin A^\star_\delta$ and $x \notin B^\star_\delta$. Hence, there exist $U_1, U_2 \in \tau(x)$, such that $A \cap int(cl(U_1)) \in \mathcal{I}$ and $B \cap int(cl(U_2)) \in \mathcal{I}$. Therefore, $A \cap int(cl(U_1 \cap U_2)) \in \mathcal{I}$ and $B \cap int(cl(U_1 \cap U_2)) \in \mathcal{I}$. Thus $(A \cup B) \cap int(cl(U_1 \cap U_2)) \in \mathcal{I}$ and so $x \notin (A \cup B)^\star_\delta$. Consequently, $(A \cup B)^\star_\delta = A^\star_\delta \cup B^\star_\delta$.

(viii) $\cup_j A^\star_\delta = (\cup_j A)^\star_\delta$ follows directly from (vii).

(ix) $(A \cap B)^\star_\delta \subseteq A^\star_\delta \cap B^\star_\delta$ follows directly from (ii).

(x) Since $(A - B) \subseteq A$ then, from (ii), $(A - B)^\star_\delta \subseteq A^\star_\delta$ and so $(A - B)^\star_\delta - B^\star_\delta \subseteq A^\star_\delta - B^\star_\delta$. Since $A = (A - B) \cup (A \cap B)$, thus from (vii) $A^\star_\delta = (A - B)^\star_\delta \cup (A \cap B)^\star_\delta$. Therefore, from (ix), $A^\star_\delta \subseteq (A - B)^\star_\delta \cup (A^\star_\delta \cap B^\star_\delta) \subseteq (A - B)^\star_\delta \cup B^\star_\delta$. Hence, $(A^\star_\delta - B^\star_\delta) \subseteq ((A - B)^\star_\delta \cup B^\star_\delta) - B^\star_\delta$ and so $(A^\star_\delta - B^\star_\delta) \subseteq (A - B)^\star_\delta - B^\star_\delta$. Consequently, $A^\star_\delta - B^\star_\delta = (A - B)^\star_\delta - B^\star_\delta \subseteq (A - B)^\star_\delta$.

(xi) Let $B \in \mathcal{I}$, then $B^\star_\delta = \{x \in X \mid B \cap int(cl(U)) \notin \mathcal{I}$ for every $U \in \tau(x)\} = \emptyset$.

(xii) Let $B \in \mathcal{I}$, then $(A \cup B)^\star_\delta = A^\star_\delta \cup B^\star_\delta = A^\star_\delta$, by using (vii) and (xi). Since $A^\star_\delta - B^\star_\delta = (A - B)^\star_\delta - B^\star_\delta$, then $(A - B)^\star_\delta = A^\star_\delta$, by using (x) and (xi).

Corollary 3.1. Let $\mathcal{I}$ and $\mathcal{J}$ be any two ideals on a topological space (X, τ). Let A, B be subsets of X. Then,

(i) $cl^\star_\delta(\emptyset) = \emptyset$, $cl^\star_\delta(X) = X$.

(ii) $A \subseteq cl^\star_\delta(A)$.

(iii) If $A \subseteq B$, then $cl^\star_\delta(A) \subseteq cl^\star_\delta(B)$.

(iv) $cl^\star_\delta(A \cup B) = cl^\star_\delta(A) \cup cl^\star_\delta(B)$.

(v) $cl^\star_\delta(\cup_j A_j) = \cup_j cl^\star_\delta(A_j)$.

(vi) $cl^\star_\delta cl^\star_\delta(A) = cl^\star_\delta(A)$.

(vii) If $\mathcal{I} \subseteq \mathcal{J}$, then $cl^\star_{\mathcal{J}\delta}(A) \subseteq cl^\star_{\mathcal{I}\delta}(A)$.

(viii) $cl^\star(A) \subseteq cl^\star_\delta(A) \subseteq cl_\delta(A)$.

Definition 3.2. Let $(X, \tau, \mathcal{I})$ be an ideal topological space, $cl^\star_\delta(A) : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ be the soft closure operator. Then, there exist a topology $\tau^\star_\delta$ which is finer than τ_δ, defined by $\tau^\star_\delta = \{A \subseteq X : cl^\star_\delta(A)^c = (A)^c\}$.

Theorem 3.2. Let $\mathcal{I}$ be an ideal on a topological space (X, τ) and A, B be subsets of X. If A is regular open, then $(A \cap B^\star_\delta) = A \cap (A \cap B)^\star_\delta \subseteq (A \cap B)^\star_\delta$.

Proof. Obviously, $A \cap (A \cap B)^\star_\delta \subseteq A \cap B^\star_\delta$, by using (ix) of Theorem 3.1. Let $x \notin (A \cap B)^\star_\delta$ and $x \in A$, then there exist $U \in \tau(x)$ such that $int(cl(U)) \cap (A \cap B) \in \mathcal{I}$.

Since A is regular open set, then it is open and containing x and so $U \cap A$ is open and containing x. Hence,

$$int(cl(U \cap A)) \cap B \subseteq int(cl(U)) \cap int(cl(A)) \cap B = int(cl(U)) \cap A \cap B \in \mathcal{I}.$$

Therefore, $x \notin B^{\star}_{\delta}$ and so $A \cap B^{\star}_{\delta} \subseteq (A \cap B)^{\star}_{\delta}$. Consequently,

$$(A \cap B^{\star}_{\delta}) = A \cap (A \cap B)^{\star}_{\delta} \subseteq (A \cap B)^{\star}_{\delta}.$$

Definition 3.3. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\delta\beta\mathcal{I}$-open, if $A \subseteq cl(int(cl^{\star}_{\delta}(A)))$ and its complement is called $\delta\beta\mathcal{I}$-closed. In other words, a subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\delta\beta\mathcal{I}$-closed, if $int(cl(int^{\star}_{\delta}(A))) \subseteq A$.

In view of Corollary 3.1 (viii), the following lemma is true.

Lemma 3.1. In an ideal topological space $(X, \tau, \mathcal{I})$, the following implication hold

$$\beta\mathcal{I} - \text{open} \Longrightarrow \delta\beta\mathcal{I} - \text{open} \Longrightarrow \delta\beta - \text{open}.$$

None of these implications are reversible as shown in the next example.

Example 3.1. Let $X = \{a, b, c, d, e\}$ be a universe set, $\mathcal{I} = \{\{a\}, \{c\}, \{a, c\}, \emptyset\}$ and $\tau = \{\{d\}, \{e\}, \{a, d\}, \{d, e\}, \{a, d, e\}, \{b, c, e\}, \{b, c, d, e\}, \emptyset, X\}$. Then,

(i) The set $\{a, c\}$ is $\delta\beta$-open and it is not $\delta\beta\mathcal{I}$-open.

(ii) The set $\{b, d\}$ is $\delta\beta\mathcal{I}$-open set and it is not $\beta\mathcal{I}$-open.

The next two lemmas, whose proofs are omitted.

Lemma 3.2. Let $\mathcal{I}$ be an ideal on a topological space (X, τ), then the following statements hold:

(i) If $\mathcal{I} = \{\emptyset\}$, then $A^{\star}_{\delta} = cl_{\delta}(A)$ and so the concepts $\delta\beta$-open and $\delta\beta\mathcal{I}$-open are coincide.

(ii) If $\mathcal{I}$=$\mathcal{P}(X)$, then $A^{\star}_{\delta} = \emptyset$ and so the concepts semi-open and $\delta\beta\mathcal{I}$-open are coincide.

Lemma 3.3. Let $\mathcal{I}$ be an ideal on a topological space (X, τ) and A be subset of X, then

(i) If $cl^{\star}_{\delta}(A)$ is $\delta\beta\mathcal{I}$-open set, then A is $\delta\beta\mathcal{I}$-open.

(ii) If A is $\tau^{\star}_{\delta}$-closed and $\delta\beta\mathcal{I}$-open set, then A is semi-open.

Theorem 3.3. The arbitrary union of $\delta\beta\mathcal{I}$-open sets is $\delta\beta\mathcal{I}$-open, but the intersection of two $\delta\beta\mathcal{I}$-open sets needs not to be $\delta\beta\mathcal{I}$-open, in general.

Proof. Let A_j be $\delta\beta\mathcal{I}$-open sets for every j, then $A_j \subseteq cl(int(cl^\star_\delta(A_j)))$ for every j. Hence, $\cup_j A_j \subseteq \cup_j cl(int(cl^\star_\delta(A_j))) \subseteq cl(int(cl^\star_\delta(\cup_j A_j)))$ by using (v) of Corollary 3.1. Consequently, $\cup_j A_j$ is $\delta\beta\mathcal{I}$-open. To complete the proof, we shall use next example.

Example 3.2. Based on what mentioned in the Example 3.1. The sets $\{b, c\}, \{c, e\}$ are $\delta\beta\mathcal{I}$-open sets but their intersection $\{c\}$ is not $\delta\beta\mathcal{I}$-open.

Theorem 3.4. The intersection of $\delta\beta\mathcal{I}$-open set and regular open set is $\delta\beta\mathcal{I}$-open.

Proof. Let A be regular open set and B be $\delta\beta\mathcal{I}$-open set, then $A = int(cl(A))$ and $B \subseteq cl(int(cl^\star_\delta(B)))$, and so $A \cap B \subseteq int(cl(A)) \cap cl(int(cl^\star_\delta(B)))$. Since A is an open set, then $(A \cap B) \subseteq cl[int(cl(A)) \cap int(cl^\star_\delta(B))] = cl(int[int(cl(A)) \cap cl^\star_\delta(B)])$. Since A is a regular open set, then $A \cap B \subseteq cl(int[A \cap cl^\star_\delta(B)]) \subseteq cl(int(cl^\star_\delta(A \cap B)))$ from Theorem 3.2. Hence, $(A \cap B)$ is $\delta\beta\mathcal{I}$-open.

Theorem 3.5. Let $\mathcal{I}$ and $\mathcal{J}$ be any two ideals on a topological space (X, τ) with $\mathcal{I} \subseteq \mathcal{J}$. If a subset A of X is $\delta\beta\mathcal{J}$-open, then it is $\delta\beta\mathcal{I}$-open.

Proof. Follows directly from (vii) Corollary 3.1.

4. Approximation Spaces Based On $\delta\beta\mathcal{I}$-Open Sets

This section aims to generalize $\delta\beta$-approximation space attributed to [2] to $\delta\beta\mathcal{I}$-approximation space via ideal. $\delta\beta\mathcal{I}$-upper approximation and $\delta\beta\mathcal{I}$-lower approximation are presented. Several of their basic properties and the interrelatedness between them are obtained. Furthermore, some of counter examples for comparison between the current approach and the approach reported in [2] are provided.

Definition 4.1. Let X be a finite nonempty universe and $\mathcal{I}$ be any ideal on X. The pair $(X, R_{\delta\beta\mathcal{I}})$ is called a $\delta\beta\mathcal{I}$-approximation space, where $R_{\delta\beta\mathcal{I}}$ is a binary relation on X used to get subbase for a topology on X, which generate the class of $\delta\beta\mathcal{I}$-open sets.

Example 4.1. Let $X = \{a, b, c, d, e\}$ be a universe, $\mathcal{I} = \{\{a\}, \{c\}, \{a, c\}, \emptyset\}$ and $R = \{(a, a), (a, e), (b, c), (b, d), (c, e), (d, a), (d, e), (e, e)\}$ be a binary relation on X thus $aR = dR = \{a, e\}, bR = \{c, d\}$ and $cR = eR = \{e\}$. Then,
$\{\emptyset, \{d\}, \{e\}, \{a, e\}, \{b, d\}, \{b, e\}, \{c, d\}, \{a, b, e\}, \{a, d, e\}, \{d, e\}, \{b, c, d\}, \{b, d, e\},$
$\{c, d, e\}, \{b, c, d, e\}, \{a, b, d, e\}, \{a, c, d, e\}, X\}$
is the class of $\delta\beta\mathcal{I}$-open sets generated by R. Consequently, $(X, R_{\delta\beta\mathcal{I}})$ is a $\delta\beta\mathcal{I}$-approximation space.

Definition 4.2. Let $(X, R_{\delta\beta\mathcal{I}})$ be $\delta\beta\mathcal{I}$-approximation space and A be a nonempty subset of X. Then, for any ideal $\mathcal{I}$ on X, $\delta\beta\mathcal{I}$-upper approximation and $\delta\beta\mathcal{I}$-lower approximation are defined as follows:

(i) $\overline{R}_{\delta\beta\mathcal{I}}(A) = \cap\{V \mid V$ is a $\delta\beta\mathcal{I}$-closed and $A \subseteq V\}$;

(ii) $\underline{R}_{\delta\beta\mathcal{I}}(A) = \cup\{U \mid U$ is a $\delta\beta\mathcal{I}$-open and $U \subseteq A\}$.

Definition 4.3. Let $(X, R_{\delta\beta\mathcal{I}})$ be $\delta\beta\mathcal{I}$-approximation space and A be a nonempty subset of X. Then, $\delta\beta\mathcal{I}$-accuracy measure of A is defined as

$$\alpha_{R_{\delta\beta\mathcal{I}}}(A) = \frac{|\underline{R}_{\delta\beta\mathcal{I}}(A)|}{|\overline{R}_{\delta\beta\mathcal{I}}(A)|}.$$

Example 4.2. If $X = \{a, b, c, d, e\}$ is a universe, $\mathcal{I} = \{\{\text{a}\}, \{\text{c}\}, \{\text{a, c}\}, \emptyset\}$ and $R = \{$(a, a), (a, e), (b, c), (b, d), (c, e), (d, a), (d, e), (e, e)$\}$, then Table 6.1 shows that $\delta\beta\mathcal{I}$-upper approximation, $\delta\beta\mathcal{I}$-lower approximation and $\delta\beta\mathcal{I}$-accuracy measure for any nonempty subset A of X. Additionally, it establishes comparison between the $\delta\beta\mathcal{I}$-accuracy measure and $\delta\beta$-accuracy measure due to [2] for a set A.

Corollary 4.1. For any subset A of $\delta\beta\mathcal{I}$-approximation space $(X, R_{\delta\beta\mathcal{I}})$, it holds that $\alpha_{R_{\delta\beta\mathcal{I}}}(A) \leq \alpha_{R_{\delta\beta}}$.

Some of the fundamental properties of $\delta\beta\mathcal{I}$-approximation space will be shown in the next theorems.

Theorem 4.1. Let $(X, R_{\delta\beta\mathcal{I}})$ be a $\delta\beta\mathcal{I}$-approximation space and $\mathcal{I}$, $\mathcal{J}$ be two ideals on X with $\mathcal{I} \subseteq \mathcal{J}$. Then, for any set A the following statements hold

(i) $\underline{R}_{\delta\beta\mathcal{J}}(A) \subseteq \underline{R}_{\delta\beta\mathcal{I}}(A)$.

(ii) $\overline{R}_{\delta\beta\mathcal{I}}(A) \subseteq \overline{R}_{\delta\beta\mathcal{J}}(A)$.

Proof. It is obvious in view of Theorem 3.5.

Theorem 4.2. Let $(X, R_{\delta\beta\mathcal{I}})$ be a $\delta\beta\mathcal{I}$-approximation space and A, B be subsets of X, then for any ideal $\mathcal{I}$ the following statements hold

(i) $\underline{R}_{\delta\beta\mathcal{I}}(A) \subseteq \underline{R}_{\delta\beta}(A) \subseteq A \subseteq \overline{R}_{\delta\beta}(A) \subseteq \overline{R}_{\delta\beta\mathcal{I}}(A)$.

(ii) $\overline{R}_{\delta\beta\mathcal{I}}(\emptyset) = \emptyset = \underline{R}_{\delta\beta\mathcal{I}}(\emptyset)$ and $\overline{R}_{\delta\beta\mathcal{I}}(X) = X = \underline{R}_{\delta\beta\mathcal{I}}(X)$.

(iii) If $A \subseteq B$, then $\underline{R}_{\delta\beta\mathcal{I}}(A) \subseteq \underline{R}_{\delta\beta\mathcal{I}}(B)$ and $\overline{R}_{\delta\beta\mathcal{I}}(A) \subseteq \overline{R}_{\delta\beta\mathcal{I}}(B)$.

Proof. (i) Follows directly from Lemma 3.1 and Definition 4.2.

(ii) Obvious from Definition 4.2.

(iii) Let $x \in \underline{R}_{\delta\beta\mathcal{I}}(A)$, then there exist $\delta\beta\mathcal{I}$-open U such that $x \in U \subseteq A$. Since $A \subseteq B$, then $x \in \underline{R}_{\delta\beta\mathcal{I}}(B)$. Also, $x \notin \overline{R}_{\delta\beta\mathcal{I}}(B)$, then there exist $\delta\beta\mathcal{I}$-closed V such that $x \notin V$ and $B \subseteq V$. Since $A \subseteq B$ so $x \notin \overline{R}_{\delta\beta\mathcal{I}}(A)$.

Theorem 4.3. Let A be a subset of a $\delta\beta\mathcal{I}$-approximation space $(X, R_{\delta\beta\mathcal{I}})$, then for any ideal $\mathcal{I}$ the following statements hold

(i) $\underline{R}_{\delta\beta\mathcal{I}}(X - A) = X - \overline{R}_{\delta\beta\mathcal{I}}(A)$ and $\overline{R}_{\delta\beta\mathcal{I}}(X - A) = X - \underline{R}_{\delta\beta\mathcal{I}}(A)$.

(ii) $\underline{R}_{\delta\beta\mathcal{I}}\underline{R}_{\delta\beta\mathcal{I}}(A) = \underline{R}_{\delta\beta\mathcal{I}}(A)$ and $\overline{R}_{\delta\beta\mathcal{I}}\overline{R}_{\delta\beta\mathcal{I}}(A) = \overline{R}_{\delta\beta\mathcal{I}}(A)$.

(iii) $\underline{R}_{\delta\beta\mathcal{I}}\underline{R}_{\delta\beta\mathcal{I}}(A) \subseteq \overline{R}_{\delta\beta\mathcal{I}}\underline{R}_{\delta\beta\mathcal{I}}(A)$ and $\underline{R}_{\delta\beta\mathcal{I}}\overline{R}_{\delta\beta\mathcal{I}}(A) \subseteq \overline{R}_{\delta\beta\mathcal{I}}\overline{R}_{\delta\beta\mathcal{I}}(A)$.

Proof. (i) Obvious.

(ii) $\underline{R}_{\delta\beta\mathcal{I}}\underline{R}_{\delta\beta\mathcal{I}}(A) = \cup\{U \mid U$ is a $\delta\beta\mathcal{I}$-open and $U \subseteq \underline{R}_{\delta\beta\mathcal{I}}(A) \subseteq A\} = \cup\{U \mid U$ is a $\delta\beta - \mathcal{I}$-open and $U \subseteq A\} = \underline{R}_{\delta\beta\mathcal{I}}(A)$. Similarly, $\overline{R}_{\delta\beta\mathcal{I}}\overline{R}_{\delta\beta\mathcal{I}}(A) = \overline{R}_{\delta\beta\mathcal{I}}(A)$.

(iii) Immediate consequence of (i) of Theorem 4.2 and (ii) of this Theorem.

Example 4.3. In Table 6.1, the following is observable:
(i) If $A = \{d\}$, then $\overline{R}_{\delta\beta\mathcal{I}}\underline{R}_{\delta\beta\mathcal{I}}(A) \nsubseteq \underline{R}_{\delta\beta\mathcal{I}}\underline{R}_{\delta\beta\mathcal{I}}(A)$.

(ii) If $A = \{a\}$, then $\overline{R}_{\delta\beta\mathcal{I}}\overline{R}_{\delta\beta\mathcal{I}}(A) \nsubseteq \underline{R}_{\delta\beta\mathcal{I}}\overline{R}_{\delta\beta\mathcal{I}}(A)$.

Theorem 4.4. Let A, B be subsets of a $\delta\beta\mathcal{I}$-approximation space $(X, R_{\delta\beta\mathcal{I}})$, then for any ideal $\mathcal{I}$ the following statements hold

(i) $\underline{R}_{\delta\beta\mathcal{I}}(A \cup B) \supseteq \underline{R}_{\delta\beta\mathcal{I}}(A) \cup \underline{R}_{\delta\beta\mathcal{I}}(B)$ and $\overline{R}_{\delta\beta\mathcal{I}}(A \cup B) \supseteq \overline{R}_{\delta\beta\mathcal{I}}(A) \cup \overline{R}_{\delta\beta\mathcal{I}}(A)$.

(ii) $\underline{R}_{\delta\beta\mathcal{I}}(A \cap B) \subseteq \underline{R}_{\delta\beta\mathcal{I}}(A) \cap \underline{R}_{\delta\beta\mathcal{I}}(B)$ and $\overline{R}_{\delta\beta\mathcal{I}}(A \cap B) \subseteq \overline{R}_{\delta\beta\mathcal{I}}(A) \cap \overline{R}_{\delta\beta\mathcal{I}}(A)$.

Proof. The proof is obvious from Theorem 4.2(iii).

Example 4.4. Example 4.1 and Table 6.1 show that the inclusion in Theorem 4.4 can not be replaced by equality relation

(i) If $A = \{a, e\}$ and $B = \{b, c\}$, then $\underline{R}_{\delta\beta\mathcal{I}}(A) = \{a, e\}$, $\underline{R}_{\delta\beta\mathcal{I}}(B) = \emptyset$ and $\underline{R}_{\delta\beta\mathcal{I}}(A \cup B) = \{a, b, e\}$. Hence, $\underline{R}_{\delta\beta\mathcal{I}}(A \cup B) \nsubseteq \underline{R}_{\delta\beta\mathcal{I}}(A) \cup \underline{R}_{\delta\beta\mathcal{I}}(B)$.

(ii) If $A = \{d\}$ and $B = \{e\}$, then $\overline{R}_{\delta\beta\mathcal{I}}(A) = \{c, d\}, \overline{R}_{\delta\beta\mathcal{I}}(B) = \{a, e\}$ and $\overline{R}_{\delta\beta\mathcal{I}}(A \cup B) = X$. Hence, $\overline{R}_{\delta\beta\mathcal{I}}(A \cup B) \nsubseteq \overline{R}_{\delta\beta\mathcal{I}}(A) \cup \overline{R}_{\delta\beta\mathcal{I}}(A)$.

(iii) If $A = \{b, d\}$ and $B = \{b, e\}$, then $\underline{R}_{\delta\beta\mathcal{I}}(A) = \{b, d\}$, $\underline{R}_{\delta\beta\mathcal{I}}(B) = \{b, e\}$ and $\underline{R}_{\delta\beta\mathcal{I}}(A \cap B) = \emptyset$. Hence, $\underline{R}_{\delta\beta\mathcal{I}}(A) \cap \underline{R}_{\delta\beta\mathcal{I}}(B) \nsubseteq \underline{R}_{\delta\beta\mathcal{I}}(A \cap B)$.

(iv) If $A = \{e\}$ and $B = \{a, b\}$, then $\overline{R}_{\delta\beta\mathcal{I}}(A) = \{a, e\}, \overline{R}_{\delta\beta\mathcal{I}}(B) = \{a, b\}$ and $\overline{R}_{\delta\beta\mathcal{I}}(A \cap B) = \emptyset$. Hence, $\overline{R}_{\delta\beta\mathcal{I}}(A) \cap \overline{R}_{\delta\beta\mathcal{I}}(A) \nsubseteq \overline{R}_{\delta\beta\mathcal{I}}(A \cap B)$.

Definition 4.4. Let A be a subset of a $\delta\beta\mathcal{I}$-approximation space $(X, R_{\delta\beta\mathcal{I}})$, then for any ideal $\mathcal{I}$ the following statements hold

(i) $\delta\beta\mathcal{I}$-external edge of A, $\delta\beta\mathcal{I} - \overline{edg}(A) = \overline{R}_{\delta\beta\mathcal{I}}(A) - A$.

(ii) $\delta\beta\mathcal{I}$-internal edge of A, $\delta\beta\mathcal{I} - \underline{edg}(A) = A - \underline{R}_{\delta\beta\mathcal{I}}(A)$.

(iii) $\delta\beta\mathcal{I}$-boundary of A, $\delta\beta\mathcal{I} - b(A) = \overline{R}_{\delta\beta\mathcal{I}}(A) - \underline{R}_{\delta\beta\mathcal{I}}(A)$.

(iv) $\delta\beta\mathcal{I}$-exterior of A, $\delta\beta\mathcal{I} - ext(A) = X - \overline{R}_{\delta\beta\mathcal{I}}(A)$.

Lemma 4.1. Let A be a subset of a $\delta\beta\mathcal{I}$-approximation space $(X, R_{\delta\beta\mathcal{I}})$, then for any ideal $\mathcal{I}$ the following statements hold

(i) $\delta\beta - \overline{edg}(A) \subseteq \delta\beta\mathcal{I} - \overline{edg}(A)$.
(ii) $\delta\beta - \underline{edg}(A) \subseteq \delta\beta\mathcal{I} - \underline{edg}(A)$.
(iii) $\delta\beta - b(A) \subseteq \delta\beta\mathcal{I} - b(A)$.
(iv) $\delta\beta\mathcal{I} - ext(A) \subseteq \delta\beta - ext(A)$.

Example 4.5. Table 6.2 shows the relation between $\delta\beta\mathcal{I}$-boundary and $\delta\beta$-boundary, for any set A.

Lemma 4.2. Let A be a subset of a $\delta\beta\mathcal{I}$-approximation space $(X, R_{\delta\beta\mathcal{I}})$, then for any ideal $\mathcal{I}$ the following statements hold
(i) $\delta\beta\mathcal{I} - b(A) = \delta\beta\mathcal{I} - \underline{edg}(A) \cup \delta\beta\mathcal{I} - \overline{edg}(A)$.
(ii) $\overline{R}_{\delta\beta\mathcal{I}}(A) - \underline{R}_{\delta\beta}(A) = \delta\beta - \underline{edg}(A) \cup \delta\beta\mathcal{I} - \overline{edg}(A)$.
(iii) $\overline{R}_{\delta\beta}(A) - \underline{R}_{\delta\beta\mathcal{I}}(A) = \delta\beta - \overline{edg}(A) \cup \delta\beta\mathcal{I} - \underline{edg}(A)$.

Proof. (i) In view of Theorem 4.2 and Definition 4.4, $\delta\beta\mathcal{I} - b(A) = \overline{R}_{\delta\beta\mathcal{I}}(A) - \underline{R}_{\delta\beta\mathcal{I}}(A) = (\overline{R}_{\delta\beta\mathcal{I}}(A) - A) \cup (A - \underline{R}_{\delta\beta\mathcal{I}}(A)) = \delta\beta\mathcal{I} - \underline{edg}(A) \cup \delta\beta\mathcal{I} - \overline{edg}(A)$. By the same manner we can prove (ii) and (iii).

Lemma 4.3. Let A be a subset of a $\delta\beta\mathcal{I}$-approximation space $(X, R_{\delta\beta\mathcal{I}})$, then for any ideal $\mathcal{I}$ the following statements hold
(i) $\delta\beta\mathcal{I} - \overline{edg}(A) = \delta\beta - \overline{edg}(A) \cup (\overline{R}_{\delta\beta\mathcal{I}}(A) - \overline{R}_{\delta\beta}(A))$.
(ii) $\delta\beta\mathcal{I} - \underline{edg}(A) = \delta\beta - \underline{edg}(A) \cup (\underline{R}_{\delta\beta}(A) - \underline{R}_{\delta\beta\mathcal{I}}(A))$.
(iii) $\delta\beta - \overline{edg}(A) = \delta\beta\mathcal{I} - \overline{edg}(A) \cup (\overline{R}_{\delta\beta}(A) - \overline{R}_{\delta\beta\mathcal{I}}(A))$.

Proof. (i) In view of Theorem 4.2 and Definition 4.4, $\delta\beta\mathcal{I} - \overline{edg}(A) = (\overline{R}_{\delta\beta\mathcal{I}}(A) - A) = (\overline{R}_{\delta\beta\mathcal{I}}(A) - (A \cap \overline{R}_{\delta\beta}(A)) = (\overline{R}_{\delta\beta\mathcal{I}}(A) - A) \cup (\overline{R}_{\delta\beta\mathcal{I}}(A) - \overline{R}_{\delta\beta}(A)) = \delta\beta - \overline{edg}(A) \cup (\overline{R}_{\delta\beta\mathcal{I}}(A) - \overline{R}_{\delta\beta}(A))$. By the same manner we can prove (ii) and (iii).

Theorem 4.5. Let $(X, R_{\delta\beta\mathcal{I}})$ be a $\delta\beta\mathcal{I}$-approximation space and $\mathcal{I}$, $\mathcal{J}$ be two ideals on X with $\mathcal{I} \subseteq \mathcal{J}$. Then, for any set A, $\delta\beta\mathcal{J} - b(A) \subseteq \delta\beta\mathcal{I} - b(A)$.

Proof. Let $x \in \delta\beta\mathcal{J} - b(A)$, then $x \in \overline{R}_{\delta\beta\mathcal{J}}(A) - \underline{R}_{\delta\beta\mathcal{J}}(A)$ and so $x \in \overline{R}_{\delta\beta\mathcal{J}}(A)$ and $x \notin \underline{R}_{\delta\beta\mathcal{J}}(A)$. Since $\underline{R}_{\delta\beta\mathcal{J}}(A) = X - \overline{R}_{\delta\beta\mathcal{J}}(X - A)$, then $x \in \overline{R}_{\delta\beta\mathcal{J}}(X - A)$. Since $\mathcal{I} \subseteq \mathcal{J}$, then $x \in \overline{R}_{\delta\beta\mathcal{I}}(A)$ and $x \in \overline{R}_{\delta\beta\mathcal{I}}(X - A)$ from Theorem 4.1. Hence, $x \in \overline{R}_{\delta\beta\mathcal{I}}(A)$ and $x \notin \underline{R}_{\delta\beta\mathcal{I}}(A)$ and so $x \in \delta\beta\mathcal{I} - b(A)$. Consequently, $\delta\beta\mathcal{J} - b(A) \subseteq \delta\beta\mathcal{I} - b(A)$.

Definition 4.5. Let A be a subset of a $\delta\beta\mathcal{I}$-approximation space $(X, R_{\delta\beta\mathcal{I}})$ and $\mathcal{I}$ be an ideal on X. $\delta\beta\mathcal{I}$-strong memberships, $\underline{\in}_{\delta\beta\mathcal{I}}$ (resp. $\delta\beta\mathcal{I}$-weak memberships, $\overline{\in}_{\delta\beta\mathcal{I}}$) are defined as
(i) $x \underline{\in}_{\delta\beta\mathcal{I}}(A)$ if and only if $x \in \underline{R}_{\delta\beta\mathcal{I}}(A)$.
(ii) $x \overline{\in}_{\delta\beta\mathcal{I}}(A)$ if and only if $x \in \overline{R}_{\delta\beta\mathcal{I}}(A)$.

In view of (i) of Theorem 4.2, the following lemma is obvious and then the proof is omitted.

Lemma 4.4. Let A be a subset of a $\delta\beta\mathcal{I}$-approximation space $(X, R_{\delta\beta\mathcal{I}})$ and $\mathcal{I}$ be an ideal on X. Then,

(i) $x\underline{\in}_{\delta\beta\mathcal{I}}(A) \Longrightarrow x\underline{\in}_{\delta\beta}(A)$.

(ii) $x\overline{\in}_{\delta\beta}(A) \Longrightarrow x\overline{\in}_{\delta\beta\mathcal{I}}(A)$.

(iii) $x\underline{\in}_{\delta\beta\mathcal{I}}(A) \Longrightarrow x\overline{\in}_{\delta\beta\mathcal{I}}(A)$.

The converse may not be true in general as seen in the following example.

Example 4.6. In Example 4.1 and Table 6.1, we have

(i) Let $A = \{a, b\}$, then $a, b\underline{\in}_{\delta\beta}(A)$ and $a, b\underline{\notin}_{\delta\beta\mathcal{I}}(A)$.

(ii) Let $A = \{e\}$, then $a\overline{\in}_{\delta\beta\mathcal{I}}(A)$ and $a\overline{\notin}_{\delta\beta}(A)$.

(iii) Let $A = \{d\}$, then $c\overline{\in}_{\delta\beta\mathcal{I}}(A)$ and $c\underline{\notin}_{\delta\beta\mathcal{I}}(A)$.

Definition 4.6. Let $(X, R_{\delta\beta\mathcal{I}})$ be $\delta\beta\mathcal{I}$-approximation space and $A, B \subseteq X$. Then, for any ideal $\mathcal{I}$, we say that

(i) A is $\delta\beta\mathcal{I}$ roughly bottom included in B, symbolically $A\underline{\subset}_{\delta\beta\mathcal{I}}B$, if $\underline{R}_{\delta\beta\mathcal{I}}(A)\underline{\subset}_{\delta\beta\mathcal{I}}\underline{R}_{\delta\beta\mathcal{I}}(B)$.

(ii) A is $\delta\beta\mathcal{I}$ roughly top included in B, symbolically $A\overline{\subset}_{\delta\beta\mathcal{I}}B$, if $\overline{R}_{\delta\beta\mathcal{I}}(A)\overline{\subset}_{\delta\beta\mathcal{I}}\overline{R}_{\delta\beta\mathcal{I}}(B)$.

(iii) A is $\delta\beta\mathcal{I}$ roughly included in B, symbolically $A\overline{\underline{\subset}}_{\delta\beta\mathcal{I}}B$, if $A\underline{\subset}_{\delta\beta\mathcal{I}}B$ and $A\overline{\subset}_{\delta\beta\mathcal{I}}B$.

Example 4.7. In Example 4.1 and Table 6.1, it is clear that $\{a, c, e\}\underline{\subset}_{\delta\beta\mathcal{I}}\{a, b, d, e\}$, $\{a\}\overline{\subset}_{\delta\beta\mathcal{I}}\{e\}$ and $\{d\}\overline{\underline{\subset}}_{\delta\beta\mathcal{I}}\{a, d\}$.

Definition 4.7. Let $(X, R_{\delta\beta\mathcal{I}})$ be a $\delta\beta\mathcal{I}$-approximation space and $\mathcal{I}$ be any ideal on X. Then, the subsets A, B of X are said to be

(i) $\delta\beta\mathcal{I}$ roughly bottom equals, symbolically $A\bar{\sim}_{\delta\beta\mathcal{I}}B$, if $\underline{R}_{\delta\beta\mathcal{I}}(A) = \underline{R}_{\delta\beta\mathcal{I}}(B)$.

(ii) $\delta\beta\mathcal{I}$ roughly top equals, symbolically $A \simeq_{\delta\beta\mathcal{I}} B$, if $\overline{R}_{\delta\beta\mathcal{I}}(A) = \overline{R}_{\delta\beta\mathcal{I}}(B)$.

(iii) $\delta\beta\mathcal{I}$ roughly equals, symbolically $A \approx_{\delta\beta\mathcal{I}} B$, if $A\bar{\sim}_{\delta\beta\mathcal{I}}B$ and $A \simeq_{\delta\beta\mathcal{I}} B$.

Example 4.8. In Example 4.1 and Table 6.3, we have $\{d\}\bar{\sim}_{\delta\beta\mathcal{I}}\{a, d\}$ and $\{a, d, e\} \simeq_{\delta\beta\mathcal{I}} \{c, d, e\}$.

Definition 4.8. Let $(X, R_{\delta\beta\mathcal{I}})$ be $\delta\beta\mathcal{I}$-approximation space and $\mathcal{I}$ be any ideal on X. Then, a subset A of X is called

(i) $R_{\delta\beta\mathcal{I}}$-definable or $R_{\delta\beta\mathcal{I}}$-exact, if $\underline{R}_{\delta\beta\mathcal{I}}(A) = \overline{R}_{\delta\beta\mathcal{I}}(A)$ or $\delta\beta\mathcal{I}b(A) = \emptyset$.

(ii) $R_{\delta\beta\mathcal{I}}$-rough, if $\underline{R}_{\delta\beta\mathcal{I}}(A) \neq \overline{R}_{\delta\beta\mathcal{I}}(A)$ or $\delta\beta\mathcal{I}b(A) \neq \emptyset$.

Example 4.9. As in Example 4.1 and Table 6.1, let $(X, R_{\delta\beta\mathcal{I}})$ be $\delta\beta\mathcal{I}$-approximation space, we have the sets $\{a, e\}$, $\{c, d\}$, $\{a, b, e\}$ and $\{b, c, d\}$ are $R_{\delta\beta\mathcal{I}}$-exact and other sets are $R_{\delta\beta\mathcal{I}}$-rough.

Lemma 4.5. Let $(X, R_{\delta\beta\mathcal{I}})$ be $\delta\beta\mathcal{I}$-approximation space. Then,

(i) every $\delta\beta\mathcal{I}$-exact is $\delta\beta$-exact.

(ii) every $\delta\beta$-rough is $\delta\beta\mathcal{I}$-rough.

Proof. Follows directly from Theorem 4.2.

The converse of Lemma 4.5 may not be true in general as seen in the following example.

Example 4.10. In Example 4.1 and Table 6.1, then the set $\{c\}$ is $\delta\beta$-exact, but it is not $\delta\beta\mathcal{I}$-exact.

Definition 4.9. Let $(X, R_{\delta\beta\mathcal{I}})$ be $\delta\beta\mathcal{I}$-approximation space. Then, for any ideal $\mathcal{I}$ a subset A of X is called

(i) Roughly $R_{\delta\beta\mathcal{I}}$-definable, if $\underline{R}_{\delta\beta\mathcal{I}}(A) \neq \emptyset$ and $\overline{R}_{\delta\beta\mathcal{I}}(A) \neq X$.

(ii) Internally $R_{\delta\beta\mathcal{I}}$-undefinable, if $\underline{R}_{\delta\beta\mathcal{I}}(A) = \emptyset$ and $\overline{R}_{\delta\beta\mathcal{I}}(A) \neq X$.

(iii) Externally R$_{\delta\beta\mathcal{I}}$-undefinable, if $\underline{R}_{\delta\beta\mathcal{I}}(A) \neq \emptyset$ and $\overline{R}_{\delta\beta\mathcal{I}}(A) = X$.

(iv) Totally $R_{\delta\beta\mathcal{I}}$-undefinable, if $\underline{R}_{\delta\beta\mathcal{I}}(A) = \emptyset$ and $\overline{R}_{\delta\beta\mathcal{I}}(A) = X$.

Example 4.11. In Example 4.1 and Table 6.1, then the sets $\{a\}, \{b\}, \{c\}, \{b, c\}, \{a, b\}, \{a, c\}$, $\{a, b, c\}$ are internally $R_{\delta\beta\mathcal{I}}$-undefinable. The sets $\{d, e\}, \{a, d, e\}, \{c, d, e\}, \{b, d, e\}, \{a, c, d, e\}, \{a, b, d, e\}, \{b, c, d, e\}$ are externally $R_{\delta\beta\mathcal{I}}$-undefinable. The sets $\{a, e\}, \{c, d\}$, $\{a, b, e\}, \{b, c, d\}$ are $R_{\delta\beta\mathcal{I}}$-definable. All other non empty proper subset are roughly $R_{\delta\beta\mathcal{I}}$-definable.

Corollary 4.2. For any $\delta\beta\mathcal{I}$-approximation space $(X, R_{\delta\beta\mathcal{I}})$. The following statements are hold:

(i) Roughly $R_{\delta\beta\mathcal{I}}$-definable is roughly $R_{\delta\beta}$-definable.

(ii) Internally $R_{\delta\beta}$-undefinable is internally $R_{\delta\beta\mathcal{I}}$-undefinable.

(iii) Externally $R_{\delta\beta}$-undefinable is externally $R_{\delta\beta\mathcal{I}}$-undefinable.

(iv) Totally $R_{\delta\beta}$-undefinable is totally $R_{\delta\beta\mathcal{I}}$-undefinable.

5. Tables

Table 5.1:

A	$\underline{R}_{\delta\beta\mathcal{I}}$(A)	$\overline{R}_{\delta\beta\mathcal{I}}$(A)	$\alpha_{\delta\beta\mathcal{I}}$	$\underline{R}_{\delta\beta}$(A)	$\overline{R}_{\delta\beta}$(A)	$\alpha_{\delta\beta}$
$\{a\}$	$\emptyset$	$\{a\}$	0	$\{a\}$	$\{a\}$	1
$\{b\}$	$\emptyset$	$\{b\}$	0	$\emptyset$	$\{b\}$	0
$\{c\}$	$\emptyset$	$\{c\}$	0	$\{c\}$	$\{c\}$	1
$\{d\}$	$\{d\}$	$\{c, d\}$	$\frac{1}{2}$	$\{d\}$	$\{d\}$	1
$\{e\}$	$\{e\}$	$\{a, e\}$	$\frac{1}{2}$	$\{e\}$	$\{e\}$	1
$\{a, b\}$	$\emptyset$	$\{a, b\}$	0	$\{a, b\}$	$\{a, b\}$	1
$\{a, c\}$	$\emptyset$	$\{a, c\}$	0	$\{a, c\}$	$\{a, c\}$	1
$\{a, d\}$	$\{d\}$	$\{a, c, d\}$	$\frac{1}{3}$	$\{a, d\}$	$\{a, d\}$	1
$\{a, e\}$	$\{a, e\}$	$\{a, e\}$	1	$\{a, e\}$	$\{a, e\}$	1
$\{b, c\}$	$\emptyset$	$\{b, c\}$	0	$\{b, c\}$	$\{b, c\}$	1
$\{b, d\}$	$\{b, d\}$	$\{b, c, d\}$	$\frac{2}{3}$	$\{b, d\}$	$\{b, d\}$	1
$\{b, e\}$	$\{b, e\}$	$\{a, b, e\}$	$\frac{2}{3}$	$\{b, e\}$	$\{b, e\}$	1
$\{c, d\}$	$\{c, d\}$	$\{c, d\}$	1	$\{c, d\}$	$\{c, d\}$	1
$\{c, e\}$	$\{e\}$	$\{a, c, e\}$	$\frac{1}{3}$	$\{c, e\}$	$\{c, e\}$	1
$\{d, e\}$	$\{d, e\}$	X	$\frac{2}{5}$	$\{d, e\}$	$\{d, e\}$	1
$\{a, b, c\}$	$\emptyset$	$\{a, b, c\}$	0	$\{a, b, c\}$	$\{a, b, c\}$	1
$\{a, b, d\}$	$\{b, d\}$	$\{a, b, c, d\}$	$\frac{1}{2}$	$\{a, b, d\}$	$\{a, b, d\}$	1
$\{a, b, e\}$	$\{a, b, e\}$	$\{a, b, e\}$	1	$\{a, b, e\}$	$\{a, b, e\}$	1
$\{a, c, d\}$	$\{c, d\}$	$\{a, c, d\}$	$\frac{2}{3}$	$\{a, c, d\}$	$\{a, c, d\}$	1
$\{a, c, e\}$	$\{a, e\}$	$\{a, c, e\}$	$\frac{2}{3}$	$\{a, c, e\}$	$\{a, c, e\}$	1
$\{a, d, e\}$	$\{a, d, e\}$	X	$\frac{3}{5}$	$\{a, d, e\}$	$\{a, d, e\}$	1
$\{b, c, d\}$	$\{b, c, d\}$	$\{b, c, d\}$	1	$\{b, c, d\}$	$\{b, c, d\}$	1
$\{b, c, e\}$	$\{b, e\}$	$\{a, b, c, e\}$	$\frac{1}{2}$	$\{b, c, e\}$	$\{b, c, e\}$	1
$\{c, d, e\}$	$\{c, d, e\}$	X	$\frac{3}{5}$	$\{c, d, e\}$	$\{c, d, e\}$	1
$\{b, d, e\}$	$\{b, d, e\}$	X	$\frac{3}{5}$	$\{b, d, e\}$	$\{b, d, e\}$	1
$\{a, b, c, d\}$	$\{b, c, d\}$	$\{a, b, c, d\}$	$\frac{3}{4}$	$\{a, b, c, d\}$	$\{a, b, c, d\}$	1
$\{a, b, c, e\}$	$\{a, b, e\}$	$\{a, b, c, e\}$	$\frac{3}{4}$	$\{a, b, c, e\}$	$\{a, b, c, e\}$	1
$\{a, b, d, e\}$	$\{a, b, d, e\}$	X	$\frac{4}{5}$	$\{a, b, d, e\}$	$\{a, b, d, e\}$	1
$\{a, c, d, e\}$	$\{a, c, d, e\}$	X	$\frac{4}{5}$	$\{a, c, d, e\}$	X	$\frac{4}{5}$
$\{b, c, d, e\}$	$\{b, c, d, e\}$	X	$\frac{4}{5}$	$\{b, c, d, e\}$	$\{b, c, d, e\}$	1

Table 5.2:

A	$\delta\beta\mathcal{I}$-b(A)	$\delta\beta$-b(A)
$\{a\}$	$\{a\}$	$\emptyset$
$\{b\}$	$\{b\}$	$\{b\}$
$\{c\}$	$\{c\}$	$\emptyset$
$\{d\}$	$\{c\}$	$\emptyset$
$\{e\}$	$\{a\}$	$\emptyset$
$\{a,b\}$	$\{a,b\}$	$\emptyset$
$\{a,c\}$	$\{a,c\}$	$\emptyset$
$\{a,d\}$	$\{a,c\}$	$\emptyset$
$\{a,e\}$	$\emptyset$	$\emptyset$
$\{b,c\}$	$\{b,c\}$	$\emptyset$
$\{b,d\}$	$\{c\}$	$\emptyset$
$\{b,e\}$	$\{a\}$	$\emptyset$
$\{c,d\}$	$\emptyset$	$\emptyset$
$\{c,e\}$	$\{a,c\}$	$\emptyset$
$\{d,e\}$	$\{a,b,c\}$	$\emptyset$
$\{a,b,c\}$	$\{a,b,c\}$	$\emptyset$
$\{a,b,d\}$	$\{a,c\}$	$\emptyset$
$\{a,b,e\}$	$\emptyset$	$\emptyset$
$\{a,c,d\}$	$\{a\}$	$\emptyset$
$\{a,c,e\}$	$\{c\}$	$\emptyset$
$\{a,d,e\}$	$\{b,c\}$	$\emptyset$
$\{b,c,d\}$	$\emptyset$	$\emptyset$
$\{b,c,e\}$	$\{a,c\}$	$\emptyset$
$\{c,d,e\}$	$\{a,b\}$	$\emptyset$
$\{b,d,e\}$	$\{a,c\}$	$\emptyset$
$\{a,b,c,d\}$	$\{a\}$	$\emptyset$
$\{a,b,c,e\}$	$\{c\}$	$\emptyset$
$\{a,b,d,e\}$	$\{c\}$	$\emptyset$
$\{a,c,d,e\}$	$\{b\}$	$\{b\}$
$\{b,c,d,e\}$	$\{a\}$	$\emptyset$

Table 5.3:

$\delta\beta\mathcal{I}$ roughly bottom equals	$\delta\beta\mathcal{I}$ roughly top equals
$\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$,	
$\{a,c\}$, $\{b,c\}$, $\{a,b,c\}$	$\{d\}$, $\{c,d\}$
$\{d\}$, $\{a,d\}$	$\{d,e\},\{a,d,e\},\{c,d,e\},\{b,d,e\},\{a,c,d,e\},\{a,b,d,e\},\{b,c,d,e\}$
$\{e\}$, $\{c,e\}$	$\{e\}$, $\{a,e\}$
$\{b,d\}$, $\{a,b,d\}$	$\{a,d\}$, $\{a,c,d\}$
$\{b,e\}$, $\{b,c,e\}$	$\{b,d\}$, $\{b,c,d\}$
$\{a,e\}$, $\{a,c,e\}$	$\{b,e\}$, $\{a,b,e\}$
$\{c,d\}$, $\{a,c,d\}$	$\{c,e\}$, $\{a,c,e\}$
$\{a,b,e\}$, $\{a,b,c,e\}$	$\{a,b,d\}$, $\{a,b,c,d\}$
$\{b,c,d\}$, $\{a,b,c,d\}$	$\{b,c,e\}$, $\{a,b,c,e\}$

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REFERENCES

1. M. E. Abd El-Monsef; S. N. El-Deeb, R. A. Mahmoud, β-opensets and β-continuous mappings, Bull. Fac. Sci. Assiut Univ., 12(1983), 77–90.
2. H. M. Abu-Donia, A. S. Salama, Generalization of Pawlaks rough approximation spaces by using $\delta\beta$-open sets, International Journal of Approximate Reasoning, 53(7)(2012), 1094–1105.
3. D. Andrijević, Semi-preopensets, Mat. Vesnik., 38(1986), 24–32.
4. D. Ciucci; T. Mihalydeak, Z. E.Csajbk, On exactness, definability and vagueness in partial approximation spaces, Tech. Sci. 18(3)(2015), 203-212.
5. Deepmala, L. N. Mishra, Differential operators over modules and rings as a path to the generalized differential geometry, FACTA Univ. (NIŠ) Ser. Math. Inform., 30(5)(2015), 753–764.
6. Deepmala, A Study on Fixed Point Theorems for Nonlinear Contractions and its Applications, Ph. D. Thesis, Pt. Ravishankar Shukla University, Raipur 492 010, Chhatisgarh, India, (2014).
7. E. Ekici, On e-open sets and $(D,S)^*$-sets and decompositions of continuous functions, Math. Moravica, 13(1)(2009), 29–36.
8. M. E. El-Shafei, A. M. Kozae, M. Abo-elhamayel, Semi ordered topologicalapproximations of rough sets, Ann. of fuzzy sets, fuzzy logic and fuzzy syst., 2(2013) 61–72.

9. T. R. Hamlett, D. Rose, $*$-topological properties, Internat J. Math. and Math. Sci., 13(3)(1990), 507–512.
10. E. Hatir, T. Noiri, On $\delta\beta$-continuous functions, Chaos, Solitons Fractals, 42(2009), 205–211.
11. E. Hatir, T. Noiri, Decompositions of continuity and complete continuity, Acta Math. Hungary, 11(3)(2006), 281–287.
12. E. Hatir, T. Noiri, On descompositions of continuity via idealization, Acta Math. Hungar., 96(2002), 341–349.
13. E. Hayashi, Topologies defined by local properties, Mathematische Annalen, 156(1964), 205–215.
14. D. Jankovic, T. R. Hamlett, New topologies from old via ideals, Amer. Math. Monthly, 97(1990), 295–310.
15. K. Kuratowski, Topology, Vol. I, Academic Press, New York, (1966).
16. N. Levine, Semi open sets and semi continuity topological spaces, Amer. Math. Monthly, 70(1963), 24–32.
17. T. Y. Lin, An overview of rough set theory from the Point of View of RelationalDatabases, Bull. Int. Rough Set Soc., 1(1)(1997) 30–34.
18. V. N. Mishra, Some Problems on Approximations of Functions in Banach Spaces, Ph. D. Thesis, Indian Institute of Technology, Roorkee 247 667, Uttarakhand, India, (2007).
19. Z. Pawlak, Rough Sets, Internat. J. Comput. Inform. Sci., 11(1982), 341–356.
20. Z. Pawlak, Rough Sets, Theoretical Aspects of Reasoning about Data, Kluwer Academic, Boston, (1991).
21. M. H. Stone, Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc., 41(1937), 375–481.
22. R. Vaidyanathaswamy, The localisation theory in set Topology, Proc. Indian Acad. Sci. Math. Sci., 20(1945), 51–61.
23. N. V. Velicko, H-closed topological spaces, Amer. Math. Soc. Transl., 78(1968), 103–118.
24. W. Zhu, Topological approaches to covering rough sets, Inf. Sci. 177(15)(2007), 1499-1508.

A. E. Radwan
Ain Shams University
Faculty of science
Department of Mathematics
Cairo, Egypt
a_radwan@sci.asu.edu.eg

Rodyna A. Hosny
Zagazig University
Faculty of Science
Department of Mathematics

Zagazig, Egypt
rodynahosny@yahoo.com

A. M. Abd El-latif
Northern Borders University
Faculty of Arts and Sciences
Department of Mathematics
Rafha, KSA
alaa_8560@yahoo.com , Alaa.ali@nbu.edu.sa