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COMMON HERMITIAN LEAST-RANK SOLUTION OF MATRIX EQUATIONS $A_1X_1A_1^* = B_1$ AND $A_2X_2A_2^* = B_2$ SUBJECT TO INEQUALITY RESTRICTIONS

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Abstract. In this paper, we establish a set of explicit formulas for calculating the maximal and minimal ranks and inertias of $P - X$ with respect to X, where $P \in \mathbb{C}_H^n$ is given, X is a common Hermitian least-rank solution of matrix equations $A_1XA_1^* = B_1$ and $A_2XA_2^* = B_2$. As application, we drive necessary and sufficient conditions for $X \succ P\, (\geq P,\ \prec P,\ \leq P)$ in the Löwner partial ordering. As consequence, we give necessary and sufficient conditions for the existence of common Hermitian positive (nonnegative, negative, nonpositive) definite least-rank solution to $A_1XA_1^* = B_1$ and $A_2XA_2^* = B_2$.

Keywords: Matrix equation, Rank formulas, Moore-Penrose generalized inverse, Hermitian, Least-rank solution, Inertia.

1. Introduction

Throughout this paper, $\mathbb{C}^{m\times n}$ and $\mathbb{C}_H^n$ stand for the sets of all $m \times n$ complex matrices and all $n \times n$ complex Hermitian matrices respectively. The symbols, A^*, $r(A)$, $\mathrm{Re}(A)$, stand for the conjugate transpose, the rank, and the range of A, respectively. I_m denotes the identity matrix of order m. We write $A \succ 0$ $(A \geq 0)$ if A is Hermitian positive (nonnegative) definite. Two Hermitian matrices A and B of the same size are said to satisfy the inequality $A \succ B$ $(A \geq B)$ in the Löwner partial ordering if $A - B$ is positive (nonnegative) definite. The Moore-Penrose generalized inverse of a matrix $A \in \mathbb{C}^{m\times n}$, denoted by A^+, is defined to be the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the following four matrix equations:

$$(1)\ AXA = A,\ (2)\ XAX = X,\ (3)\ (AX)^* = AX,\ (4)\ (XA)^* = XA.$$

Results on the generalized inverse and the Moore-Penrose generalized inverse can be found in [1, 2, 4, 8, 11].

Further, define E_A and F_A stand for the two orthogonal projectors $E_A = I - AA^+$, $F_A = I - A^+A$ induced by A. Their ranks are given by $r(E_A) = m - r(A)$, $r(F_A) =$

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$n - r(A)$.
The inertia of $A \in \mathbb{C}_H^m$ is defined to be the triplet $In(A) = \{i_+(A), i_-(A), i_0(A)\}$. Where $i_+(A)$, $i_-(A)$ and $i_0(A)$ are the number of positive, negative and zero eigenvalues of A counted with multiplicities, respectively. The two numbers $i_+(A)$ and $i_-(A)$ are usually called the partial inertias of A. For a matrix $A \in \mathbb{C}_H^m$, we have $r(A) = i_+(A) + i_-(A)$ and $i_0(A) = m - r(A)$.

We need the following lemmas concerning ranks and inertias of matrices in the latter part of this paper.

Lemma 1.1. *[9] Let S be a set consisting of matrices over $\mathbb{C}^{m\times n}$, and let H be a set consisting of Hermitian matrices over $\mathbb{C}_H^m$. Then,*

a) For $m = n$, S has a non singular matrix if and only if $\max\limits_{X\in S} r(X) = m$.

b) For $m = n$, all $X \in S$ are non singular if and only if $\min\limits_{X\in S} r(X) = m$.

c) $0 \in S$ if and only if $\min\limits_{X\in S} r(X) = 0$.

d) All $X \in S$ have the same rank if and only if $\max\limits_{X\in S} r(X) = \min\limits_{X\in S} r(X)$.

e) H has a matrix $X \succ 0$ ($X \prec 0$) if and only if $\max\limits_{X\in H} i_+(X) = m$ ($\max\limits_{X\in H} i_-(X) = m$).

f) H has a matrix $X \geq 0$ ($X \leq 0$) if and only if $\min\limits_{X\in H} i_-(X) = 0$ ($\min\limits_{X\in H} i_+(X) = 0$).

g) All $X \in H$ satisfy $X \succ 0$ ($X \prec 0$) if and only if $\min\limits_{X\in H} i_+(X) = m$ ($\min\limits_{X\in H} i_-(X) = m$).

h) All $X \in H$ satisfy $X \geq 0$ ($X \leq 0$) if and only if $\max\limits_{X\in H} i_-(X) = 0$ ($\max\limits_{X\in H} i_+(X) = 0$).

Lemma 1.2. *[11] Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, $D \in \mathbb{C}^{l\times k}$. Then,*

$$r\begin{bmatrix} A, & B \end{bmatrix} = r(A) + r(E_A B) = r(B) + r(E_B A),$$

$$r\begin{pmatrix} A \\ C \end{pmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C),$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_B A F_C).$$

The following formulas follow from Lemma 1.2

$$r\begin{bmatrix} A & BF_P \\ E_Q C & 0 \end{bmatrix} = r\begin{bmatrix} A & B & 0 \\ C & 0 & Q \\ 0 & P & 0 \end{bmatrix} - r(P) - r(Q),$$

$$r\begin{bmatrix} M & N \\ E_P A & E_P B \end{bmatrix} = r\begin{bmatrix} M & N & 0 \\ A & B & P \end{bmatrix} - r(P),$$

$$r\begin{bmatrix} M & AF_P \\ N & BF_P \end{bmatrix} = r\begin{bmatrix} M & A \\ N & B \\ O & P \end{bmatrix} - r(P).$$

Lemma 1.3. *[9] Let $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}^{m\times n}$ and denote $M = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}$. Then,*

$$i_\pm(M) = r(B) + i_\pm(E_B A E_B).$$

In particular,
a) If $A \geq 0$, then $i_+(M) = r[A, B]$ and $i_-(M) = r(B)$,
b) If $A \leq 0$, then $i_+(M) = r(B)$ and $i_-(M) = r[A, B]$,
c) $i_\pm(A) \leq i_\pm(M) \leq i_\pm(A) + r(B)$.

Some useful formulas derived from lemma 1.3 are given below

$$i_\pm\begin{bmatrix} A & BF_P \\ F_P B^* & 0 \end{bmatrix} = i_\pm\begin{bmatrix} A & B & 0 \\ B^* & 0 & P^* \\ 0 & P & 0 \end{bmatrix} - r(P),$$

$$i_\pm\begin{bmatrix} E_Q A E_Q & E_Q B \\ B^* E_Q & D \end{bmatrix} = i_\pm\begin{bmatrix} A & B & Q \\ B^* & D & 0 \\ Q^* & 0 & 0 \end{bmatrix} - r(Q).$$

Lemma 1.4. *[10, 12] Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$. Then,*

i) $\min\limits_{X\in\mathbb{C}^{k\times n}, Y\in\mathbb{C}^{m\times l}} r(A - BX - YC) = r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - r(B) - r(C)$.

ii) if $A \in \mathbb{C}^{m\times m}$, $A^* = -A$. Then,

$\min\limits_{X\in\mathbb{C}^{k\times m}} r(A - BX - X^*B^*) = r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - 2r(B)$.

Lemma 1.5. *[11] Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$ and $D \in \mathbb{C}^{l\times k}$ be given. Then the rank of the Shur complement $S_A = D - CA^\dagger B$ satisfies the equality*

$$r\left(D - CA^\dagger B\right) = r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} - r(A).$$

Lemma 1.6. *[11] Let A_1, A_2, B_1, B_2, C_1, C_2, and D are matrices such that expression $D - C_1A_1^\dagger B_1 - C_2A_2^\dagger B_2$ is defined. Then,*

$$r\left(D - C_1A_1^\dagger B_1 - C_2A_2^\dagger B_2\right) = r\begin{bmatrix} A_1^*A_1A_1^* & 0 & A_1^*B_1 \\ 0 & A_2^*A_2A_2^* & A_2^*B_2 \\ C_1A_1^* & C_2A_2^* & D \end{bmatrix} - r(A_1) - r(A_2).$$

Lemma 1.7. *[9] Let* $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}^{m\times n}$ *and* $D \in \mathbb{C}_H^n$. *Then,*

$$i_\pm\left(D - B^*A^\dagger B\right) = i_\pm \begin{bmatrix} A^3 & AB \\ (AB)^* & D \end{bmatrix} - i_\pm(A).$$

We consider the linear matrix equation

$$AXA^* = B \tag{1.1}$$

Where $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}_H^m$, are given and $X \in \mathbb{C}_H^n$ is unknown matrix.
Equation (1.1) is one of the best known matrix equations in matrix theory and applications. Many results have been obtained on solving rank minimization problems and many results have been obtained on rank minimizations associated with matrix equations and their solutions (see e.g. [5, 6, 7, 16]). Obviously, the concept of least-rank solution was first proposed and studied in [14, 18].

In [13] The Hermitian least-rank solution of (1.1) is the matrix X which minimizes the rank of the difference $(B - AXA^*)$ or equivalently

$$r(B - AXA^*) = \min \tag{1.2}$$

The Hermitian least-rank solution of (1.1) is the solution of the consistent equation

$$E_{T_1}(X + TM^+T^*)E_{T_1} = 0 \tag{1.3}$$

Equation (1.3) is called the normal equation associated with (1.2). Hence the general expression of the Hermitian least-rank solution of (1.1) can be written by

$$X = -TM^+T^* + T_1U + U^*T_1^*, \tag{1.4}$$

where $M = \begin{bmatrix} B & A \\ A^* & 0 \end{bmatrix}$, $T = \begin{bmatrix} 0 & I_n \end{bmatrix}$, $T_1 = TF_M$, and $U \in \mathbb{C}^{(m+n)\times n}$ is arbitrary.

Many papers on the rank, inertia, consistency and solutions of the equation (1.1) and its applications can be found in the literature, see, e.g. in [10, 15, 17, 19, 22]

2. Common Hermitian least rank solution of matrix equations $A_1XA_1^* = B_1$ and $A_2XA_2^* = B_2$ subject to inequality restrictions

Following the work of [3, 13, 20, 21, 22, 23], in this section we study the existence of a Hermitian matrix satisfying the matrix inequality $X \succ P\,(\geq P,\ \prec P,\ \leq P)$ in the löwner partial ordering.

Consider the pair of matrix equations

$$A_1XA_1^* = B_1 \text{ and } A_2XA_2^* = B_2. \tag{2.1}$$

where $A_j \in \mathbb{C}^{m_j\times n}$, $B_j \in \mathbb{C}_H^{m_j}$, $j = 1, 2$.are given matrices and $X \in \mathbb{C}_H^n$ is unknown matrix.

We need the following lemma

Lemma 2.1. [10, 16] *Let* $M = \begin{bmatrix} C_1 & 0 & A_1 \\ 0 & -C_2 & A_2 \\ A_1^* & A_2^* & 0 \end{bmatrix}$. *Then the pair of matrix equations* $A_1X_1A_1^* = C_1$ *and* $A_2X_2A_2^* = C_2$ *have a common solution* $X \in \mathbb{C}_H^n$ *if and only if* $\mathrm{Re}\left(C_j\right) \subseteq \mathrm{Re}\left(A_j\right)$ *and* $r(M) = 2r(A)$, *j =1, 2.*
where $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$. *In this case the general common Hermitian solution of* $A_1X_1A_1^* = C_1$ *and* $A_2X_2A_2^* = C_2$ *can be written in the following parametric form*

$$X = X_0 + F_AU_1 + (F_AU_1)^* + F_{A_1}U_2F_{A_2} + (F_{A_1}U_2F_{A_2})^*.$$

where X_0 *is a special solution of* $A_1X_1A_1^* = C_1$ *and* $A_2X_2A_2^* = C_2$, *and* $U_1, U_2, U_3 \in \mathbb{C}^{n\times n}$ *are arbitrary.*

It is well known that the least squares solution of matrix equation is the solution of its normal equation. Therefore the common Hermitian least-rank solution of pair of matrix equations (2.1) is the common Hermitian solution of matrix equations:

$$E_{T_{11}}XE_{T_{11}} = -E_{T_{11}}\left(T_1M_1^+T_1^*\right)E_{T_{11}} \text{ and } E_{T_{22}}XE_{T_{22}} = -E_{T_{22}}\left(T_2M_2^+T_2^*\right)E_{T_{22}}. \tag{2.2}$$

From Lemma 2.1 the general common Hermitian solution of (2.1) can be written in the following parametric form

$$X = X_0 + F_GU_1 + (F_GU_1)^* + F_{E_{T_{11}}}U_2F_{E_{T_{22}}} + \left(F_{E_{T_{11}}}U_2F_{E_{T_{22}}}\right)^*. \tag{2.3}$$

Where $G^* = \begin{bmatrix} E_{T_{11}}, & E_{T_{22}} \end{bmatrix}$ and $U_1, U_2 \in \mathbb{C}^{n\times n}$ are arbitrary.
For convenience of representation, the following notation for the collection of all common Hermitian least-rank solutions of (2.1) is adopted

$$S = \left\{X \in \mathbb{C}_H^n / E_{T_{11}}XE_{T_{11}} = -E_{T_{11}}\left(T_1M_1^+T_1^*\right)E_{T_{11}}, E_{T_{22}}XE_{T_{22}} = -E_{T_{22}}\left(T_2M_2^+T_2^*\right)E_{T_{22}}\right\}. \tag{2.4}$$

We need the following Lemma

Lemma 2.2. [20] *Let*

$$P(X,Y) = A - BX - (BX)^* - CYD - (CYD)^* . \tag{2.5}$$

Where $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{m\times p}$ *and* $D \in \mathbb{C}^{q\times m}$ *are given, and* $X \in \mathbb{C}^{n\times m}$, $Y \in \mathbb{C}^{p\times q}$ *are variable matrices. Also, let*

$$M = \begin{bmatrix} A & B & C & D^* \\ B^* & 0 & 0 & 0 \end{bmatrix}, M_1 = \begin{bmatrix} A & B & C \\ B^* & 0 & 0 \\ C^* & 0 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} A & B & D^* \\ B^* & 0 & 0 \\ D & 0 & 0 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} A & B & C & D^* \\ B^* & 0 & 0 & 0 \\ C^* & 0 & 0 & 0 \end{bmatrix}, N_2 = \begin{bmatrix} A & B & C & D^* \\ B^* & 0 & 0 & 0 \\ D & 0 & 0 & 0 \end{bmatrix}.$$

Then,

$$\max_{X,Y} r\left[P(X,Y)\right] = \min\left\{m, r(M), r(M_1), r(M_2)\right\}, \tag{2.6}$$

$$\min_{X,Y} r\left[P(X,Y)\right] = 2r(M) - 2r(B) + \max\left\{\begin{array}{c} s_+ + s_-, s_- + t_+, \\ s_+ + t_-, t_+ + t_- \end{array}\right\}, \tag{2.7}$$

$$\max_{X,Y} i_\pm\left[P(X,Y)\right] = \min\left\{i_\pm(M_1), i_\pm(M_2)\right\}, \tag{2.8}$$

$$\min_{X,Y} i_\pm\left[P(X,Y)\right] = r(M) - r(B) + \max\left\{s_\pm, t_\pm\right\}, \tag{2.9}$$

where $s_\pm = i_\pm(M_1) - r(N_1)$ *and* $t_\pm = i_\pm(M_2) - r(N_2)$.

Theorem 2.1. *Let* $A_j \in \mathbb{C}^{m_j\times n}$, $B_j \in \mathbb{C}_H^{m_j}$, $j = 1,2$ *and* $P \in \mathbb{C}_H^n$ *be given, and assume that* (2.1) *have a common Hermitian least-rank solution and* S *is as given in* (2.4). *Also, let*

$$Q_1 = \begin{bmatrix} M_1^*M_1M_1^* & 0 & 0 & M_1^*T_1^*E_{T_{11}} & 0 \\ 0 & M_2^*M_2M_2^* & 0 & 0 & M_2^*T_2^*E_{T_{22}} \\ -E_{T_{11}}T_1M_1^* & 0 & E_{T_{11}} & E_{T_{11}}PE_{T_{11}} & 0 \\ 0 & E_{T_{22}}T_2M_2^* & E_{T_{22}} & 0 & -E_{T_{22}}PE_{T_{22}} \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} M_1^*M_1M_1^* & 0 & M_1^*T_1^*E_{T_{11}} \\ -E_{T_{11}}T_1M_1^* & E_{T_{11}} & E_{T_{11}}PE_{T_{11}} \\ 0 & E_{T_{22}} & 0 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} M_2^*M_2M_2^* & 0 & M_2^*T_2^*E_{T_{22}} \\ E_{T_{22}}T_2M_2^* & E_{T_{11}} & 0 \\ 0 & E_{T_{22}} & -E_{T_{22}}PE_{T_{22}} \end{bmatrix},$$

$$Q_4 = \begin{bmatrix} M_1^3 & 0 & M_1 T_1^* E_{T_{11}} \\ 0 & 0 & E_{T_{11}} \\ E_{T_{11}} T_1 M_1^* & E_{T_{11}} & -E_{T_{11}} P E_{T_{11}} \end{bmatrix},$$

$$Q_5 = \begin{bmatrix} M_2^3 & 0 & M_2 T_2^* E_{T_{22}} \\ 0 & 0 & E_{T_{22}} \\ E_{T_{22}} T_2 M_2^* & E_{T_{22}} & -E_{T_{22}} P E_{T_{22}} \end{bmatrix}.$$

Then,

$$\max_{X\in S} r(P-X) = \min\{n, c_1, c_2, c_3\}, \tag{2.10}$$

$$\min_{X\in S} r(P-X) = 2r(Q_1) - 2r(M_1) - 2r(M_2) + \max\{s_1, s_2, s_3, s_4\}, \tag{2.11}$$

$$\max_{X\in S} i_\pm(P-X) = \min\left\{ \begin{array}{c} n + i_\pm(Q_4) - i_\pm(M_1) - r(E_{T_{11}}), \\ n + i_\pm(Q_5) - i_\pm(M_2) - r(E_{T_{22}}) \end{array} \right\}, \tag{2.12}$$

$$\min_{X\in S} i_\pm(P-X) = r(Q_1) - r(M_1) - r(M_2) + \max\left\{ \begin{array}{c} i_\pm(Q_4) - i_\pm(M_1) + r(M_1) - r(Q_2), \\ i_\pm(Q_5) - i_\pm(M_2) + r(M_2) - r(Q_3) \end{array} \right\}, \tag{2.13}$$

where

$$c_1 = 2n + r(Q_1) - r(E_{T_{11}}) - r(E_{T_{22}}) - r(G) - r(M_1) - r(M_2),$$

$$c_2 = 2n + r(Q_4) - r(M_1) - 2r(E_{T_{11}}), \quad c_3 = 2n + r(Q_5) - r(M_2) - 2r(E_{T_{22}}),$$

$$s_1 = r(Q_4) - 2r(Q_2) + r(M_1), \qquad s_2 = r(Q_5) - 2r(Q_3) + r(M_2),$$

$$s_3 = i_+(Q_4) + i_-(Q_5) - r(Q_2) - r(Q_3) + i_-(M_1) + i_+(M_2),$$

$$s_4 = i_-(Q_4) + i_+(Q_5) - r(Q_2) - r(Q_3) + i_+(M_1) + i_-(M_2).$$

Proof. Substituting (2.3) into $P - X$ yields

$$P - X = P - X_0 - F_G U_1 - (F_G U_1)^* - F_{E_{T_{11}}} U_2 F_{E_{T_{22}}} - \left(F_{E_{T_{11}}} U_2 F_{E_{T_{22}}}\right)^*. \tag{2.14}$$

Let

$$L = \begin{bmatrix} P - X_0 & F_G & F_{E_{T_{11}}} & F_{E_{T_{22}}} \\ F_G & 0 & 0 & 0 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} P - X_0 & F_G & F_{E_{T_{11}}} \\ F_G & 0 & 0 \\ F_{E_{T_{11}}} & 0 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} P - X_0 & F_G & F_{E_{T_{22}}} \\ F_G & 0 & 0 \\ F_{E_{T_{22}}} & 0 & 0 \end{bmatrix},$$

$$L_1 = \begin{bmatrix} P - X_0 & F_G & F_{E_{T_{11}}} & F_{E_{T_{22}}} \\ F_G & 0 & 0 & 0 \\ F_{E_{T_{11}}} & 0 & 0 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} P - X_0 & F_G & F_{E_{T_{11}}} & F_{E_{T_{22}}} \\ F_G & 0 & 0 & 0 \\ F_{E_{T_{22}}} & 0 & 0 & 0 \end{bmatrix}.$$

Applying Lemma 2.2 to (2.14) yields

$$\max_{X \in S} r(P - X) = \min\{n, r(L), r(G_1), r(G_2)\}, \tag{2.15}$$

$$\min_{X \in S} r(P - X) = 2r(L) - 2r(F_G) + \max\{t_1, t_2, t_3, t_4\}, \tag{2.16}$$

$$\max_{X \in S} i_{\pm}(P - X) = \min\{i_{\pm}(G_1), i_{\pm}(G_2)\}, \tag{2.17}$$

$$\min_{X \in S} i_{\pm}(P - X) = r(L) - r(F_G) + \max\left\{ \begin{matrix} i_{\pm}(G_1) - r(L_1), \\ i_{\pm}(G_2) - r(L_2) \end{matrix} \right\}, \tag{2.18}$$

Where

$$t_1 = r(G_1) - 2r(L_1), \tag{2.19}$$

$$t_2 = r(G_2) - 2r(L_2), \tag{2.20}$$

$$t_3 = i_{+}(G_1) + i_{-}(G_2) - r(L_1) - r(L_2), \tag{2.21}$$

$$t_4 = i_{-}(G_1) + i_{+}(G_2) - r(L_1) - r(L_2). \tag{2.22}$$

We will simplify $r(L)$, $r(L_1)$, $r(L_2)$, $i_{\pm}(G_1)$, $i_{\pm}(G_2)$ by applying three types of elementary block matrix operations, elementary block congruence matrix operations and Lemmas 1.2, 1.3, 1.5, 1.6 and 1.7.
It is easy to show that $R(F_G) \subset R\left(F_{E_{T_1}}\right)$ and $R(F_G) \subset R\left(F_{E_{T_2}}\right)$. Therefore, we obtain

$$r(L) = \begin{bmatrix} P - X_0 & F_G & F_{E_{T_{11}}} & F_{E_{T_{22}}} \\ F_G & 0 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} P - X_0 & F_{E_{T_{11}}} & F_{E_{T_{22}}} \\ F_G & 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} P - X_0 & I_n & I_n & 0 \\ I_n & 0 & 0 & G^* \\ 0 & E_{T_{11}} & 0 & 0 \\ 0 & 0 & E_{T_{22}} & 0 \end{bmatrix} - r(E_{T_{11}}) - r(E_{T_{22}}) - r(G)$$

$$= 2n + r\begin{bmatrix} E_{T_{11}} & 0 \\ E_{T_{22}} & E_{T_{22}}(X_0 - P)G^* \end{bmatrix} - r(E_{T_{11}}) - r(E_{T_{22}}) - r(G)$$

$$= 2n + r\begin{bmatrix} E_{T_{11}} & 0 & 0 \\ E_{T_{22}} & E_{T_{22}}(X_0 - P)E_{T_{11}} & E_{T_{22}}(X_0 - P)E_{T_{22}} \end{bmatrix} - r(E_{T_{11}}) - r(E_{T_{22}}) - r(G)$$

$$= 2n + r\begin{bmatrix} E_{T_{11}} & -E_{T_{11}}\left(X_0 - P\right)E_{T_{11}} & 0 \\ E_{T_{22}} & 0 & E_{T_{22}}\left(X_0 - P\right)E_{T_{22}} \end{bmatrix} - r\left(E_{T_{11}}\right) - r\left(E_{T_{22}}\right) - r\left(G\right)$$

$$= 2n + r\begin{bmatrix} E_{T_{11}} & -E_{T_{11}}X_0E_{T_{11}} + E_{T_{11}}PE_{T_{11}} & 0 \\ E_{T_{22}} & 0 & E_{T_{22}}X_0E_{T_{22}} - E_{T_{22}}PE_{T_{22}} \end{bmatrix}$$

$$- r\left(E_{T_{11}}\right) - r\left(E_{T_{22}}\right) - r\left(G\right) \tag{2.23}$$

$$r\begin{bmatrix} E_{T_{11}} & -E_{T_{11}}\left(T_1M_1^{\dagger}T_1^*\right)E_{T_{11}} + E_{T_{11}}PE_{T_{11}} & 0 \\ E_{T_{22}} & 0 & E_{T_{22}}\left(T_2M_2^{\dagger}T_2^*\right)E_{T_{22}} - E_{T_{22}}PE_{T_{22}} \end{bmatrix}$$

$$= r\left(\begin{array}{c} \begin{bmatrix} E_{T_{11}} & E_{T_{11}}PE_{T_{11}} & 0 \\ E_{T_{22}} & 0 & -E_{T_{22}}PE_{T_{22}} \end{bmatrix} - \begin{bmatrix} E_{T_{11}}T_1 \\ 0 \end{bmatrix} M_1^{\dagger} \begin{bmatrix} 0, & T_1^*E_{T_{11}}, & 0 \end{bmatrix} \\ - \begin{bmatrix} 0 \\ E_{T_{22}}T_2 \end{bmatrix} M_2^{\dagger} \begin{bmatrix} 0, & 0, & T_2^*E_{T_{22}} \end{bmatrix} \end{array} \right)$$

$$= r\begin{bmatrix} M_1^*M_1M_1^* & 0 & 0 & M_1^*T_1^*E_{T_{11}} & 0 \\ 0 & M_2^*M_2M_2^* & 0 & 0 & M_2^*T_2^*E_{T_{22}} \\ -E_{T_{11}}T_1M_1^* & 0 & E_{T_{11}} & E_{T_{11}}PE_{T_{11}} & 0 \\ 0 & E_{T_{22}}T_2M_2^* & E_{T_{22}} & 0 & -E_{T_{22}}PE_{T_{22}} \end{bmatrix} - r\left(M_1\right) - r\left(M_2\right)$$

$$= r\left(Q_1\right) - r\left(M_1\right) - r\left(M_2\right) \tag{2.24}$$

Substituting (2.24) into (2.23) yields

$$r\left(L\right) = 2n + r\left(Q_1\right) - r\left(E_{T_{11}}\right) - r\left(E_{T_{22}}\right) - r\left(G\right) - r\left(M_1\right) - r\left(M_2\right), \tag{2.25}$$

$$r\left(L_1\right) = r\begin{bmatrix} P - X_0 & F_G & F_{E_{T_{11}}} & F_{E_{T_{22}}} \\ F_G & 0 & 0 & 0 \\ F_{E_{T_{11}}} & 0 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} P - X_0 & F_{E_{T_{11}}} & F_{E_{T_{22}}} \\ F_{E_{T_{11}}} & 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} P - X_0 & I_n & I_n & 0 \\ I_n & 0 & 0 & E_{T_{11}} \\ 0 & E_{T_{11}} & 0 & 0 \\ 0 & 0 & E_{T_{22}} & 0 \end{bmatrix} - 2r\left(E_{T_{11}}\right) - r\left(E_{T_{22}}\right)$$

$$= 2n + r\begin{bmatrix} E_{T_{11}} & -E_{T_{11}}\left(X_0 - P\right)E_{T_{11}} \\ E_{T_{22}} & 0 \end{bmatrix} - 2r\left(E_{T_{11}}\right) - r\left(E_{T_{22}}\right)$$

$$= 2n + r\begin{bmatrix} E_{T_{11}} & -E_{T_{11}}X_0E_{T_{11}} + E_{T_{11}}PE_{T_{11}} \\ E_{T_{22}} & 0 \end{bmatrix} - 2r\left(E_{T_{11}}\right) - r\left(E_{T_{22}}\right)$$

(2.26)
$$= 2n+r\begin{bmatrix} E_{T_{11}} & -E_{T_{11}}\left(T_1M_1^{\dagger}T_1^*\right)E_{T_{11}} + E_{T_{11}}PE_{T_{11}} \\ E_{T_{22}} & 0 \end{bmatrix} - 2r\left(E_{T_{11}}\right) - r\left(E_{T_{22}}\right)$$

$$r\begin{bmatrix} E_{T_{11}} & -E_{T_{11}}\left(T_1M_1^{\dagger}T_1^*\right)E_{T_{11}} + E_{T_{11}}PE_{T_{11}} \\ E_{T_{22}} & 0 \end{bmatrix}$$

$$= r\left(\begin{bmatrix} E_{T_{11}} & E_{T_{11}}PE_{T_{11}} \\ E_{T_{22}} & 0 \end{bmatrix} - \begin{bmatrix} -E_{T_{11}}T_1 \\ 0 \end{bmatrix} M_1^{\dagger}\begin{bmatrix} 0, & T_1^*E_{T_{11}} \end{bmatrix}\right)$$

$$= r\begin{bmatrix} M_1^*M_1M_1^* & 0 & M_1^*T_1^*E_{T_{11}} \\ -E_{T_{11}}T_1M_1^* & E_{T_{11}} & E_{T_{11}}PE_{T_{11}} \\ 0 & E_{T_{22}} & 0 \end{bmatrix} - r\left(M_1\right)$$

(2.27)
$$= r\left(Q_2\right) - r\left(M_1\right)$$

Substituting (2.27) into (2.26) yields

(2.28)
$$r\left(L_1\right) = 2n + r\left(Q_2\right) - 2r\left(E_{T_{11}}\right) - r\left(E_{T_{22}}\right) - r\left(M_1\right),$$

$$r\left(L_2\right) = r\begin{bmatrix} P - X_0 & F_G & F_{E_{T_{11}}} & F_{E_{T_{22}}} \\ F_G & 0 & 0 & 0 \\ F_{E_{T_{22}}} & 0 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} P - X_0 & F_{E_{T_{11}}} & F_{E_{T_{22}}} \\ F_{E_{T_{22}}} & 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} P - X_0 & I_n & I_n & 0 \\ I_n & 0 & 0 & E_{T_{22}} \\ 0 & E_{T_{11}} & 0 & 0 \\ 0 & 0 & E_{T_{22}} & 0 \end{bmatrix} - r\left(E_{T_{11}}\right) - 2r\left(E_{T_{22}}\right)$$

$$= 2n+r\begin{bmatrix} E_{T_{11}} & 0 \\ E_{T_{22}} & E_{T_{22}}\left(X_0 - P\right)E_{T_{22}} \end{bmatrix} - r\left(E_{T_{11}}\right) - 2r\left(E_{T_{22}}\right)$$

$$= 2n+r\begin{bmatrix} E_{T_{11}} & 0 \\ E_{T_{22}} & E_{T_{22}}X_0E_{T_{22}} - E_{T_{22}}PE_{T_{22}} \end{bmatrix} - r\left(E_{T_{11}}\right) - 2r\left(E_{T_{22}}\right)$$

(2.29)
$$= 2n+r\begin{bmatrix} E_{T_{11}} & 0 \\ E_{T_{22}} & -E_{T_{22}}\left(T_2M_2^{\dagger}T_2^*\right)E_{T_{22}} - E_{T_{22}}PE_{T_{22}} \end{bmatrix} - r\left(E_{T_{11}}\right) - 2r\left(E_{T_{22}}\right)$$

$$r\begin{bmatrix} E_{T_{11}} & 0 \\ E_{T_{22}} & -E_{T_{22}}\left(T_2M_2^{\dagger}T_2^*\right)E_{T_{22}} - E_{T_{22}}PE_{T_{22}} \end{bmatrix}$$

$$= r\left(\begin{bmatrix} E_{T_{11}} & 0 \\ E_{T_{22}} & -E_{T_{22}}PE_{T_{22}} \end{bmatrix} - \begin{bmatrix} 0 \\ E_{T_{22}}T_2 \end{bmatrix} M_2^{\dagger}\begin{bmatrix} 0, & T_2^*E_{T_{22}} \end{bmatrix}\right)$$

$$= r\begin{bmatrix} M_2^*M_2M_2^* & 0 & M_2^*T_2^*E_{T_{22}} \\ E_{T_{22}}T_2M_2^* & E_{T_{11}} & 0 \\ 0 & E_{T_{22}} & -E_{T_{22}}PE_{T_{22}} \end{bmatrix} - r\left(M_2\right)$$

$$= r\left(Q_3\right) - r\left(M_2\right) \tag{2.30}$$

Substituting (2.30) into (2.29) yields

$$r\left(L_2\right) = 2n + r\left(Q_3\right) - r\left(E_{T_{11}}\right) - 2r\left(E_{T_{22}}\right) - r\left(M_2\right), \tag{2.31}$$

$$i_{\pm}\left(G_1\right) = i_{\pm}\begin{bmatrix} P - X_0 & F_G & F_{E_{T_{11}}} \\ F_G & 0 & 0 \\ F_{E_{T_{11}}} & 0 & 0 \end{bmatrix} = i_{\pm}\begin{bmatrix} P - X_0 & F_{E_{T_{11}}} \\ F_{E_{T_{11}}} & 0 \end{bmatrix}$$

$$= i_{\pm}\begin{bmatrix} P - X_0 & I_n & 0 \\ I_n & 0 & E_{T_{11}} \\ 0 & E_{T_{11}} & 0 \end{bmatrix} - r\left(E_{T_{11}}\right)$$

$$= i_{\pm}\begin{bmatrix} 0 & I_n & \frac{1}{2}\left(X_0 - P\right)E_{T_{11}} \\ I_n & 0 & E_{T_{11}} \\ \frac{1}{2}E_{T_{11}}\left(X_0 - P\right) & E_{T_{11}} & 0 \end{bmatrix} - r\left(E_{T_{11}}\right)$$

$$= n + i_{\pm}\begin{bmatrix} 0 & E_{T_{11}} \\ E_{T_{11}} & E_{T_{11}}\left(X_0 - P\right)E_{T_{11}} \end{bmatrix} - r\left(E_{T_{11}}\right)$$

$$= n + i_{\pm}\begin{bmatrix} 0 & E_{T_{11}} \\ E_{T_{11}} & -E_{T_{11}}\left(T_1M_1^{\dagger}T_1^*\right)E_{T_{11}} + E_{T_{11}}PE_{T_{11}} \end{bmatrix} - r\left(E_{T_{11}}\right)$$

$$= n + i_{\pm}\left(\begin{bmatrix} 0 & E_{T_{11}} \\ E_{T_{11}} & -E_{T_{11}}PE_{T_{11}} \end{bmatrix} - \begin{bmatrix} 0 \\ E_{T_{11}}T_1 \end{bmatrix} M_1^{\dagger}\begin{bmatrix} 0, & T_1^*E_{T_{11}} \end{bmatrix}\right)$$

$$= n + i_{\pm}\begin{bmatrix} M_1^3 & 0 & M_1T_1^*E_{T_{11}} \\ 0 & 0 & E_{T_{11}} \\ E_{T_{11}}T_1M_1^* & E_{T_{11}} & -E_{T_{11}}PE_{T_{11}} \end{bmatrix} - i_{\pm}\left(M_1\right) - r\left(E_{T_{11}}\right)$$

So

$$i_{\pm}\left(G_1\right) = n + i_{\pm}\left(Q_4\right) - i_{\pm}\left(M_1\right) - r\left(E_{T_{11}}\right), \tag{2.32}$$

$$i_{\pm}\left(G_2\right) = i_{\pm}\begin{bmatrix} P - X_0 & F_G & F_{E_{T_{22}}} \\ F_G & 0 & 0 \\ F_{E_{T_{22}}} & 0 & 0 \end{bmatrix} = i_{\pm}\begin{bmatrix} P - X_0 & F_{E_{T_{22}}} \\ F_{E_{T_{22}}} & 0 \end{bmatrix}$$

$$= i_{\pm}\begin{bmatrix} P - X_0 & I_n & 0 \\ I_n & 0 & E_{T_{22}} \\ 0 & E_{T_{22}} & 0 \end{bmatrix} - r\left(E_{T_{22}}\right)$$

$$= i_{\pm}\begin{bmatrix} 0 & I_n & \frac{1}{2}(X_0 - P)E_{T_{22}} \\ I_n & 0 & E_{T_{22}} \\ \frac{1}{2}E_{T_{22}}(X_0 - P) & E_{T_{22}} & 0 \end{bmatrix} - r(E_{T_{22}})$$

$$= n + i_{\pm}\begin{bmatrix} 0 & E_{T_{22}} \\ E_{T_{22}} & E_{T_{22}}(X_0 - P)E_{T_{22}} \end{bmatrix} - r(E_{T_{22}})$$

$$= n + i_{\pm}\begin{bmatrix} 0 & E_{T_{22}} \\ E_{T_{22}} & -E_{T_{22}}\left(T_2 M_2^{\dagger} T_2^{*}\right)E_{T_{22}} - E_{T_{22}}PE_{T_{22}} \end{bmatrix} - r(E_{T_{22}})$$

$$= n + i_{\pm}\left(\begin{bmatrix} 0 & E_{T_{22}} \\ E_{T_{22}} & -E_{T_{22}}PE_{T_{22}} \end{bmatrix} - \begin{bmatrix} 0 \\ E_{T_{22}}T_2 \end{bmatrix} M_2^{\dagger} \begin{bmatrix} 0, & T_2^{*}E_{T_{22}} \end{bmatrix}\right) - r(E_{T_{22}})$$

$$= n + i_{\pm}\begin{bmatrix} M_2^{3} & 0 & M_2 T_2^{*}E_{T_{22}} \\ 0 & 0 & E_{T_{22}} \\ E_{T_{22}}T_2 M_2^{*} & E_{T_{22}} & -E_{T_{22}}PE_{T_{22}} \end{bmatrix} - i_{\pm}(M_2) - r(E_{T_{22}})$$

So

$$i_{\pm}(G_2) = n + i_{\pm}(Q_5) - i_{\pm}(M_2) - r(E_{T_{22}}). \tag{2.33}$$

Therefore we get

$$r(G_1) = 2n + r(Q_4) - r(M_1) - 2r(E_{T_{11}}), \tag{2.34}$$

$$r(G_2) = 2n + r(Q_5) - r(M_2) - 2r(E_{T_{22}}). \tag{2.35}$$

Substituting the above results into (2.19)-(2.22) yields

$$t_1 = r(Q_1) - 2r(Q_2) + r(M_1) + 2r(E_{T_{11}}) + 2r(E_{T_{22}}) - 2n, \tag{2.36}$$

$$t_2 = r(Q_5) - 2r(Q_3) + r(M_2) + 2r(E_{T_{11}}) + 2r(E_{T_{22}}) - 2n, \tag{2.37}$$

$$t_3 = i_{+}(Q_4) - i_{-}(Q_5) - r(Q_3) - r(Q_2) + 2r(E_{T_{11}}) +$$

$$2r(E_{T_{22}}) + i_{-}(M_1) + i_{+}(M_2) - 2n, \tag{2.38}$$

$$t_4 = i_{-}(Q_4) + i_{+}(Q_5) - r(Q_3) - r(Q_2) + 2r(E_{T_{11}}) +$$

$$2r(E_{T_{22}}) + i_{+}(M_1) + i_{-}(M_2) - 2n. \tag{2.39}$$

Substituting (2.36)-(2.39) into (2.15)-(2.18) yields (2.10)-(2.13). ☐

From Theorem 2.1 and Lemma 1.1 we have the result

Theorem 2.2. *The assumption and the symbols are the same as in Theorem 2.1. Then,*
a) Eq (2.1) has a common Hermitian least-rank solution $X \geq P$ *if and only if*

$$\begin{aligned} r(Q_1) &= r(Q_2) + r(M_2) = r(Q_3) + r(M_1), \\ Q_4 &\geq 0, \quad Q_5 \geq 0, \quad M_1 \leq 0, \quad M_2 \leq 0. \end{aligned}$$

b) Eq (2.1) has a common Hermitian least-rank solution $X \leq P$ *if and only if*

$$\begin{aligned} r(Q_1) &= r(Q_2) + r(M_2) = r(Q_3) + r(M_1), \\ Q_4 &\geq 0, \quad Q_5 \geq 0, \quad M_1 \geq 0, \quad M_2 \geq 0. \end{aligned}$$

c) Eq (2.1) has a common Hermitian least-rank solution $X > P$ *if and only if*

$$i_-(Q_4) = i_-(M_1) + r(E_{T_{11}}), \quad i_-(Q_5) = i_-(M_2) + r(E_{T_{22}}).$$

d) Eq (2.1) has a common Hermitian least-rank solution $X < P$ *if and only if*

$$i_+(Q_4) = i_+(M_1) + r(E_{T_{11}}), \quad i_+(Q_5) = i_+(M_2) + r(E_{T_{22}}).$$

e) There exists a nonsingular matrix $P - X$ *such that* X *is a common Hermitian least-rank solution to (2.1) if and only if*

$$\begin{aligned} n + r(Q_1) &\geq r(E_{T_{11}}) + r(E_{T_{22}}) + r(G) + r(M_1) + r(M_2), \\ n + r(Q_4) &\geq r(M_1) + 2r(E_{T_{11}}) \quad \text{and} \quad n + r(Q_5) \geq r(M_2) + 2r(E_{T_{22}}). \end{aligned}$$

If P is the zero matrix in Theorem 2.2, we can achieve equivalent conditions for the existence of common Hermitian positive (negative, nonpositive, nonnegative)definite least-rank solution to(2.1)

Corollary 2.1. *The assumption and the symbols are the same as in Theorem 2.1. Define*

$$R_1 = \begin{bmatrix} M_1^* M_1 M_1^* & 0 & 0 & M_1^* T_1^* E_{T_{11}} & 0 \\ 0 & M_2^* M_2 M_2^* & 0 & 0 & M_2^* T_2^* E_{T_{22}} \\ -E_{T_{11}} T_1 M_1^* & 0 & E_{T_{11}} & 0 & 0 \\ 0 & E_{T_{22}} T_2 M_2^* & E_{T_{22}} & 0 & 0 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} M_1^* M_1 M_1^* & 0 & M_1^* T_1^* E_{T_{11}} \\ -E_{T_{11}} T_1 M_1^* & E_{T_{11}} & 0 \\ 0 & E_{T_{22}} & 0 \end{bmatrix},$$

$$R_3 = \begin{bmatrix} M_2^* M_2 M_2^* & 0 & M_2^* T_2^* E_{T_{22}} \\ E_{T_{22}} T_2 M_2^* & E_{T_{11}} & 0 \\ 0 & E_{T_{22}} & 0 \end{bmatrix},$$

$$R_4 = \begin{bmatrix} M_1^3 & 0 & M_1 T_1^* E_{T_{11}} \\ 0 & 0 & E_{T_{11}} \\ E_{T_{11}} T_1 M_1^* & E_{T_{11}} & 0 \end{bmatrix},$$

$$R_5=\begin{bmatrix} M_2^3 & 0 & M_2T_2^*E_{T_{22}} \\ 0 & 0 & E_{T_{22}} \\ E_{T_{22}}T_2M_2^* & E_{T_{22}} & 0 \end{bmatrix}.$$

Then,

a) Eq (2.1) has a common Hermitian positive definite least-rank solution if and only if

$$i_-(R_4)=i_-(M_1)+r(E_{T_{11}}),\quad i_-(R_5)=i_-(M_2)+r(E_{T_{22}}).$$

b) Eq (2.1) has a common Hermitian negative definite least-rank solution if and only if

$$i_+(R_4)=i_+(M_1)+r(E_{T_{11}}),\quad i_+(R_5)=i_+(M_2)+r(E_{T_{22}}).$$

c) Eq (2.1) has a common Hermitian nonpositive definite least-rank solution if and only if

$$\begin{aligned} r(R_1) &= r(R_2)+r(M_2)=r(R_3)+r(M_1), \\ R_4 &\geq 0,\quad R_5\geq 0,\quad M_1\geq 0,\quad M_2\geq 0. \end{aligned}$$

d) Eq (2.1) has a common Hermitian nonnegative definite least-rank solution if and only if

$$\begin{aligned} r(R_1) &= r(R_2)+r(M_2)=r(R_3)+r(M_1), \\ R_4 &\geq 0,\quad R_5\geq 0,\quad M_1\leq 0,\quad M_2\leq 0. \end{aligned}$$

e) There exists a nonsingular common Hermitian least-rank solution to (2.1) if and only if

$$\begin{aligned} n+r(R_1) &\geq r(E_{T_{11}})+r(E_{T_{22}})+r(G)+r(M_1)+r(M_2), \\ n+r(R_4) &\geq r(M_1)+2r(E_{T_{11}}) \quad \text{and} \quad n+r(R_5)\geq r(M_2)+2r(E_{T_{22}}). \end{aligned}$$

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