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A GENERAL FIXED POINT THEOREM FOR A PAIR OF SELF MAPPINGS WITH COMMON LIMIT RANGE PROPERTY IN G - METRIC SPACES

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Abstract. In this paper a general fixed point theorem for a pair of self mappings with the common limit range property in G - metric spaces satisfying an implicit relation is proved. In the last part of this paper, as applications, some fixed point results for mappings satisfying contractive conditions of integral type, for almost contractive mappings, for ϕ - contractive mappings and for (ϕ, ψ) - weak contractive mappings in G - metric spaces, are obtained.

Keywords: Fixed point theorem; self mapping; limit range property; contractive mapping; metric space.

1. Introduction

Let (X, d) be a metric space and S, T be two self mappings of X. In [25], Jungck defined S and T to be compatible if

$$\lim_{n\to\infty} d(TSx_n, STx_n) = 0$$

whenever (x_n) is a sequence in X, such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t,$$

for some $t \in X$.

This concept has frequently been used to prove the existence theorems in fixed point theory.

Let f, g be self mappings of a nonempty set X. A point $x \in X$ is a coincidence point of f and g if $w = fx = gx$ and w is said to be a point of coincidence of f and g. The set of all coincidence points of f and g is denoted by $C(f, g)$.

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In 1994, Pant [41] introduced the notion of pointwise R - weakly commuting mappings. It is proved in [42] that the pointwise R – weakly commutativity is equivalent to commutativity at coincidence points.

In [26] Jungck introduced the notion of weakly compatible mappings.

Definition 1.1. ([26]) Let X be a nonempty set and f, g to be self mappings of X. f and g are weakly compatible if $fgu = gfu$ for all $u \in C(f,g)$.

Hence, f and g are weakly compatible if and only if f and g are pointwise R - weakly commuting.

The study of common fixed points for noncompatible mappings is also interesting, the work in this regard has been initiated by Pant in [38], [39], [40]. Aamri and El-Moutawakil [1] introduced a generalization of noncompatible mappings.

Definition 1.2. ([1]) Let S and T be two self mappings of a metric space (X,d). We say that S and T satisfy property (EA) if there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t,$$

for some $t \in X$.

Remark 1.1. It is clear that two self mappings S and T of a metric space (X,d) will be noncompatible if there exists a sequence (x_n) in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$, but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is nonzero or nonexistent. Therefore, two noncompatible self mappings of a metric space (X,d) satisfy property (EA).

It is known [43], [44] that the notions of weakly compatible mappings and mappings satisfying property (EA) are independent.

There exists a vast literature concerning the study of fixed points for pairs of mappings satisfying the property (EA).

In 2007, Sintunavarat and Kumam [63] introduced the idea of common limit range property.

Definition 1.3. ([63]) A pair (A,S) of self mappings of a metric space (X,d) is said to satisfy the limit range property with respect to S, denoted $CLR_{(S)}$, if there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t,$$

for some $t \in S(X)$.

Thus, we can infer that a pair (A,S) satisfying the property (EA) along with the closedness of the subspace $S(X)$ always have the $CLR_{(S)}$ - property, with respect to S (see Examples 2.16, 2.17 [22]).

Some fixed point results for pairs of mappings with $CLR_{(S)}$ - property, also, are obtained in [23], [24], [27], [64] and in other papers.

In [18], [19], Dhage introduced a new class of generalized metric space, named D – metric space. Mustafa and Sims [32], [33] proved that most of the claims concerning the fundamental topological structures on D – metric spaces are incorrect and introduced an appropriate notion of generalized metric space, named G – metric space.

In fact, Mustafa, Sims and other authors studied many fixed point results for self mappings in G – metric spaces under certain conditions [34], [35], [36], [37], [62].

In *Facta Universitatis* the following papers are published: [6], [29], [57]. Other papers concerning the study of fixed points in G - metric spaces are published in [15], [21], [59], [65].

Several classical fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit relation in [45], [46] and in other papers. Recently, the method is used in the study of fixed points in metric spaces, symmetric spaces, quasi – metric spaces, ultra - metric spaces, convex metric spaces, reflexive spaces, compact metric spaces, paracompact metric spaces, in two or three metric spaces, for single valued functions, hybrid pairs of mappings and set valued mappings. The method is used in the study of fixed points for mappings satisfying a contractive/extensive condition of integral type, in fuzzy metric spaces, probabilistic metric spaces, intuitionistic metric spaces, G - metric spaces. With this method, the proofs of some fixed points theorems are more simple. Also, the method allow the study of local and global properties of fixed point structures.

The study of fixed points for mappings satisfying implicit relations in G - metric spaces is initiated in [47], [51], [52], [53].

The study of fixed points for pairs of self mappings with common limit range property in metric spaces satisfying implicit relations is initiated in [24].

The study of fixed points for a pair of self mappings with common limit range property in G - metric spaces is initiated in [6].

Definition 1.4. ([28]) An altering distance is a function $\phi : [0,\infty) \to [0,\infty)$ satisfying:

(ϕ_1) : ϕ is increasing and continuous;

(ϕ_2) : $\phi(t) = 0$ if and only if $t = 0$.

Fixed point theorems involving altering distances have been studied in [50], [60], [61] and in other papers.

Definition 1.5. An almost altering distance is a function $\psi : [0,\infty) \to [0,\infty)$ satisfying:

(ψ_1) : ψ is continuous;

(ψ_2) : $\psi(t) = 0$ if and only if $t = 0$.

Remark 1.2. Every altering distance function is an almost altering distance, the converse is not true.

Example 1.1. $\psi(t) = \begin{cases} t, & t \in [0,1] \\ \frac{1}{t}, & t \in (1,\infty) \end{cases}$

In this paper, a general fixed point theorem for a pair of self mappings with the common limit range property in G - metric spaces satisfying an implicit relation is proved.

In the last part of this paper, as application, some fixed point results for mappings satisfying contractive conditions of integral type, for almost contractive mappings, for φ - contractive mappings and for (φ, ψ) - weak contractive mappings in G - metric spaces are obtained.

2. Preliminaries

Definition 2.1. ([33]) Let X be a nonempty set and $G : X^3 \to \mathbb{R}_+$ be a function satisfying the following properties:

(G_1) : $G(x,y,z) = 0$ if $x = y = z$,

(G_2) : $0 < G(x,x,y)$, for all $x, y \in X$ with $x \neq y$,

(G_3) : $G(x,y,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$,

(G_4) : $G(x,y,z) = G(y,z,x) = G(z,x,y) = ...$ (symmetry in all three variables),

(G_5) : $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).

The function G is called a G - metric on X and the pair (X,G) is called a G - metric space.

Note that if $G(x,y,z) = 0$, then $x = y = z$.

Remark 2.1. Let (X,G) be a G - metric space. If $y = z$, then by Lemma 5.1 [47], $G(x,y,y)$ is a quasi-metric on X. Hence (X,Q), where $Q(x,y) = G(x,y,y)$ is a quasi-metric space and since every metric space is a particular case of quasi-metric space it follows that the notion of G-metric space is a generalization of a metric space.

Definition 2.2. ([33]) Let (X,G) be a G – metric space. A sequence (x_n) in X is said to be

a) G - convergent if for $\varepsilon > 0$, there is an $x \in X$ and $k \in \mathbb{N}$ such that for all $m,n \in \mathbb{N}$, $m,n \geq k$, $G(x,x_n,x_m) < \varepsilon$;

b) G - Cauchy if for $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for $m,n,p \in \mathbb{N}$, $m,n,p \geq k$, $G(x_n,x_m,x_p) < \varepsilon$, that is $G(x_n,x_m,x_p) \to 0$ as $n,m,p \to \infty$;

c) A G - metric space (X,G) is said to be G - complete if every G - Cauchy sequence in X is G - convergent.

Lemma 2.1. *([33]) Let (X, G) be a G - metric space. Then, the following conditions are equivalent:*

1) (x_n) is G - convergent to x;

2) $G(x_n, x_n, x) \to 0$ as $n \to \infty$;

3) $G(x_n, x, x) \to 0$ as $n \to \infty$;

4) $G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.

Lemma 2.2. *([33]) If (X, G) is a G - metric space, then the following conditions are equivalent:*

1) (x_n) is G - Cauchy;

2) for $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$ for all $m, n \in \mathbb{N}$, $m, n \geq k$.

Lemma 2.3. *([33]) Let (X, G) be a G - metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Definition 2.3. Let $\mathfrak{F}_{CL}$ be the set of all real continuous functions $F(t_1, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ such that:

$(F_1) : F(t, 0, t, 0, 0, t) > 0, \forall t > 0.$

$(F_2) : F(t, t, 0, 0, t, t) > 0, \forall t > 0.$

Example 2.1. $F(t_1, ..., t_6) = t_1 - k \max\{t_2, t_3, ..., t_6\}$, where $k \in [0, 1)$.

Example 2.2. $F(t_1, ..., t_6) = t_1 - at_2 - b \max\{t_3, t_4\} - c \max\{t_2, t_5, t_6\}$, where $a, b, c \geq 0$ and $a + b + c < 1$.

Example 2.3. $F(t_1, ..., t_6) = t_1 - k \max\{t_2, t_3, t_4, \frac{t_5+t_6}{2}\}$, where $k \in [0, 1)$.

Example 2.4. $F(t_1, ..., t_6) = t_1 - k \max\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\}$, where $k \in [0, 1)$.

Example 2.5. $F(t_1, ..., t_6) = t_1 - \alpha \max\{t_2, t_3, t_4\} - (1 - \alpha)(at_5 + bt_6)$, where $\alpha \in (0, 1)$ and $a, b \geq 0$, $a + b < 1$.

Example 2.6. $F(t_1, ..., t_6) = t_1 - at_2 - b(t_3 + t_4) - c \min\{t_5, t_6\}$, where $a, b, c \geq 0$ and $a + b + c < 1$.

Example 2.7. $F(t_1, ..., t_6) = t_1 - at_2 - b\frac{t_5+t_6}{1+t_3+t_4}$, where $a, b \geq 0$ and $a + 2b < 1$.

Example 2.8. $F(t_1, ..., t_6) = t_1 - \max\{ct_2, ct_3, ct_4, at_5 + bt_6\}$, where $c \in (0, 1)$, $a, b \geq 0$ and $a + b < 1$.

Other examples satisfying the conditions (F_1), (F_2) are presented in [5], [24] and in other papers.

3. Main results

Lemma 3.1. *([2]) Let f and g be two weakly compatible self mappings on a nonempty set X. If f and g have an unique point of coincidence $w = fx = gx$, for some $x \in X$, then w is the unique common fixed point of f and g.*

Lemma 3.2. *Let T, S be self mappings of a G - metric space (X, G) such that*

$$\begin{aligned} F(&\psi(G(Tx,Tx,Ty)),\psi(G(Sx,Sx,Sy)),\psi(G(Tx,Tx,Sx)),\\ &\psi(G(Ty,Ty,Sy)),\psi(G(Sx,Sx,Ty)),\psi(G(Tx,Tx,Sy)))\leq 0 \end{aligned} \tag{3.1}$$

for all $x, y \in X$, where F satisfy property (F_2) and ψ is an almost altering distance. If there exists $u, v \in X$ such that $w = Su = Tu$ and $z = Sv = Tv$, then S and T have an unique point of coincidence.

Proof. First we prove that $Tu = Sv$. By (3.1) we obtain

$$\begin{aligned} F(&\psi(G(Tu,Tu,Tv)),\psi(G(Su,Su,Sv)),\psi(G(Tu,Tu,Su)),\\ &\psi(G(Tv,Tv,Sv)),\psi(G(Su,Su,Tv)),\psi(G(Tu,Tu,Sv)))\leq 0 \end{aligned}$$

which implies

$$F(\psi(G(w,w,z)),\psi(G(w,w,z)),0,0,\psi(G(w,w,z)),\psi(G(w,w,z)))\leq 0$$

a contradiction of (F_2) if $\psi(G(w,w,z)) \neq 0$. Hence $\psi(G(w,w,z)) = 0$ which implies $w = z$. Hence, $Tu = Sv = Su = Tv = w = z$. Therefore, z is an common fixed point of coincidence of T and S.

Suppose that there exists two points of coincidence of T and S: $z_1 = Tu = Su$ and $z_2 = Tv = Sv$. By (3.1) we obtain

$$F(\psi(G(z_1,z_1,z_2)),\psi(G(z_1,z_1,z_2)),0,0,\psi(G(z_1,z_1,z_2)),\psi(G(z_1,z_1,z_2)))\leq 0,$$

a contradiction to (F_2) if $\psi(G(z_1,z_1,z_2)) \neq 0$. Hence $\psi(G(z_1,z_1,z_2)) = 0$ which implies $z_1 = z_2$. □

Theorem 3.1. *Let T, S be self mappings of a G - metric space (X, G) such that the inequality (3.1) holds for all $x, y \in X$, where $F \in \mathfrak{F}_{CL}$ and ψ is an almost altering distance. If T and S satisfy $CLR_{(S)}$ - property, then $C(T, S) \neq \emptyset$. Moreover, if T and S are weakly compatible, then T and S have an unique common fixed point.*

Proof. Since T and S satisfy $CLR_{(S)}$ - property, there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = Su,$$

for some $u \in X$.

By (3.1) we have

$$F(\psi(G(Tu,Tu,Tx_n)),\psi(G(Su,Su,Sx_n)),\psi(G(Tu,Tu,Su)),\\ \psi(G(Tx_n,Tx_n,Sx_n)),\psi(G(Su,Su,Tx_n)),\psi(G(Tu,Tu,Sx_n))) \leq 0.$$

Letting n tend to infinity we obtain

$$F(\psi(G(Tu,Tu,Su)),0,\psi(G(Tu,Tu,Su)),0,0,\psi(G(Tu,Tu,Su))) \leq 0,$$

a contradiction of (F_1) if $\psi(G(Tu,Tu,Su)) \neq 0$. Hence, $\psi(G(Tu,Tu,Su)) = 0$, which implies $Tu = Su = z$. Hence, $C(T,S) \neq \emptyset$ and z is a point of coincidence of T and S. By Lemma 3.2, z is the unique point of coincidence of T and S. Moreover, if T and S are weakly compatible, then by Lemma 3.1, z is the unique common fixed point of T and S. □

Example 3.1. Let $X = [0,\infty)$ and let $G : X^3 \to \mathbb{R}_+$ be the G - metric defined as follows

$$G(x,y,z) = \max\{|x-y|,|y-z|,|x-z|\}$$

for all $x,y,z \in X$. Then (X,G) is a G - metric space.

Define the self mappings T and S by $Tx = x$ and $Sx = 2x$. Let $x_n = \{\frac{1}{n}\}$. We have $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = S0 = 0 \in X$.

Hence, the pair (T,S) satisfy (CLR_S) - property.

Let

$$F(\psi(G(Tx,Tx,Ty)),\psi(G(Sx,Sx,Sy)),\psi(G(Tx,Tx,Sx)),\\ \psi(G(Sx,Sx,Sy)),\psi(G(Sx,Sx,Ty)),\psi(G(Tx,Tx,Sy))) =\\ \psi(G(Tx,Tx,Ty)) - k\max\{\psi(G(Sx,Sy,Sy)),\psi(G(Tx,Tx,Sx)),\\ \psi(G(Sx,Sx,Sy)),\psi(G(Sx,Sx,Ty)),\psi(G(Tx,Tx,Sy))\}$$

where $\psi(t) = 2t$ and $k \in \left[\frac{1}{2},1\right)$.

Since

$$G(Tx,Tx,Ty) = |Tx-Ty| = |x-y|$$

and

$$G(Sx,Sx,Sy) = 2|x-y|$$

and

$$\psi(G(Tx,Tx,Ty)) = 2|x-y|$$

and

$$\psi(G(Sx,Sy,Sy)) = 4|\ x-y\ |,$$

then

$$\psi(G(Tx,Tx,Ty)) \leq k\psi(G(Sx,Sy,Sy))$$

which implies

$$\psi(G(Tx,Tx,Ty)) \quad \leq \quad k\max\{\psi(G(Sx,Sy,Sy)),\psi(G(Tx,Tx,Sx)),\psi(G(Ty,Ty,Sy)),\\ \psi(G(Sx,Sx,Ty)),\psi(G(Tx,Tx,Sy))\}.$$

On the other hand, if $Tx = Sx$, then $x = 0$ which implies $TS0 = ST0 = \{0\}$. Hence T and S are weakly compatible. By Theorem 3.1 and Example 2.1, T and S have an unique common fixed point which is $x = 0$.

For $\psi(t) = t$, we obtain

Theorem 3.2. *Let T, S be self mappings of a G - metric space (X, G) such that:*

$$F(G(Tx, Tx, Ty), G(Sx, Sx, Sy), G(Tx, Tx, Sx), G(Ty, Ty, Sy), G(Sx, Sx, Ty), G(Tx, Tx, Sy)) \leq 0 \tag{3.2}$$

for all $x, y \in X$ and $F \in \mathfrak{F}_{CL}$. If T and S satisfy $CLR_{(S)}$ - property, then $C(T, S) \neq \emptyset$. Moreover, if T and S are weakly compatible, then T and S have an unique common fixed point.

Theorem 3.3. *Let T, S be self mappings of a G - metric space (X, G) such that:*

$$F(\psi(G(Tx, Ty, Ty)), \psi(G(Sx, Sy, Sy)), \psi(G(Tx, Sx, Sx)), \psi(G(Ty, Sy, Sy)), \psi(G(Sx, Ty, Ty)), \psi(G(Tx, Sy, Sy))) \leq 0 \tag{3.3}$$

for all $x, y \in X$, $F \in \mathfrak{F}_{CL}$ and ψ is an almost altering distance. If T and S satisfy $CLR_{(S)}$ - property, then $C(T, S) \neq \emptyset$. Moreover, if T and S are weakly compatible, then T and S have an unique common fixed point.

Proof. The proof is similar to the proof of Theorem 3.2. □

Example 3.2. Let $X = [1, \infty)$ and let $G : X^3 \to \mathbb{R}_+$ be the G - metric defined as follows

$$G(x, y, z) = \max\{|x - y|, |x - z|, |y - z|\}$$

for all $x, y, z \in X$. Then (X, G) is a G - metric space.

Define the self mappings T and S by $Tx = x$ and $Sx = x^2$. Let $x_n = \{1 + \frac{1}{n}\}$. Then we have $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = 1 = S1 \in X$.

Hence, the pair (T, S) satisfy (CLR_S) - property.

Let

$$\begin{aligned} &F(\psi(G(Tx, Tx, Ty)), \psi(G(Sx, Sx, Sy)), \psi(G(Tx, Sx, Sx)), \\ &\psi(G(Ty, Sy, Sy)), \psi(G(Sx, Tx, Ty)), \psi(G(Tx, Sy, Sy))) = \\ &\psi(G(Tx, Ty, Ty)) - k \max\{\psi(G(Sx, Sy, Sy)), \psi(G(Tx, Sx, Sx)), \\ &\psi(G(Ty, Sy, Sy)), \psi(G(Sx, Ty, Ty)), \psi(G(Tx, Sy, Sy))\} \end{aligned}$$

where $\psi(t) = 2t$ and $k \in (0, 1)$.

Since

$$G(Tx, Ty, Ty) = |x - y|$$

and

$$G(Sx, Sy, Sy) = |x^2 - y^2| = |x - y| \cdot |x + y|$$

and

$$\psi(G(Tx, Ty, Ty)) = 2|x - y|$$

and

$$\psi(G(Sx, Sy, Sy)) = |x^2 - y^2| = |x - y| \cdot |x + y|,$$

then

$$\psi(G(Tx, Ty, Ty)) \leq k\psi(G(Sx, Sy, Sy))$$

which implies

$$\psi(G(Tx,Ty,Ty)) \leq k\max\{\psi(G(Sx,Sy,Sy)),\psi(G(Tx,Sx,Sx)),\psi(G(Ty,Sy,Sy)),\psi(G(Sx,Ty,Ty)),\psi(G(Tx,Sy,Sy))\}.$$

On the other hand, if $Tx = Sx$, then $x = 1 \in X$ which implies $ST1 = TS1$. Hence T and S are weakly compatible. By Theorem 3.3 and Example 2.1, T and S have an unique common fixed point which is $x = 1$.

If $\psi(t) = t$, by Theorem 3.3 we obtain

Theorem 3.4. *Let T, S be self mappings of a G - metric space (X, G) such that:*

$$F(G(Tx,Ty,Ty),G(Sx,Sy,Sy),G(Tx,Sx,Sx),G(Ty,Sy,Sy),G(Sx,Ty,Ty),G(Tx,Sy,Sy)) \leq 0 \tag{3.4}$$

for all $x, y \in X$, where $F \in \mathfrak{F}_{CL}$. If T and S satisfy $CLR_{(S)}$ - property, then $C(T,S) \neq \emptyset$. Moreover, if T and S are weakly compatible, then T and S have an unique common fixed point.

Corollary 3.1. *Let T, S be self mappings of a G - metric space (X, G) such that*

$$G(Tx,Ty,Ty) \leq k\max\{G(Sx,Sy,Sy),G(Tx,Sx,Sx),G(Ty,Sy,Sy),G(Sx,Ty,Ty),G(Tx,Sy,Sy)\}, \tag{3.5}$$

where $k \in [0,1)$, for all $x, y \in X$. If T and S satisfy $CLR_{(S)}$ - property, then $C(T,S) \neq \emptyset$. If T and S are weakly compatible, then T and S have an unique common fixed point.

Proof. The proof it follows by Theorem 3.4 and Example 2.1. □

Theorem 3.5. *Let f, g be self maps of a G - metric space (X, G) satisfying the inequality*

$$G(fx,fy,fz) \leq k\max\{G(gx,gy,gz),G(gx,fx,fx),G(gx,fy,fy),G(gz,fz,fz),G(gy,fy,fy),G(gy,fx,fx),G(gy,fz,fz),G(gz,fz,fz),G(gz,fx,fx),G(gz,fy,fy)\}, \tag{3.6}$$

for all $x, y, z \in X$, where $k \in [0,1)$.

If f and g satisfy $CLR_{(g)}$ - property, then f and g have an unique common fixed point.

Proof. If $z = y$ be, then by (3.6) we obtain

$$G(fx,fy,fy) \leq k\max\{G(gx,gy,gy),G(fx,gx,gx),G(fy,gy,gy),G(gx,fy,fy),G(fx,gy,gy)\} \leq 0, \tag{3.7}$$

where $k \in [0,1)$, for all $x, y \in X$, which is inequality (3.5). Hence, Theorem 3.5 it follows from Corollary 3.1. □

Remark 3.1. This result is similar to the results of Theorem 5.1 [55], where $k \in \left[0, \frac{1}{4}\right)$.

4. Applications

4.1. Fixed points for mappings satisfying contractive conditions of integral type

In [16], Branciari established the following theorem which opened the way to the study of fixed points for mappings satisfying contractive conditions of integral type.

Theorem 4.1. *([16]) Let (X, G) be a complete metric space, $c \in (0, 1)$ and $f : X \to X$ such that for all $x, y \in X$*

$$\int_0^{d(fx,fy)} h(t)dt \leq c \int_0^{d(x,y)} h(t)dt,$$

whenever $h : [0, \infty) \to [0, \infty)$ is a Lebesgue measurable mapping which is summable (i.e. with finite integral) on each compact subset of $[0, \infty)$, such that, $\int_0^{\varepsilon} h(t)dt > 0$, for each $\varepsilon > 0$, . Then, f has an unique fixed point $z \in X$ such that for all $x \in X$, $z = \lim_{n\to\infty} f^n x$.

Theorem 4.1 has been extended to a pair of compatible mappings in [30].

Theorem 4.2. *([30]) Let f, g be compatible mappings of a complete G - metric space (X, G), with g - continuous satisfying the following conditions:*

(1) $f(X) \subset g(X)$,

(2) $\int_0^{d(fx,gy)} h(t)dt \leq c \int_0^{d(x,y)} h(t)dt$*, for some $c \in (0, 1)$, whenever $x, y \in X$ and $h(t)$ is as in Theorem 4.1. Then f and g have an unique common fixed point.*

Some fixed point results for mappings satisfying contractive conditions of integral type are obtained in [49], [50], [58] and in other papers.

Lemma 4.1. *Let $h : [0, \infty) \to [0, \infty)$ be as in Theorem 4.1. Then $\psi(t) = \int_0^t h(x)dx$ is an almost altering distance.*

Proof. The proof it follows from Lemma 2.5 [50]. □

Theorem 4.3. *Let T, S be self compatible mappings of a G - metric space (X, G) such that*

$$F\left(\int_0^{G(Tx,Tx,Ty)} h(t)dt, \int_0^{G(Sx,Sx,Sy)} h(t)dt, \int_0^{G(Tx,Tx,Sx)} h(t)dt, \int_0^{G(Ty,Ty,Sy)} h(t)dt, \int_0^{G(Sx,Sx,Ty)} h(t)dt, \int_0^{G(Tx,Tx,Sy)} h(t)dt\right) \leq 0 \tag{4.1}$$

for all $x, y \in X$, *where* $F \in \mathfrak{F}_{LC}$ *and* $h(t)$ *is as in Theorem 4.1.*

If T *and* S *satisfy* $CLR_{(S)}$ *- property, then* $C(T,S) \neq \emptyset$. *Moreover, if* T *and* S *are weakly compatible, then* T *and* S *have an unique common fixed point.*

Proof. By Lemma 4.1, $\psi(t) = \int_0^t h(x)dx$ is an almost altering distance. By (4.1) we obtain

$$F(\psi(G(Tx,Tx,Ty)), \psi(G(Sx,Sx,Sy)), \psi(G(Tx,Tx,Sx)),$$
$$\psi(G(Ty,Ty,Sy)), \psi(G(Sx,Sx,Ty)), \psi(G(Tx,Tx,Sy))) \leq 0,$$

which is the inequality (3.1). Hence, the conditions of Theorem 3.1 are satisfied. Theorem 4.3 it follows from Theorem 3.1. □

Similarly, from Theorem 3.3 we obtain

Theorem 4.4. *Let* T *and* S *be self mappings of a* G *- metric space* (X,G) *such that*

$$\begin{aligned}(4.2)\qquad & F\left(\int_0^{G(Tx,Ty,Ty)} h(t)dt, \int_0^{G(Sx,Sy,Sy)} h(t)dt, \int_0^{G(Tx,Sx,Sx)} h(t)dt,\right.\\ & \left.\int_0^{G(Ty,Sy,Sy)} h(t)dt, \int_0^{G(Sx,Ty,Ty)} h(t)dt, \int_0^{G(Tx,Sy,Sy)} h(t)dt\right) \leq 0\end{aligned}$$

for all $x, y \in X$, *where* $F \in \mathfrak{F}_{LC}$ *and* $h(t)$ *is as in Theorem 4.1.*

If T *and* S *satisfy* $CLR_{(S)}$ *- property, then* $C(T,S) \neq \emptyset$. *Moreover, if* T *and* S *are weakly compatible, then* T *and* S *have an unique common fixed point.*

Corollary 4.1. *([6]) Let* f, g *be weakly compatible self mappings of a* G *- metric space* (X,G) *such that*

$$(4.3)\qquad \int_0^{G(fx,fy,fz)} h(t)dt \leq \alpha \int_0^{\mathfrak{L}(x,y,z)} h(t)dt$$

for all $x, y, z \in X$, $\alpha \in [0,1)$, $h(t)$ *as in Theorem 4.1 and*

$$\mathfrak{L}(x,y,z) = \max\{G(gx,gy,gz), G(gx,fx,fx), G(gy,fy,fy), G(gz,fz,fz)\}.$$

If f *and* g *satisfy* $CLR_{(g)}$ *- property, then* f *and* g *have an unique common fixed point.*

Proof. Let $y = z$ be. Then by (4.3) we obtain

$$\begin{aligned}&\int_0^{G(fx,fy,fy)} h(t)dt \leq \alpha \int_0^{\max\{G(gx,gy,gy),G(gx,fx,fx),G(gy,fy,fy)\}} h(t)dt\\ &\leq \alpha \max\{\int_0^{G(gx,gy,gy)} h(t)dt, \int_0^{G(fx,fx,gx)} h(t)dt, \int_0^{G(fy,gy,gy)} h(t)dt,\\ &\int_0^{G(fy,fy,gx)} h(t)dt, \int_0^{G(fx,gy,gy)} h(t)dt\} \leq 0.\end{aligned}$$

Then by Theorem 4.4 and Example 2.1, f and g have an unique common fixed point. □

4.2. Fixed points for almost contractive mappings in G - metric spaces

Definition 4.1. Let (X, d) be a metric space. A mapping $T : (X, d) \to (X, d)$ is called weak contractive [10], [12] or almost contractive [11] if there exists $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx).$$

The following theorem is proved in [14].

Theorem 4.5. *Let (X, d) be a metric space and $T, S : (X, d) \to (X, d)$ be mappings for which there exists $a \in (0, 1)$ and $L \geq 0$ such that*

$$d(Tx, Ty) \leq ad(Sx, Sy) + Ld(Sy, Tx),$$

for all $x, y \in X$.

If $T(X) \subset S(X)$ and $S(X)$ is a complete subspace of X, then T and S have an unique point of coincidence. Moreover, if T and S are weakly compatible, then T and S have an unique common fixed point.

A similar result is obtained if

$$d(Tx, Ty) \leq ad(Sx, Sy) + L\min\{d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Tx, Sy)\}$$

where $a \in (0, 1)$ and $L \geq 0$.

In [7], a similar result is obtained if

$$d(Tx, Ty) \leq \delta m(x, y) + L\min\{d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Tx, Sy)\},$$

where $\delta \in (0, 1)$, $L \geq 0$ and

$$m(x, y) = \max\left\{d(Sx, Sy), \frac{d(Tx, Sx) + d(Ty, Sy)}{2}, \frac{d(Sx, Ty) + d(Tx, Sy)}{2}\right\}.$$

A general fixed point theorem for almost contractive mappings is published in [48].

The following functions $F(t_1, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfy the conditions $(F_1), (F_2)$.

Example 4.1. $F(t_1, ..., t_6) = t_1 - \delta\max\left\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\right\} - L\min\{t_3, t_4, t_5, t_6\}$, where $\delta \in (0, 1)$ and $L \geq 0$.

Example 4.2. $F(t_1, ..., t_6) = t_1 - at_2 - L\min\{t_3, t_4, t_5, t_6\}$, where $a \in (0, 1)$ and $L \geq 0$.

Example 4.3. $F(t_1, ..., t_6) = t_1 - k\max\left\{t_2, t_3, t_4, \frac{t_5+t_6}{2}\right\} - L\min\{t_3, t_4, t_5, t_6\}$, where $k \in (0, 1)$ and $L \geq 0$.

Example 4.4. $F(t_1, ..., t_6) = t_1 - k \max\{t_2, t_3, t_4, t_5, t_6\} - L \min\{t_3, t_4, t_5, t_6\}$, where $k \in (0, 1)$ and $L \geq 0$.

Example 4.5. $F(t_1, ..., t_6) = t_1 - k \max\left\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\right\} - L \min\{t_3, t_4, \sqrt{t_4 t_5}, \sqrt{t_5 t_6}\}$, where $k \in (0, 1)$ and $L \geq 0$.

Example 4.6. $F(t_1, ..., t_6) = t_1 - k \max\left\{t_2, t_3, \sqrt{t_4 t_5}, \sqrt{t_5 t_6}\right\} - L \min\{t_3, t_4, t_5, t_6\}$, where $k \in (0, 1)$ and $L \geq 0$.

Example 4.7. $F(t_1, ..., t_6) = t_1 - \max\{t_2, k(t_3 + t_4), k(t_5 + t_6)\} - L \min\{t_3, t_4, t_5, t_6\}$, where $k \in (0, 1)$ and $L \geq 0$.

Example 4.8. $F(t_1, ..., t_6) = t_1 - \max\left\{t_2, \alpha t_3, \alpha t_4, \frac{\alpha(t_5+t_6)}{2}\right\} - L \min\{t_3, t_4, t_5, t_6\}$, where $k \in (0, 1)$ and $L \geq 0$.

By Theorem 3.1 and Example 4.1 we obtain

Theorem 4.6. *Let T, S be self mappings of a G - metric space (X, G) such that*

$$\begin{aligned}
\psi(G(Tx, Tx, Ty)) \quad \leq \quad & \delta \max\{\psi(G(Sx, Sx, Sy)), \frac{\psi(G(Tx, Tx, Sx)) + \psi(G(Ty, Ty, Sy))}{2}, \\
& \frac{\psi(G(Sx, Sx, Ty)) + \psi(G(Tx, Tx, Sy))}{2}\} + \\
& + L \min\{\psi(G(Tx, Tx, Sx)), \psi(G(Ty, Ty, Sy)), \\
& \psi(G(Sx, Sx, Ty)), \psi(G(Tx, Tx, Sy))\},
\end{aligned}$$

where $\delta \in (0, 1)$ and $L \geq 0$, for all $x, y \in X$. If T and S satisfy $CLR_{(S)}$ - property, then $C(T, S) \neq \emptyset$. Moreover, if T and S are weakly compatible, then T and S have an unique common fixed point.

By Example 4.1 and Theorem 4.6 we obtain

Theorem 4.7. *Let T, S be self mappings of a G - metric space (X, G) such that*

$$\begin{aligned}
\int_0^{G(Tx,Tx,Ty)} h(t)dt \quad \leq \quad & \delta \max\{\int_0^{G(Sx,Sx,Sy)} h(t)dt, \\
& \frac{\int_0^{G(Tx,Tx,Sx)} h(t)dt + \int_0^{G(Ty,Ty,Sy)} h(t)dt}{2}, \\
& \frac{\int_0^{G(Sx,Sx,Ty)} h(t)dt + \int_0^{G(Tx,Tx,Sy)} h(t)dt}{2}\} + \\
& + L \min\{\int_0^{G(Tx,Tx,Sx)} h(t)dt, \int_0^{G(Ty,Ty,Sy)} h(t)dt, \\
& \int_0^{G(Sx,Sx,Ty)} h(t)dt, \int_0^{G(Tx,Tx,Sy)} h(t)dt\},
\end{aligned}$$

for all $x, y \in X$, $\delta \in (0, 1)$, $L \geq 0$ and $h(t)$ as in Theorem 4.1. If T and S satisfy $CLR_{(S)}$ - property, then $C(T, S) \neq \emptyset$. Moreover, if T and S are weakly compatible, then T and S have an unique common fixed point.

4.3. Fixed points for mappings satisfying φ - contractive conditions

As in [31], let ϕ be the set of all real nondecreasing continuous functions $\varphi : [0,\infty) \to [0,\infty)$ with $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t \in [0,\infty)$. If $\varphi \in \phi$, then

1) $\varphi(t) < t$ for all $t \in (0,\infty)$,

2) $\varphi(0) = 0$.

The following functions $F(t_1,...,t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfy the conditions $(F_1), (F_2)$.

Example 4.9. $F(t_1,...,t_6) = t_1 - \varphi \max\{t_2,t_3,t_4,t_5,t_6\}$.

Example 4.10. $F(t_1,...,t_6) = t_1 - \varphi \max\left\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\right\}$.

Example 4.11. $F(t_1,...,t_6) = t_1 - \varphi \max\left\{t_2,\frac{t_3+t_4}{2},\frac{t_5+t_6}{2}\right\}$.

Example 4.12. $F(t_1,...,t_6) = t_1 - \varphi \max\left\{t_2,\sqrt{t_3t_4},\sqrt{t_5t_6},\sqrt{t_3t_5},\sqrt{t_4t_6}\right\}$.

Example 4.13. $F(t_1,...,t_6) = t_1 - \varphi(at_2 + bt_3 + ct_4 + dt_5 + et_6)$, where $a,b,c,d,e \ge 0$ and $a+b+c+d+e \le 1$.

Example 4.14. $F(t_1,...,t_6) = t_1 - \varphi\left(at_2 + \frac{b\sqrt{t_5t_6}}{1+t_3+t_4}\right)$, where $a,b \ge 0$ and $a+b \le 1$.

Example 4.15. $F(t_1,...,t_6) = t_1 - \varphi\left(at_2 + b\max\{t_3,t_4\} + c\max\left\{\frac{t_3+t_4}{2},\frac{t_5+t_6}{2}\right\}\right)$, where $a,b,c \ge 0$ and $a+b+c \le 1$.

Example 4.16. $F(t_1,...,t_6) = t_1 - \varphi(at_2 + b\max\{2t_4+t_5, 2t_4+t_6, t_3+t_5+t_6\})$, where $a,b \ge 0$ and $a+b \le 1$.

By Theorem 4.4 and Example 4.9 we obtain

Theorem 4.8. *Let T,S be self mappings of a G - metric space (X,G) such that*

$$\begin{aligned}\psi(G(Tx,Tx,Ty)) \quad \le \quad & \varphi(\max\{\psi(G(Sx,Sx,Sy)),\psi(G(Tx,Tx,Sx)),\\ & \psi(G(Ty,Ty,Sy)),\psi(G(Sx,Sx,Ty)),\psi(G(Tx,Tx,Sy))\},\end{aligned}$$

for all $x,y \in X$, $\varphi \in \phi$ and ψ is an almost altering distance. If T and S satisfy $CLR_{(S)}$ - property, then $C(T,S) \neq \emptyset$. Moreover, if T and S are weakly compatible, then T and S have an unique common fixed point.

By Theorem 4.6 and Example 4.9 we obtain

Theorem 4.9. *Let T and S be self mappings of a G - metric space (X,G) such that*

$$\begin{aligned}&\int_0^{G(Tx,Tx,Ty)} h(t)dt \le \varphi(\max\{\int_0^{G(Sx,Sx,Sy)} h(t)dt, \int_0^{G(Tx,Tx,Sx)} h(t)dt,\\ &\int_0^{G(Ty,Ty,Sy)} h(t)dt, \int_0^{G(Sx,Sx,Ty)} h(t)dt, \int_0^{G(Tx,Tx,Sy)} h(t)dt\},\end{aligned}$$

for all $x,y \in X$, $\varphi \in \phi$ and $h(t)$ as in Theorem 4.1. If T and S satisfy $CLR_{(S)}$ - property, then $C(T,S) \neq \emptyset$. Moreover, if T and S are weakly compatible, then T and S have an unique common fixed point.

Corollary 4.2. *(Theorem 3.3 [6]) Let (X,G) be a G - metric space and the pair (T,S) of self mappings of X is weakly compatible such that*

$$\int_0^{G(Tx,Ty,Tz)} h(t)dt \leq \varphi\left(\int_0^{\mathfrak{L}(x,y,z)} h(t)dt\right),$$

for all $x,y,z \in X$, $\varphi \in \phi$ and $h(t)$ as in Theorem 4.1, where

$$\mathfrak{L}(x,y,z) = \max\{G(Sx,Sy,Sz), G(Sx,Tx,Tx), G(Sy,Ty,Ty), G(Sz,Tz,Tz)\}.$$

If the pairs (T,S) satisfy $CLR_{(S)}$ - property, then T and S have an unique common fixed point.

Proof. If $x = z$, then

$$\begin{aligned}
&\int_0^{G(Tx,Tx,Ty)} h(t)dt \leq \varphi\left(\int_0^{\max\{G(Sx,Sx,Sy),G(Tx,Tx,Sx),G(Ty,Ty,Sy)\}} h(t)dt\right) \\
&\leq \varphi\left(\max\left\{\int_0^{G(Sx,Sx,Sy)} h(t)dt, \int_0^{G(Tx,Tx,Sx)} h(t)dt, \int_0^{G(Ty,Ty,Sy)} h(t)dt\right\}\right) \\
&\leq \varphi\left(\max\left\{\int_0^{G(Sx,Sx,Sy)} h(t)dt, \int_0^{G(Tx,Tx,Sx)} h(t)dt,\right.\right. \\
&\left.\int_0^{G(Ty,Ty,Sy)} h(t)dt\right\}, \int_0^{G(Sx,Sx,Ty)} h(t)dt, \int_0^{G(Tx,Tx,Sy)} h(t)dt\}.
\end{aligned}$$

Then by Theorem 4.8, T and S have an unique common fixed point. □

4.4. Fixed point for (φ,ψ) - weakly contractive mappings

In 1997, Alber and Guerre - Delabierre [4] defined the concept of weak contraction as a generalization of contraction and established the existence of fixed points for a self mapping in Hilbert spaces. Rhoades [57] extended this concept in metric spaces. In [9], the authors studied the existence of fixed points for a pair of (φ,ψ) - weakly contractive mappings.

New results are obtained in [11], [17], [20], [54], [56] and in other papers. In [3] and [8], the study of common fixed points of (φ,ψ) - weakly contractions with $(E.A)$ - properties is initiated.

Also, some fixed points theorems for mappings with common limit range property satisfying (φ,ψ) - weakly contractive conditions are proved in [25] and [64].

Definition 4.2. 1) Let Ψ be the set of all functions $\psi : [0,\infty) \to [0,\infty)$ satisfying

a) ψ is continuous,

b) $\psi(0) = 0$ and $\psi(t) > 0, \forall t > 0$.

2) Let Φ be the set of all functions $\phi : [0,\infty) \to [0,\infty)$ satisfying

a) ϕ is lower semi - continuous,

b) $\phi(0) = 0$ and $\phi(t) > 0, \forall t > 0$.

The following functions $F(t_1, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfy the conditions $(F_1), (F_2)$.

Example 4.17. $F(t_1, ..., t_6) = \psi(t_1) - \psi(\max\left\{t_2, t_3, t_4, \frac{t_5+t_6}{2}\right\}) + \phi(\max\{t_3, t_4, t_5\})$.

Example 4.18. $F(t_1, ..., t_6) = \psi(t_1) - \psi(\max\{t_2, t_3, t_4, t_5, t_6\}) + \phi(\max\left\{t_2, t_3, t_4, \frac{t_5+t_6}{2}\right\})$.

Example 4.19. $F(t_1, ..., t_6) = \psi(t_1) - \psi(\max\left\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\right\}) + \phi(\max\{t_2, t_3, t_4, t_5, t_6\})$.

Example 4.20. $F(t_1, ..., t_6) = \psi(t_1) - \psi(\max\left\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\right\}) + \phi(\max\left\{t_3, t_4, \frac{t_5+t_6}{2}\right\})$.

Example 4.21. $F(t_1, ..., t_6) = \psi(t_1) - \psi(\max\left\{t_2, t_3, t_4, \frac{t_5+t_6}{2}\right\}) + \phi(\max\left\{\sqrt{t_3t_6}, \sqrt{t_2t_5}, \sqrt{t_5t_6}\right\})$.

Example 4.22. $F(t_1, ..., t_6) = \psi(t_1) - \psi(\max\left\{\sqrt{t_3t_6}, \sqrt{t_2t_5}, \sqrt{t_5t_6}\right\}) + \phi(\max\{t_2, t_3, t_4, t_5, t_6\})$.

Example 4.23. $F(t_1, ..., t_6) = \psi(t_1) - \psi\left(\frac{\sqrt{t_3t_6}+\sqrt{t_4t_6}+\sqrt{t_2t_6}}{1+\sqrt{t_3t_4}+\sqrt{t_4t_6}+\sqrt{t_2t_3}}\right) + \phi(\max\{t_2, t_3, t_4, t_5, t_6\})$.

Example 4.24. $F(t_1, ..., t_6) = \psi(t_1) - \psi\left(\sqrt{t_2t_5} + \sqrt{t_2t_6} + \sqrt{t_3t_6} + \sqrt{t_4t_5}\right) +$
$+ \phi(\max\{t_2, t_3, t_4, t_5, t_6\})$.

By Theorem 3.2 and Example 4.17 we obtain the following

Theorem 4.10. *Let T and S be self mappings of a G - metric space (X, G) such that*

$$G(Tx, Tx, Ty) \le \psi(M_1(x, y)) - \phi(M_2(x, y)),$$

for all $x, y \in X$, where

$$M_1(x, y) = \max\{G(Sx, Sx, Sy), G(Tx, Tx, Sx), G(Ty, Ty, Sy), \frac{G(Sx,Sx,Ty)+G(Tx,Tx,Sy)}{2}\},$$

$$M_2(x, y) = \max\{G(Tx, Tx, Sx), G(Ty, Ty, Sy), G(Sx, Sx, Ty), G(Tx, Tx, Sy)\},$$

$\psi \in \Psi$ and $\varphi \in \Phi$.

If T and S satisfy $CLR_{(S)}$ - property, then $C(T, S) \neq \emptyset$. Moreover, if T and S are weakly compatible, then T and S have an unique common fixed point.

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