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SOME RESULT IN ORDERED METRIC SPACES FOR RATIONAL TYPE EXPRESSIONS

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Abstract. In this paper, we prove some fixed point theorems for mappings involving rational expression in the framework of metric spaces endowed with a partial order using a class of pairs of functions satisfying certain assumptions. Our results generalize and extend some known results which appeared in [6], [14], [15].
Keywords: Fixed point, metric space, contractive condition, partially ordered set, altering distance function.

1. Introduction

Fixed point theory is one of the well-known traditional theories in mathematics that has a broad set of applications. In 1922, Polish mathematician Stephan Banach published his famous contraction principle. Since then, this principle has been extended and generalized in several ways either by using the contractive condition or by imposing some additional conditions on an ambient space. From inspiration of this work, several mathematicians heavily studied this field. For example, the work of Kannan [19] ,Chatterjea [7], Berinde [4], Ciric [12], Geraghty [15], Meir and Keeler [21], Suzuki [25] and so forth.

On the other hand, a number of generalizations of metric space have been done and one such generalization is partially ordered metric space, that is, metric spaces endowed with a partial ordering. The theory originated at a relatively later point of time. An early result in this direction was established by Turinici in ordered metrizable uniform spaces [26]. Application of fixed point result in partially ordered metric spaces was made subsequentially, for example, by Ran and Reurings [23] to solving matrix equations and by Nito and Rodriguez-Lopez [22] to obtain solutions of certain partial differential equations with periodic boundary conditions.

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Recently, fixed point theory has developed in partially ordered metric spaces and many mathematicians have obtained several fixed point, common fixed point theorems in the setting of partially ordered metric spaces (see e.g.[1, 2, 5, 6, 8]-[11, 16]-[18]).

The aim of this paper is to establish some fixed point theorems satisfying generalized contraction mapping of rational type using a class of pairs of functions satisfying certain assumptions. The main result of this paper is generalizes and extends the main result of Cabrera et al [6]. Furthermore, our result generalized and extends the corresponding result of [14] and [15] on the context of ordered metric spaces.

2. Preliminaries

Das and Gupta [13] were the Pioneers in proving fixed point theorems using contractive conditions involving rational expressions. They proved the following fixed point theorem..

Theorem 2.1. *[13] Let (X, d) be a complete metric space and $T : X \longrightarrow X$ a mapping such that there exist $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ satisfying*

(2.1) $$d(Tx, Ty) \leq \alpha \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y) \text{ for all } x, y \in X.$$

In [6], Cabrera , Harjani and Sadarangani proved the above theorem in the context of partially ordered metric spaces.

Definition 2.1. Let $(X, \leq)$ is a partially ordered set and $T : X \longrightarrow X$ is said to be monotone non-decreasing if for all $x, y \in X$,

(2.2) $$x \leq y \Longrightarrow Tx \leq Ty.$$

Theorem 2.2. *[6] Let $(X, \leq)$ is a partially ordered set and suppose that there exist a metric d on X such that (X, d) be a complete metric space. Let $T : X \longrightarrow X$ be a continuous and non-decreasing mapping such that (2.1) is satisfied for all $x, y \in X$ with $x \leq y$. If there exist $x_\circ \in X$ such that $x_\circ \leq Tx_\circ$, then T has a fixed point.*

Theorem 2.3. *[6] Let $(X, \leq)$ is a partially ordered set and suppose that there exist a metric d on X such that (X, d) be a complete metric space. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to x$ for all $n \in N$. Let $T : X \longrightarrow X$ be a non-decreasing mapping such that (2.1) is satisfied for all $x, y \in X$ with $x \leq y$. If there exist $x_\circ \in X$ such that $x_\circ \leq Tx_\circ$, then T has a fixed point.*

Theorem 2.4. *[6] In addition to the hypothesis of Theorem 2.2 or Theorem 2.3, suppose that for every $x, y \in X$, there exist $u \in X$ such that $u \leq x$ and $u \leq y$. Then T has a unique fixed point.*

Khan et al. [20] initiated the use of control function that alter distance between two points in a metric space, which they called an altering distance function

Definition 2.2. [20] A function $\phi : [0, \infty) \to [0, \infty)$ is called altering distance function if the following conditions are satisfied: (1) ϕ is monotone increasing and continuous, (2) $\varphi(t) = 0$ if and only if $t = 0$.

In this paper, we consider the following class of pairs of functions .

Definition 2.3. A pair of functions (φ, ϕ) is said to belong to the class F, if they satisfy the following conditions: (i) $\varphi, \phi : [0, \infty) \to [0, \infty)$; (ii) for $t, s \in [0, \infty)$ $\varphi(t) \leq \phi(s)$ then $t \leq s$; (iii) for (t_n) and (s_n) sequence in $[0, \infty)$ such that x if $\varphi(t_n) \leq \phi(s_n)$ for any $n \in N$, then $a = 0$.

Remark 2.1. Note that, if $(\varphi, \phi) \in F$ and $\varphi(t) \leq \phi(t)$, then t=0, since we can take $t_n = s_n = t$ for any $n \in N$ and by (iii)we deduce t=0.

Now, we present some interesting examples of pairs of functions belonging to the class .

Example 2.1. [24] Let $\phi : [0, \infty) \to [0, \infty)$ be a continuous and increasing function such that $\phi(t) = 0$ if and only if t=0 (these functions are known in the literature as altering distance functions). Let $\phi : [0, \infty) \to [0, \infty)$ be a non-decreasing function such that $\phi(t) = 0$ if and only if t=0 and suppose that $\phi \leq \varphi$. Then the pair $(\varphi, \varphi - \phi) \in F$. In fact, it is clear that $(\varphi, \varphi - \phi)$ satisfy (i).

To prove (ii), suppose that $t, s \in [0, \infty)$ and $\varphi(t) \leq (\varphi - \phi)(s)$. Then, from

$$\varphi(t) \leq \varphi(s) - \phi(s) \leq \phi(s).$$

and taking into account the increasing character of φ, we can deduce that $t \leq s$.

In order to prove (iii), we suppose that

$$\varphi(t_n) \leq \varphi(s_n) - \phi(s_n) \leq \phi(s_n) \tag{(A)}$$

for any $n \in N$, where $t_n, s_n \in (0, \infty)$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = a$, Taking $\to \infty$ in (A), we infer that $\lim_{n\to\infty} \phi(s_n) = 0$. Let us suppose that $a > 0$. Since $\lim_{n\to\infty} s_n = a > 0$, we can find $\epsilon > 0$ and a subsequence (s_{n_k}) of (s_n) such that $s_{n_k} > \epsilon$ for any $k \in N$. As ϕ is nondecreasing, we have $\phi(s_{n_k}) > \phi(\epsilon)$ for any $k \in N$ and, consequently, $\lim_{n\to\infty} \phi(s_{n_k}) \geq \phi(\epsilon)$. This contradicts the fact that $\lim_{n\to\infty} \phi(s_{n_k}) = 0$. Therefore, $a > 0$. This proves that $(\varphi, \varphi - \phi)$. An interesting particular case is when φ is the identity mapping, $\varphi = 1_{[0,\infty)}$ and $\phi : [0, \infty) \to [0, \infty)$ is a nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$ and for any $t \in [0, \infty)$.

Example 2.2. [24] Let S be the class of functions defined by

$$S = \{\alpha : [0, \infty) \to [0, 1) : \{\alpha(t_n) \to 1 \Rightarrow t_n \to 0\}\}.$$

Let us consider the pairs of functions $(1_{[0,\infty)}, \alpha 1_{[0,\infty)})$, where $\alpha \in S$ and $\alpha 1_{[0,\infty)}$ is defined by

$$\alpha 1_{[0,\infty)}(t) = \alpha t, \text{ for } t \in [0, \infty).$$

Then $(1_{[0,\infty)}, \alpha 1_{[0,\infty)}) \in F$ It is clear that the pairs $(1_{[0,\infty)}, \alpha 1_{[0,\infty)})$, with $\alpha \in S$ satisfy(i). To prove (ii), from $1_{[0,\infty)}(t) \leq \alpha 1_{[0,\infty)}(s)$ for $t, s \in [0, \infty)$, we infer, since $\alpha : [0, \infty) \to [0, 1)$, that $t \leq \alpha(s)s < s$ and, consequently, $(1_{[0,\infty)}, \alpha 1_{[0,\infty)})$ satisfies (ii). In order to prove (iii), we suppose that $1_{[0,\infty)}(t_n) = t_n \leq \alpha 1_{[0,\infty)}(s_n) = \alpha(s_n)s_n$ for any $n \in N$, where $t_n, s_n \in [0, \infty)$, $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = a$. Let us suppose that $a > 0$. Since $\lim_{n\to\infty} s_n = a > 0$, we can find a subsequence (s_{n_k}) of (s_n) such that $s_{n_k} > 0$ for any $k \in N$. Now, as $t_n \leq \alpha(s_n)s_n \leq s_n$ for any $n \in N$ in particular, we have $t_{n_k} \leq \alpha(s_{n_k})s_{n_k} \leq s_{n_k}$ for any $k \in N$ and since $s_{n_k} > 0$ for any $k \in N$

$$\frac{t_{n_k}}{s_{n_k}} \leq \alpha(s_{n_k}) \leq 1.$$

Taking $k \to \infty$ in the last inequality, we obtain $\lim_{k\to\infty} \alpha(s_{n_k}) = 1$. Finally, since $\alpha \in S$, we infer that $\lim_{k\to\infty} s_{n_k} = 0$ and this contradicts the fact that $\lim_{n\to\infty} \alpha(s_n) = a$. Therefore, $a = 0$.This proves that $(1_{[0,\infty)}, \alpha 1_{[0,\infty)}) \in F$ for $\alpha \in S$.

Remark 2.2. Suppose that $g : [0, \infty) \to [0, \infty)$ is an increasing function and $(\varphi, \phi) \in F$. Then it is easily seen that the pair $(g \circ \varphi, \varphi \circ g) \in F$.

3. Main Result

Theorem 3.1. *Let $(X, \leq)$ is a partially ordered set and suppose that there exist a metric d on X such that be a complete metric space. Let $T : X \to X$ be a non-decreasing mapping such that there exists a pair of functions $(\varphi, \phi) \in F$ satisfying*

$$\varphi(d(Tx, Ty)) \leq \max\left\{\phi(d(x, y)), \phi\left(\frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}\right)\right\}, \tag{3.1}$$

for all $x, y \in X$ with $x \leq y$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \leq u$, for all $n \in N$. If there exist $x_\circ \in X$ such that $x_\circ \leq Tx_\circ$, then T has a fixed point.

Proof. If $x_\circ = Tx_\circ$, then we have the result. Therefore, we suppose that $x_\circ < Tx_\circ$, we construct a sequence $\{x_n\}$ in X such that

$$x_{n+1} = Tx_n \text{ fore very } n \geq 0. \tag{3.2}$$

Since T is non-decreasing, we obtain by induction that

$$x_\circ < Tx_\circ = Tx_1 = x_2 \leq ... \leq Tx_{n-1} = x_n \leq Tx_{n-1} = x_{n+1} \leq .... \tag{3.3}$$

If there exists $n \geq 1$ such that $x_n = x_{n+1}$, then from (3.2), $x_{n+1} = Tx_n = x_n$, that is x_n is a fixed point of T, and the proof is finished. Now suppose that $x_n \neq x_{n+1}$, that is $d(x_n, x_{n+1}) \neq 0$, for all $n \geq 1$. Since $x_{n-1} < x_n$ for all $n \geq 1$, from (3.1), we have

$$\begin{aligned} \varphi(d(x_{n+1}, x_n)) &= \varphi(d(Tx_n, Tx_{n-1})) \\ &\leq \max\left\{\phi(d(x_n, x_{n-1})), \phi\left(\frac{d(x_{n-1}, Tx_{n-1})[1 + d(x_n, Tx_n)]}{1 + d(x_n, x_{n-1})}\right)\right\} \end{aligned}$$

$$= \max\left\{\phi(d(x_n, x_{n-1})), \phi\left(\frac{d(x_{n-1}, x_n)[1 + d(x_n, x_{n+1})]}{1 + d(x_n, x_{n-1})}\right)\right\}. \tag{3.4}$$

Now, we distinguish two cases. **Case I.** Consider

$$\max\left\{\phi(d(x_n, x_{n-1})), \phi\left(\frac{d(x_{n-1}, x_n)[1 + d(x_n, x_{n+1})]}{1 + d(x_n, x_{n-1})}\right)\right\} = \phi(d(x_n, x_{n-1})). \tag{3.5}$$

In this case from (3.4), we have

$$\varphi(d(x_{n+1}, x_n)) \leq \phi(d(x_n, x_{n-1})) \tag{3.6}$$

Since $(\varphi, \phi) \in F$, we deduce that

$$d(x_{n+1}, x_n) \leq d(x_n, x_{n-1})$$

Case II. If

$$\max\left\{\phi(d(x_n, x_{n-1})), \phi\left(\frac{d(x_{n-1}, x_n)[1 + d(x_n, x_{n+1})]}{1 + d(x_n, x_{n-1})}\right)\right\}$$

$$= \phi\left(\frac{d(x_{n-1}, x_n)[1 + d(x_n, x_{n+1})]}{1 + d(x_n, x_{n-1})}\right) \tag{3.7}$$

In this case from (3.4) and since $(\varphi, \phi) \in F$, we get

$$d(x_n, x_{n-1}) \leq \frac{d(x_{n-1}, x_n)[1 + d(x_n, x_{n+1})]}{1 + d(x_n, x_{n-1})}$$

Since $d(x_n, x_{n+1}) \neq 0$, from the last inequality it follows that

$$d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}).$$

From both cases, we conclude that the sequence $\{d(x_{n+1}, x_n)\}$ is a decreasing sequence of non-negative real numbers and is bounded below, there exists $r \geq 0$ such that

$$d(x_{n+1}, x_n) \to r \text{ as } r \to \infty. \tag{3.8}$$

Now, we shall show that $r = 0$. Denote $A = \{n \in N : n$ satisfies (3.5)$\}$, $B = \{n \in N : n$ satisfies (3.7)$\}$ We note that the following.

(1) If $Card\ A = \infty$, then from (3.4), we can find infinitely natural numbers n satisfying inequality (3.6) and since $\lim_{n\to\infty} d(x_{n+1}, x_n) = d(x_n, x_{n-1}) = r$ and $(\varphi, \phi) \in F$, we have $r = 0$. (2) If $Card\ B = \infty$, then from (3.4), we can find infinitely many $n \in N$ such that

$$\varphi(d(x_n, x_{n-1})) \leq \phi(\frac{d(x_{n-1}, x_n)[1 + d(x_n, x_{n+1})]}{1 + d(x_n, x_{n-1})}).$$

Since $(\varphi, \phi) \in F$ and using the similar argument to the one used in case (2), we obtain

$$d(x_n, x_{n-1}) \leq \frac{d(x_{n-1}, x_n)[1 + d(x_n, x_{n+1})]}{1 + d(x_n, x_{n-1})} \tag{3.9}$$

for infinitely many $n \in N$ Letting $n \to \infty$ in (3.9) and taking into account that $\lim_{n\to\infty} d(x_{n+1}, x_n) = r$, we deduce that $r \leq r\frac{1+r}{1+r}$. And consequently, we obtain $r = 0$. Therefore

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0 \tag{3.10}$$

Next, we will show that $\{x_n\}$ is a Cauchy sequence. In contrary case since ,by Lemma 2.1 of [8], we can find $\epsilon > 0$ and subsequences $\{x_{n(k)}\}, \{x_{m(k)}\}$ of $\{x_n\}$ satisfying (i) $n(k) \geq m(k) > k$ for all positive integer k; (ii) $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$. Assuming that $n(k)$ is the smallest such positive integer, we get $n(k) \geq m(k) > k$, $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$ and $d(x_{m(k)}, x_{n(k)-1}) < \epsilon$. Now,

$$\epsilon \leq d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}),$$

that is

$$\epsilon \leq d(x_{m(k)}, x_{n(k)}) \leq \epsilon + d(x_{n(k)-1}, x_{n(k)}),$$

Letting $k \to \infty$ in the above inequality and using (3.10), we have

$$d(x_{m(k)}, x_{n(k)}) = \epsilon \tag{3.11}$$

Again

$$\begin{aligned} d(x_{m(k)}, x_{n(k)}) &\leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}), \\ d(x_{m(k)-1}, x_{n(k)-1}) &\leq d(x_{m(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)-1}) \end{aligned}$$

Letting $k \to \infty$ in the above two inequalities and using (3.10) and (3.11), we have

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \epsilon. \tag{3.12}$$

Now using contractive condition (3.1), we get

$$\begin{aligned} \varphi(d(x_{m(k)}, x_{n(k)})) &= \varphi(d(Tx_{m(k)-1}, Tx_{n(k)-1})) \\ &\leq \max\left\{ \begin{array}{c} \phi(d(x_{m(k)-1}, x_{n(k)-1})), \\ \phi\left(\frac{d(x_{n(k)-1}, Tx_{n(k)-1})[1+d(x_{m(k)-1}, Tx_{m(k)-1})]}{1+d(x_{m(k)-1}, x_{n(k)-1})}\right) \end{array} \right\} \end{aligned}$$

$$= \max\left\{\phi(d(x_{m(k)-1}, x_{n(k)-1})), \phi\left(\frac{d(x_{n(k)-1}, x_{n(k)})[1 + d(x_{m(k)-1}, x_{m(k)})]}{1 + d(x_{m(k)-1}, x_{n(k)-1})}\right)\right\}. \tag{3.13}$$

Put

$$C = \{k \in N : \varphi(d(x_{m(k)}, x_{n(k)})) \leq \phi(d(x_{m(k)-1}, x_{n(k)-1})).$$

$$D = \{k \in N : \varphi(d(x_{m(k)}, x_{n(k)})) \leq \phi\left(\frac{d(x_{n(k)-1}, x_{n(k)})[1 + d(x_{m(k)-1}, x_{m(k)})]}{1 + d(x_{m(k)-1}, x_{n(k)-1})}\right). \tag{3.14}$$

By (3.13), we have *Card* $C = \infty$ or *Card* $D = \infty$. Let us suppose that *Card* $C = \infty$. Then there exists infinitely many $k \in N$ such that

$$\varphi(d(x_{m(k)}, x_{n(k)})) \leq \phi(d(x_{m(k)-1}, x_{n(k)-1})).$$

And since $(\varphi, \phi) \in F$, we have by letting $k \to \infty$

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) \leq \lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}).$$

We infer from (3.11) that $\epsilon = 0$. This is a contradiction. On the other hand, if *Card* $D = \infty$, then we can find infinitely many $k \in N$ such that

$$\varphi(d(x_{m(k)}, x_{n(k)})) \leq \phi\left(\frac{d(x_{n(k)-1}, x_{n(k)})[1 + d(x_{m(k)-1}, x_{m(k)})]}{1 + d(x_{m(k)-1}, x_{n(k)-1})}\right).$$

And since $(\varphi, \phi) \in F$, we obtain from the above inequality

$$d(x_{m(k)}, x_{n(k)})) \leq \frac{d(x_{n(k)-1}, x_{n(k)})[1 + d(x_{m(k)-1}, x_{m(k)})]}{1 + d(x_{m(k)-1}, x_{n(k)-1})}$$

Taking $k \to \infty$ and using (3.10) and (3.12) we obtain $\epsilon \leq 0$, which is a contradiction. Therefore, in both the cases, we obtain a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Next, we will show that u is a fixed point of T. Since $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \leq u$. By the contractive condition (3.1), we obtain

$$\varphi(d(Tu, Tx_n)) \leq \max\left\{\phi(d(u, x_n)), \phi\left(\frac{d(x_n, Tx_n)[1 + d(u, Tu)]}{1 + d(u, x_n)}\right)\right\}, \tag{3.15}$$

for any $n \in N$. Now we distinguish two cases. (1)There exist infinitely many $n \in N$ such that

$$\varphi(d(Tu, Tx_n)) \leq \phi(d(u, x_n))$$

since $(\varphi, \phi) \in F$, we obtain $d(Tu, Tx_n) \leq d(u, x_n)$. For infinitely many $n \in N$. Since $\lim_{n\to\infty} x_n = u$, letting $n \to \infty$ in the last inequality, we obtain

$$\lim_{n\to\infty} Tx_n = Tu, \tag{3.16}$$

where, to simplify our assumptions, we will denote the subsequence by the same symbol Tx_n. By (3.2),

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tu. \tag{3.17}$$

$x_n \to u$ in X this means that $Tu = u$. Therefore u is a fixed point of T. (2)There exist infinitely many $n \in N$ such that

$$\varphi(d(Tu, Tx_n)) \leq \phi\left(\frac{d(x_n, Tx_n)[1 + d(u, Tu)]}{1 + d(x_n, u)}\right).$$

Again to simplify our considerations, we will denote the subsequence by the same symbol Tx_n. Since $(\varphi, \phi) \in F$, we deduce that

$$d(Tu, Tx_n)) \leq \frac{d(x_n, Tx_n)[1 + d(u, Tu)]}{1 + d(x_n, u)} \leq \frac{d(x_n, x_{n+1})[1 + d(u, Tu)]}{1 + d(x_n, u)},$$

for any $n \in N$. Taking $n \to \infty$ and by using (3.10), we infer (3.16). From the above case, we deduce that u is a fixed point of T. Therefore, in both cases we proved that u is a fixed point of in T. This complete the proof of the theorem. □

By Theorem 3.1, we obtain the following corollaries.

Corollary 3.1. *Let $(X, \leq)$ is a partially ordered set and suppose that there exist a metric d on X such that be a complete metric space. Let $T : X \to X$ be a non-decreasing mapping such that there exists a pair of functions $(\varphi, \phi) \in F$ satisfying*

$$\varphi(d(Tx, Ty)) \leq \phi(d(x, y)).$$

for any $x, y \in X$ with $x \leq y$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \leq u$, for all $n \in N$. If there exist $x_\circ \in X$ such that $x_\circ \leq Tx_\circ$, then T has a fixed point.

Corollary 3.2. *Let $(X, \leq)$ is a partially ordered set and suppose that there exist a metric d on X such that be a complete metric space. Let $T : X \to X$ be a non-decreasing mapping such that there exists a pair of functions $(\varphi, \phi) \in F$ satisfying*

$$\varphi(d(Tx, Ty)) \leq \phi\left(\frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}\right).$$

for any $x, y \in X$ with $x \leq y$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \leq u$, for all $n \in N$. If there exist $x_\circ \in X$ such that $x_\circ \leq Tx_\circ$, then T has a fixed point.

Remark 3.1. The main result of [6] is Theorem 2.2, 2.3 and 2.4. Notice that the contractive condition appearing in these theorems

$$d(Tx,Ty) \leq \alpha d(x,y) + \beta \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)},$$

for any $x, y \in X$ with $x \leq y$ where $\alpha, \beta > 0$ and $\alpha + \beta < 1$ for any $x, y \in X$, implies that

$$\begin{aligned} d(Tx,Ty) &\leq (\alpha+\beta)\max\{d(x,y), \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}\} \\ &\leq \max\{(\alpha+\beta)d(x,y), (\alpha+\beta)\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}\} \end{aligned}$$

for any $x, y \in X$ with $x \leq y$. This condition is a particular case of the contractive condition appearing in Theorem 3.1 with the pair of functions $(\varphi, \phi) \in F$, given by $\varphi = 1_{[0,\infty)}$ and $\phi = (\alpha+\beta)1_{[0,\infty)}$. Furthermore, we relaxed the requirement of the continuity of mapping to prove the results. Therefore, the following corollary is a particular case of Theorem 3.1.

Corollary 3.3. *Let $(X, \leq)$ is a partially ordered set and suppose that there exist a metric d on X such that be a complete metric space. Let $T : X \to X$ be a non-decreasing mapping such that there exists a pair of functions $(\varphi, \phi) \in F$ satisfying*

$$d(Tx,Ty) \leq \max\{(\alpha+\beta)d(x,y), (\alpha+\beta)\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}\}$$

for any $x, y \in X$ with $x \leq y$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \leq u$, for all $n \in N$. If there exist $x_\circ \in X$ such that $x_\circ \leq Tx_\circ$, then T has a fixed point.

Taking into account Example 2.1, we have the following corollary.

Corollary 3.4. *Let $(X, \leq)$ is a partially ordered set and suppose that there exist a metric d on X such that be a complete metric space. Let $T : X \to X$ be a non-decreasing mapping such that there exists a pair of functions $(\varphi, \phi) \in F$ satisfying*

$$\varphi(d(Tx,Ty)) \leq \max\left\{ \begin{array}{c} \varphi(d(x,y)) - \phi(d(x,y)), \varphi\left(\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}\right) \\ -\phi\left(\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}\right) \end{array} \right\},$$

for any $x, y \in X$ with $x \leq y$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \leq u$, for all $n \in N$. If there exist $x_\circ \in X$ such that $x_\circ \leq Tx_\circ$, then T has a fixed point.

Corollary 3.4 has the following consequences.

Corollary 3.5. *Let $(X, \leq)$ is a partially ordered set and suppose that there exist a metric d on X such that be a complete metric space. Let $T : X \to X$ be a non-decreasing mapping such that there exists a pair of functions $(\varphi, \phi) \in F$ satisfying*

$$\varphi(d(Tx,Ty)) \leq \varphi(d(x,y)) - \phi(d(x,y)),$$

for any $x, y \in X$ with $x \leq y$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \leq u$, for all $n \in N$. If there exist $x_\circ \in X$ such that $x_\circ \leq Tx_\circ$, then T has a fixed point.

Remark 3.2. Corollary 3.5 is an extension of the fixed point theorem of the following theorem in the setting of ordered metric space, which appear in [14].

Theorem 3.2. *Let (X, d) be a complete metric space and $T : X \to X$ is a mapping satisfying*

$$\varphi(d(Tx, Ty)) \leq \varphi(d(x, y)) - \phi(d(x, y)),$$

for any $x, y \in X$ where φ and ϕ satisfy the same conditions as in Corollary 3.5. Then T has a fixed point.

Corollary 3.6. *Let $(X, \leq)$ is a partially ordered set and suppose that there exist a metric d on X such that be a complete metric space. Let $T : X \to X$ be a non-decreasing mapping such that there exists a pair of functions $(\varphi, \phi) \in F$ satisfying*

$$\varphi(d(Tx, Ty)) \leq \varphi\left(\frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}\right) - \phi\left(\frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}\right)$$

for any $x, y \in X$ with $x \leq y$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \leq u$, for all $n \in N$. If there exist $x_\circ \in X$ such that $x_\circ \leq Tx_\circ$, then T has a fixed point.

Taking into account Example 2.2, we have the following corollary.

Corollary 3.7. *Let $(X, \leq)$ is a partially ordered set and suppose that there exist a metric d on X such that be a complete metric space. Let $T : X \to X$ be a non-decreasing mapping such that there exists $\alpha \in S$ (see Example 2.2) satisfying*

$$d(Tx, Ty) \leq \max\left\{ \begin{array}{c} \alpha(d(x, y))d(x, y), \\ \alpha\left(\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}\right).\alpha\left(\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}\right) \end{array} \right\},$$

for any $x, y \in X$ with $x \leq y$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \leq u$, for all $n \in N$. If there exist $x_\circ \in X$ such that $x_\circ \leq Tx_\circ$, then T has a fixed point.

A consequence of Corollary 3.7 is the following corollary.

Corollary 3.8. *Let $(X, \leq)$ is a partially ordered set and suppose that there exist a metric d on X such that be a complete metric space. Let $T : X \to X$ be a non-decreasing mapping such that there exists $\alpha \in S$ (see Example 2.2) satisfying*

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y),$$

for any $x, y \in X$ with $x \leq y$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \leq u$, for all $n \in N$. If there exist $x_\circ \in X$ such that $x_\circ \leq Tx_\circ$, then T has a fixed point.

Remark 3.3. Corollary 3.8 is the extension of the fixed point theorem of the following fixed point theorem in the setting of ordered metric space, appear in [15].

Theorem 3.3. *[15] Let (X, d) be a complete metric space and $T : X \to X$ is a mapping satisfying*

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y),$$

for any $x, y \in X$ where $\alpha \in S$. Then T has a fixed point. Now, we shall prove the uniqueness of the fixed point as in the following theorem.

Theorem 3.4. *In addition to the hypotheses of Theorem 3.1 assume that for all $x, y \in X$ there exists $z \in X$ such that $x \leq z$ and $y \leq z$. Then T has a unique fixed point.*

Proof. Suppose that x^* and y^* are two fixed points of T. By assumption, there exists $z \in X$ such that $x^* \leq z$ and $y^* \leq z$. Now, proceeding similarly to the proof of Theorem 3.1., we can define the sequence in X as follows

$$Tz_n = z_{n+1},\ z_\circ = z\ \forall n \in N.$$

Since T is non-decreasing we have

$$z \leq z_n \leq z_{n+1} \text{ and } d(z_n, z_{n+1}) \to 0 \text{ as } n \to \infty.$$

As $x^* \leq z_n$, putting $x^* = x$, $z_n = y$ in (3.1), we get

$$\varphi(d(Tx^*, Tz_n)) \leq \max\left\{\phi(d(x^*, z_n)), \phi\left(\frac{d(z_n, Tz_n)[1 + d(x^*, Tx^*)]}{1 + d(x^*, z_n)}\right)\right\},$$

that is

$$\varphi(d(x^*, z_{n+1})) \leq \max\left\{\phi(d(x^*, z_n)), \phi\left(\frac{d(z_n, z_{n+1})}{1 + d(x^*, z_n)}\right)\right\},$$

Put

$$E = \{n \in N : \varphi(d(x^*, z_{n+1})) \leq \phi(d(x^*, z_n)).$$

$$G = \{n \in N : \varphi(d(x^*, z_{n+1})) \leq \phi\left(\frac{d(z_n, z_{n+1})}{1 + d(x^*, z_n)}\right).$$

we have $Card\ E = \infty$ or $Card\ G = \infty$. Suppose that $Card\ E = \infty$, then there exists infinitely many $n \in N$ such that

$$\varphi(d(x^*, z_{n+1})) \leq \phi(d(x^*, z_n)).\ \forall n \in N.$$

It follows that the sequence $\{d(x^*, z_n)\}$ is non-increasing and it has a limit $l \geq 0$. From the above inequality and since $\lim_{n\to\infty} d(x^*, z_{n+1}) = \lim_{n\to\infty} d(x^*, z_n) = l$ and $(\varphi, \phi) \in F$, we obtain $l = 0$. Hence

$$\lim_{n\to\infty} d(x^*, z_{n+1}) = 0. \tag{3.18}$$

If *Card* $G = \infty$, then there exists infinitely many $n \in N$ such that

$$\varphi(d(x^*, z_{n+1})) \leq \phi\left(\frac{d(z_n, z_{n+1})}{1 + d(x^*, z_n)}\right) \ \forall n \in N,$$

then since $(\varphi, \phi) \in F$, we have

$$d(z_n, z_{n+1}) \leq \frac{d(z_n, z_{n+1})}{1 + d(x^*, z_n)} \ \forall n \in N.$$

Letting $n \to \infty$ and as $d(x^*, z_{n+1}) \to \infty$ as $n \to \infty$, we get

$$\lim_{n\to\infty} d(x^*, z_{n+1}) = 0. \tag{3.19}$$

From (3.18) and (3.19) we have $\lim_{n\to\infty} d(x^*, z_n) = 0$. In the same way it can be deduced that $\lim_{n\to\infty} d(y^*, z_n) = 0$. Therefore passing to the limit in $d(x^*, y^*) \leq d(x^*, z_n) + d(z_n, y^*)$ as $n \to \infty$ we obtain $d(x^*, y^*) = 0$. Hence $x^* = y^*$. That is, the fixed point is unique. □

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