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ON APPROXIMATION OF FIXED POINTS OF MEAN NONEXPANSIVE MAPPINGS IN CAT(0) SPACES

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Abstract. A new iterative algorithm for approximating fixed points of mean nonexpansive mappings in CAT(0) spaces is introduced. As a result, a $\triangle$-convergence theorem is established. The result we obtain improves and extends several recent results in the literature. Finally, some numerical examples are presented to illustrate the main result and to compare the new algorithm with some existing ones.
Keywords: Iterative algorithm; CAT(0) space; weak convergence; $\triangle$-convergence; mean nonexpansive mapping

1. Introduction

Fixed point theory of metric spaces was initiated by the celebrated *Banach contraction principle* which states that every contraction on a complete metric space has a unique fixed point; moreover, the fixed point can be approximated by Picard's iterates. Perhaps the most influential fixed point theorem in metric fixed point theory is the theorem due to F. E. Browder and D. Gohde; in 1965, F. E. Browder [10] and D. Gohde [9] independently proved that every nonexpansive self-mapping of a closed, convex, and bounded subset of a uniformly convex Banach space has a fixed point. Fixed point theory in Cartan-Alexandrov-Toponogov spaces, or briefly in CAT(0) spaces, was first studied by W. A. Kirk (see [30, 29]. Among other things, he proved that every nonexpansive mapping defined on a bounded closed convex subset of a complete CAT(0) space has a fixed point. Since then the fixed point theorems for various mappings in a CAT(0) space have been developed rapidly and numerous papers have appeared (see for example [1, 2, 31, 15, 6, 17, 18] and the references therein).

As a generalization of nonexpansive mappings, in 1975, Zhang [26] introduced the concept of a *mean nonexpansive mapping* in Banach spaces and proved the existence and uniqueness of fixed points for this type of mappings in Banach spaces with the normal structure. The mean nonexpansive mappings were extensively studied

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by Wu and Zhang [7], and by Yang and Cui [32]. In 2010, Nakprasit [13] provided an example of a mapping that is mean nonexpansive but not Suzuki-generalized nonexpansive and showed that increasing mean nonexpansiveness implies Suzuki-generalized nonexpansiveness. In 2012, Ouahab [3] proved a fixed point theorem for strong semigroups of mean nonexpansive mappings in uniformly convex Banach spaces. In this paper, we shall study mean nonexpansive mappings in the context of CAT(0) spaces.

Let (X, d) be a metric space and x, y be two fixed elements in X such that $d(x, y) = l$. A geodesic path from x to y is an isometry $c : [0, l] \to c([0, 1]) \subset X$ such that $c(0) = x$, $c(l) = y$. The image of a geodesic path between two points is called a geodesic segment. A metric space (X, d) is called a geodesic space if every two points of X are joined by a geodesic segment. A geodesic triangle represented by $\triangle(x, y, z)$ in a geodesic space consists of three points x, y, z and the three segments joining each pair of the points. A comparison triangle of a geodesic triangle $\triangle(x, y, z)$, denoted by $\overline{\triangle}(x, y, z)$ or $\triangle(\overline{x}, \overline{y}, \overline{z})$, is a triangle in the Euclidean space $\mathbb{R}^2$ such that $d(x, y) = d_{\mathbb{R}^2}(\overline{x}, \overline{y})$, $d(x, z) = d_{\mathbb{R}^2}(\overline{x}, \overline{z})$, and $d(y, z) = d_{\mathbb{R}^2}(\overline{y}, \overline{z})$. This is obtainable by using the triangle inequality, and it is unique up to isometry on $\mathbb{R}^2$. Bridson and Haefliger [16] have shown that such a triangle always exists. A geodesic segment joining two points x, y in a geodesic space X is represented by $[x, y]$. Every point z in the segment is represented by $\alpha x \oplus (1 - \alpha)y$, where $\alpha \in [0, 1]$, that is, $[x, y] := \{\alpha x \oplus (1 - \alpha)y : \quad \alpha \in [0, 1]\}$. A subset $\mathcal{C}$ of a metric space X is called convex if for all $x, y \in \mathcal{C}$, $[x, y] \subset \mathcal{C}$. A geodesic space is called a CAT(0) space if for every geodesic triangle $\triangle$ and its comparison $\overline{\triangle}$, the following inequality is satisfied: $d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y})$ for all $x, y \in \triangle$ and $\overline{x}, \overline{y} \in \overline{\triangle}$. Complete CAT(0) spaces are often called Hadamard spaces (see [28, 24, 25]. Examples of CAT(0) spaces include the $\mathbb{R}$-tree, Hadamard manifolds, and the Hilbert ball equipped with the hyperbolic metric. For more details on these spaces, see for example [19, 14, 8]. A geodesic space (X, d) is called hyperbolic (see [12, 23]) if, for any $x, y, z \in X$,

$$d(\frac{1}{2}z \oplus \frac{1}{2}x, \frac{1}{2}z \oplus \frac{1}{2}y) \leq \frac{1}{2}d(x, y).$$

The class of hyperbolic spaces include the normed spaces, CAT(0) spaces, and some others. Bashir Ali in [4] presented an example of a hyperbolic space that is not a normed space. Therefore, the class of hyperbolic spaces is more general than the class of normed spaces.

Let $\mathcal{C}$ be a nonempty subset of a CAT(0) spaces (X, d). A self-mapping $T : \mathcal{C} \to \mathcal{C}$ is called *nonexpansive* if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in \mathcal{C}$. The mapping T is called *quasi-nonexpansive* if $Fix(T) = \{x \in \mathcal{C} : Tx = x\} \neq \emptyset$ and $d(Tx, p) \leq d(x, p)$ for all $x \in \mathcal{C}$ and $p \in Fix(T)$.

In 2015, Zhou and Cui in [11] introduced an iterative algorithm to approximate fixed points of mean nonexpansive mappings in CAT(0) spaces; this algorithm is defined in the following way:

$$\begin{cases} x_1 \in \mathcal{C}, \\ x_{n+1} = (1-t_n)x_n \oplus t_n T(y_n), \\ y_n = (1-s_n)x_n \oplus s_n T(x_n), \quad n \geq 1, \end{cases}$$

where $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are some sequences in $(0,1)$.

In this paper, we introduce a new iterative algorithm for approximating fixed points of mean nonexpansive mappings in CAT(0) spaces. Under suitable conditions, we prove the $\triangle$-convergence theorem for our algorithm. The results we obtain improve and extend several recent results in the literature; they also complement many known existing results. We then provide some numerical examples to illustrate our main result. In this way, we display the efficiency of our proposed algorithm.

2. Preliminaries

Throughout this article, (X,d) will stand for a metric space. We denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}_{n=1}^{\infty}$ converges weakly to x, and $x_n \to x$ to indicate that the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to x.
We start by recalling some basic definitions.

Definition 2.1. Let $\mathcal{C}$ be a nonempty subset of (X,d). A mapping $T:\mathcal{C}\to\mathcal{C}$ is said to be nonexpansive if

$$d(Tx,Ty) \leq d(x,y), \quad \forall x,y \in \mathcal{C}.$$

Definition 2.2. Let $\mathcal{C}$ be a nonempty subset of (X,d). A mapping $T:\mathcal{C}\to\mathcal{C}$ is said to be mean nonexpansive if

$$d(Tx,Ty) \leq a\,d(x,y) + b\,d(x,Ty), \quad \forall x,y \in \mathcal{C},$$

where a and b are two nonnegative real numbers such that $a+b\leq 1$.

Obviously, every nonexpansive mapping is a mean nonexpansive mapping (with $a=1$ and $b=0$). Note that a mean nonexpansive mapping is not necessarily continuous as the following example shows, so that mean nonexpansive mappings are not necessarily nonexpansive.

Example 2.1. Suppose that $T:[0,1]\to[0,1]$ is a mapping defined by

$$Tx = \begin{cases} \dfrac{x}{5} + \dfrac{5}{12} & x \in [0,\frac{1}{2}); \\ \frac{x}{6} + \frac{5}{12} & x \in [\frac{1}{2},1]. \end{cases}$$

Then T is mean nonexpansive with $a = \frac{1}{3}, b = \frac{2}{3}$, but not continuous at $x = \frac{1}{2}$. Thus, T is not a nonexpansive mapping.

Example 2.2. Suppose that $T : [0, 1] \to [0, 1]$ is a mapping defined by

$$Tx = \begin{cases} \dfrac{1-x}{3} & x \in [0,1] \text{ is rational;} \\ \frac{1+x}{5} & x \in [0,1] \text{ is irrational.} \end{cases}$$

Then T is mean nonexpansive with $a = \frac{1}{3}, b = \frac{2}{3}$, but not continuous at any point in $[0, 1]$ except $x = \frac{1}{4}$, the fixed point of T.

In 2008, Suzuki [27] introduced Suzuki-generalized nonexpansive mappings in Banach spaces.

Definition 2.3. Let $\mathcal{C}$ be a nonempty subset of (X, d). A mapping $T : \mathcal{C} \to \mathcal{C}$ is said to be Suzuki-generalized nonexpansive if

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \ \text{ implies } \ d(Tx, Ty) \leq d(x, y)$$

for all $x, y \in \mathcal{C}$.

In [13], Nakprasit provided an example of a mapping that is mean nonexpansive but not Suzuki-generalized nonexpansive and showed that increasing mean nonexpansive mappings are Suzuki-generalized nonexpansive.
We now turn to some known facts regarding CAT(0) spaces.

Lemma 2.1. *([20], Lemma 2.5) Let (X, d) be a CAT(0) space. Then*

$$d((1-\alpha)x \oplus \alpha y, z)^2 \leq (1-\alpha)d(x, z)^2 + \alpha d(y, z)^2 - \alpha(1-\alpha)d(x, y)^2$$

for all $\alpha \in [0, 1]$ and $x, y, z \in X$.

Lemma 2.2. *([5], Lemma 4.5) Let x be a given point in a CAT(0) space (X, d) and $\{t_n\}$ be a sequence in a closed interval $[a, b]$ with $0 < a \leq b < 1$ and $0 < a(1-b) \leq \dfrac{1}{2}$. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in X such that*

1. $\limsup_{n\to\infty} d(x_n, x) \leq r$,
2. $\limsup_{n\to\infty} d(y_n, x) \leq r$,
3. $\limsup_{n\to\infty} d((1-t_n)x_n \oplus t_n y_n, x) = r$

for some $r \geq 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Theorem 2.1. *([11], Theorem 3.1) Let $\mathcal{C}$ be a nonempty bounded closed convex subset of a complete CAT(0) space (X, d) and $T : \mathcal{C} \to \mathcal{C}$ be a mean nonexpansive mapping with $b < 1$. Then T has a fixed point.*

Theorem 2.2. *([11], Theorem 3.2) Let (X, d) be a complete CAT(0) space and $\mathcal{C}$ be a nonempty bounded closed convex subset of X. Let $T : \mathcal{C} \to \mathcal{C}$ be a mean nonexpansive mapping with $b < 1$, and let $\{x_n\} \subset \mathcal{C}$ be an approximate fixed point sequence (i.e., $\lim_{n\to\infty} d(x_n, Tx_n) = 0$) and $\{x_n\} \rightharpoonup \omega$. Then $T(\omega) = \omega$.*

Definition 2.4. Let $\{x_n\}$ be a bounded sequence in a CAT(0) space (X, d).

1. The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) := \inf_{x\in X}\{r(x, \{x_n\})\},$$

   where $r(x, \{x_n\}) := \limsup_{n\to\infty} d(x_n, x)$.

2. The asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) := \{x \in X : \ r(x, \{x_n\}) = r(\{x_n\})\}.$$

In 2006, Dhompongsa et al proved that $A(\{x_n\})$ consists of exactly one point for each bounded sequence $\{x_n\}$ in a CAT(0) space (see Proposition 7 in [22]). We recall that a bounded sequence $\{x_n\}$ in X is said to be regular if $r(\{x_n\}) = r(\{u_n\})$ for every subsequence $\{u_n\}$ of $\{x_n\}$. It is known that every bounded sequence in a Banach space has a regular subsequence. It is now time to give the concept of $\triangle$-convergence in a CAT(0) space.

Definition 2.5. [31] Let (X, d) be a CAT(0) space. A sequence $\{x_n\}$ in X is said to $\triangle$-converge to $x \in X$ if and only if x is the unique asymptotic center of all subsequences of $\{x_n\}$. In this case, we write $\triangle - \lim_{n\to\infty} x_n = x$ and x is called the $\triangle$-limit of $\{x_n\}$.

Proposition 2.1. *([5], Proposition 3.12). Let $\{x_n\}$ be a bounded sequence in a CAT(0) space (X, d) and let $\mathcal{C} \subset X$ be a closed convex subset which contains $\{x_n\}$. Then,*

- *$\triangle - \lim_{n\to\infty} x_n = x$ implies $\{x_n\} \rightharpoonup x$;*

- *if $\{x_n\}$ is regular, then $\{x_n\} \rightharpoonup x$ implies $\triangle - \lim_{n\to\infty} x_n = x$.*

Lemma 2.3. *The following assertions in a CAT(0) space hold:*

- *[20] Every bounded sequence in a complete $CAT(0)$ space has a $\triangle$-convergent subsequence.*

- *[21] If $\{x_n\}$ is a bounded sequence in a closed convex subset $\mathcal{C}$ of a complete CAT(0) space (X,d), then the asymptotic center of $\{x_n\}$ is in $\mathcal{C}$.*

- *[20] If $\{x_n\}$ is a bounded sequence in a complete CAT(0) space (X,d) with $A(\{x_n\}) = \{p\}$, $\{\nu_n\}$ is a subsequence of $\{x_n\}$ with $A(\{\nu_n\}) = \{\nu\}$, and the sequence $\{d(x_n,\nu)\}$ converges, then $p = \nu$.*

Lemma 2.4. *([11], Lemma 4.4) Let $\mathcal{C}$ be a nonempty closed convex subset of a complete CAT(0) space (X,d) and $T : \mathcal{C} \to \mathcal{C}$ be a mean nonexpansive mapping. If $\{x_n\}$ is a sequence in $\mathcal{C}$ such that $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$ and $\triangle-\lim_{n\to\infty} x_n = p$, then $T(p) = p$.*

Remark 2.1. By Lemma 2.4 and Proposition 2.1 (ii), if $\{x_n\}$ in Theorem 2.2 is regular, then the condition $b < 1$ in Theorem 2.2 can be removed.

3. Weak Convergence Theorem

We begin this section by proving a $\triangle$-convergence theorem for mean nonexpansive mappings in CAT(0) spaces. Here we introduce a new iterative algorithm to approximate the fixed point of our mapping. We shall then compare our algorithm with that of Zhou and Cui [11].

Theorem 3.1. *Let (X,d) be a complete CAT(0) space, $\mathcal{C}$ be a nonempty, bounded closed convex subset of (X,d) and $T : \mathcal{C} \to \mathcal{C}$ be a mean nonexpansive mapping with $b < 1$. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $(0,1)$, also $\{\alpha_n\}$ be a sequence in a closed interval $[r,s]$ with $0 < r \le s < 1$ and $0 < r(1-s) \le \frac{1}{2}$. Then $\{x_n\}_{n=1}^{\infty}$ which is defined by*

$$(3.1) \qquad \begin{cases} x_1 \in \mathcal{C}, \\ z_n = T((1-\alpha_n)x_n \oplus \alpha_n T(x_n)), \\ y_n = T((1-\beta_n)z_n \oplus \beta_n T(z_n)), \\ x_{n+1} = T((1-\gamma_n)T(z_n) \oplus \gamma_n T(y_n)), \end{cases}$$

is $\triangle$-convergent to some point $p \in Fix(T)$.

Proof. By using Theorem 2.1, we get $Fix(T) \neq \emptyset$. Next, we will divide the proof into three steps.

Step 1. First, we will prove that $\lim_{n\to\infty} d(x_n,p)$ exists for each $p \in Fix(T)$, where $\{x_n\}$ is defined by (3.3). For this purpose, let $p \in Fix(T)$, using the fact that $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$ we obtain

$$
\begin{aligned}
d(z_n,p) &= d(T((1-\alpha_n)x_n \oplus \alpha_n T(x_n)),p) \\
&\leq a\Bigg[d((1-\alpha_n)x_n \oplus \alpha_n T(x_n),p)\Bigg] \\
&\quad +b\Bigg[d((1-\alpha_n)x_n \oplus \alpha_n T(x_n),p)\Bigg] \\
&\leq d((1-\alpha_n)x_n \oplus \alpha_n T(x_n),p) \\
&\leq (1-\alpha_n)d(x_n,p) + \alpha_n d(T(x_n),p) \\
&\leq (1-\alpha_n)d(x_n,p) + \alpha_n a d(x_n,p) + \alpha_n b d(x_n,p) \\
&\leq (1-\alpha_n)d(x_n,p) + \alpha_n d(x_n,p) \\
&\leq d(x_n,p) \qquad (3.2)
\end{aligned}
$$

for all $n \in \mathbb{N}$. Also, we have

$$
\begin{aligned}
d(y_n,p) &= d(T((1-\beta_n)z_n \oplus \beta_n T(z_n)),p) \\
&\leq a\Bigg[d((1-\beta_n)z_n \oplus \beta_n T(z_n),p)\Bigg] \\
&\quad +b\Bigg[d((1-\beta_n)z_n \oplus \beta_n T(z_n),p)\Bigg] \\
&\leq d((1-\beta_n)z_n \oplus \beta_n T(z_n),p) \\
&\leq (1-\beta_n)d(z_n,p) + \beta_n d(T(z_n),p) \\
&\leq (1-\beta_n)d(z_n,p) + \beta_n a d(z_n,p) + \beta_n b d(z_n,p) \\
&\leq (1-\beta_n)d(z_n,p) + \beta_n d(z_n,p) \\
&\leq d(z_n,p) \\
&\leq d(x_n,p) \qquad (3.3)
\end{aligned}
$$

for all $n \in \mathbb{N}$. From (3.3), (3.4) and (3.5) and using the fact that $\{\gamma_n\}_{n=1}^{\infty} \subset (0,1)$, we conclude that

$$\begin{aligned}
d(x_{n+1},p) &= d(T((1-\gamma_n)T(z_n)\oplus\gamma_n T(y_n)),p)\\
&\le a\Big[d((1-\gamma_n)T(z_n)\oplus\gamma_n T(y_n)),p)\Big]\\
&\quad +b\Big[d((1-\gamma_n)T(z_n)\oplus\gamma_n T(y_n)),p)\Big]\\
&\le d((1-\gamma_n)T(z_n)\oplus\gamma_n T(y_n),p)\\
&\le (1-\gamma_n)d(T(z_n),p)+\gamma_n d(T(y_n),p)\\
&\le (1-\gamma_n)ad(z_n,p)+(1-\gamma_n)bd(z_n,p)+\gamma_n ad(y_n,p)+\gamma_n bd(y_n,p)\\
&\le (1-\gamma_n)d(z_n,p)+\gamma_n d(y_n,p)\\
&\le d(x_n,p) \qquad (3.4)
\end{aligned}$$

Consequently, we have $d(x_{n+1},p)\le d(x_n,p)$ for all $n\ge 1$. This implies that $\{x_n\}$ is bounded and decreasing. Hence, $\lim_{n\to\infty} d(x_n,p)$ exists. Thus, $\{x_n\}$ is bounded.

Step 2. In this step, we will prove that $\lim_{n\to\infty} d(x_n,T(x_n))=0$. Without loss of generality, we may assume that

$$\mathbf{r} := \lim_{n\to\infty} d(x_n,p). \qquad (3.5)$$

Therefore,

$$\begin{aligned}
\limsup_{n\to\infty} d(T(x_n),p) &\le \limsup_{n\to\infty}\Big[ad(x_n,p)+bd(x_n,p)\Big]\\
&\le \limsup_{n\to\infty} d(x_n,p)\\
&\le \mathbf{r} \qquad (3.6)
\end{aligned}$$

from (3.2), we conclude that

$$\limsup_{n\to\infty} d(z_n,p)\le \mathbf{r} \qquad (3.7)$$

now, we can write

$$
\begin{aligned}
\mathbf{r} = \limsup_{n\to\infty} d(x_{n+1},p) &= \limsup_{n\to\infty} d(T((1-\gamma_n)T(z_n)\oplus\gamma_n T(y_n)),p)\\
&\le a\Big[d((1-\gamma_n)T(z_n)\oplus\gamma_n T(y_n)),p)\Big]\\
&\quad +b\Big[d((1-\gamma_n)T(z_n)\oplus\gamma_n T(y_n)),p)\Big]\\
&\le d((1-\gamma_n)T(z_n)\oplus\gamma_n T(y_n),p)\\
&\le (1-\gamma_n)d(T(z_n),p)+\gamma_n d(T(y_n),p)\\
&\le (1-\gamma_n)ad(z_n,p)\\
&\quad +(1-\gamma_n)bd(z_n,p)+\gamma_n ad(y_n,p)+\gamma_n bd(y_n,p)\\
&\le (1-\gamma_n)d(z_n,p)+\gamma_n d(y_n,p)\\
&\le (1-\gamma_n)d(z_n,p)+\gamma_n d(z_n,p)\\
&\le d(z_n,p),
\end{aligned}
$$

which implies that

$$
\mathbf{r} \le \limsup_{n\to\infty} d(z_n,p). \tag{3.8}
$$

From (3.7) and (3.8), we have

$$
\begin{aligned}
\mathbf{r} &= \limsup_{n\to\infty} d(z_n,p)\\
&= \limsup_{n\to\infty} d(T((1-\alpha_n)x_n\oplus\alpha_n T(x_n)),p).\\
&\le a\limsup_{n\to\infty}\Big[d((1-\alpha_n)x_n\oplus\alpha_n T(x_n),p)\Big]\\
&\quad +b\limsup_{n\to\infty}\Big[d((1-\alpha_n)x_n\oplus\alpha_n T(x_n),p)\Big]\\
&\le \limsup_{n\to\infty} d((1-\alpha_n)x_n\oplus\alpha_n T(x_n),p) \qquad (3.9)
\end{aligned}
$$

Also

$$\begin{aligned}
\limsup_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n T(x_n), p) &\le \limsup_{n\to\infty}\Big[(1-\alpha_n)d(x_n,p) + \alpha_n d(T(x_n),p)\Big] \\
&\le \limsup_{n\to\infty}\Big[(1-\alpha_n)d(x_n,p) \qquad (3.10)\\
&\qquad +\alpha_n a d((x_n),p) + \alpha_n b d((x_n),p)\Big] \\
&\le \limsup_{n\to\infty}\Big[(1-\alpha_n)d(x_n,p) + \alpha_n d((x_n),p)\Big] \\
&\le \limsup_{n\to\infty} d(x_n,p) = \mathbf{r} \qquad (3.11)
\end{aligned}$$

From (3.9) and (3.11), we have

$$\limsup_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n T(x_n), p) = \mathbf{r} \tag{3.12}$$

By using Lemma 2.2 with (3.5), (3.6) and (3.12), we have

$$\lim_{n\to\infty} d(x_n, T(x_n)) = 0. \tag{3.13}$$

Therefore, Step 2 is proved.

Step 3. Define

$$\Omega_{\triangle}(x_n) := \bigcup_{\{\nu_n\}\subseteq\{x_n\}} A(\{\nu_n\}) \subseteq Fix(T).$$

We claim that the sequence $\{x_n\}$ $\triangle$-converges to a fixed point of T and $\Omega_{\triangle}(x_n)$ consists of exactly one point. Assume that $\nu \in \Omega_{\triangle}(x_n)$. From the definition of $\Omega_{\triangle}(x_n)$, there is a subsequence $\{\nu_n\}$ of $\{x_n\}$ such that $A(\{\nu_n\}) = \{\nu\}$. From assertion $(\mathcal{A}_1)$ in Lemma 2.3, there exists a subsequence $\{\rho_n\}$ of $\{\nu_n\}$ such that $\triangle - \lim_{n\to\infty} \rho_n = \rho \in \mathcal{C}$. Using Lemma 2.4, we conclude that $\rho \in Fix(T)$. Since $\{d(\nu_n, \rho)\}$ converges, by assertion $(\mathcal{A}_2)$ in Lemma 2.3, we obtain $\nu = \rho$. Therefore, $\Omega_{\triangle}(x_n) \subseteq Fix(T)$. Finally, we show that $\Omega_{\triangle}(x_n)$ consists of exactly one point. Let $\{\nu_n\}$ be a subsequence of $\{x_n\}$ such that $A(\{\nu_n\}) = \{\nu\}$ and let $A(\{x_n\}) = \{x\}$. We have already seen that $\nu = \rho \in Fix(T)$. Since $\{d(x_n, \rho)\}$ converges, by assertion $(\mathcal{A}_3)$ in Lemma 2.3, we have $x = \rho \in Fix(T)$, that is, $\Omega_{\triangle}(x_n) = x$. This completes the proof. □

4. Numerical Experiments and Comparison

In this section, we supply a numerical example of a mean nonexpansive mapping satisfying the conditions of Theorem 3.1, and some numerical experiment results to explain the conclusion of our algorithm.

Example 4.1. Consider $X = \mathbb{R}$ with its usual metric, so X is also a complete CAT(0) space. Let $\mathcal{C} = [-1, 1]$ which clearly is a bounded closed convex subset of X. Define the mapping $T : \mathcal{C} \longrightarrow \mathcal{C}$ by

$$Tx = \begin{cases} \dfrac{x}{5} + \dfrac{5}{12} & x \in [-1, \frac{1}{2}); \\ \frac{x}{6} + \frac{5}{12} & x \in [\frac{1}{2}, 1]. \end{cases}$$

T is discontinuous at $x = 0.5$; consequently, T is neither nonexpansive nor contractive. Now, we prove that T is mean nonexpansive.

Case 1: $x, y \in [-1, \frac{1}{2})$. By the definition of T,

$$\begin{aligned} d(T(x), T(y)) &= \frac{1}{4} d(\frac{4}{5}x, \frac{4}{5}y) \\ &= \frac{1}{4} d(x - \frac{y}{5} + \frac{y}{5} - \frac{x}{5}, y - x + x - \frac{y}{5}) \\ &\leq d(x, y) + \frac{1}{4} d(-\frac{y}{5}, -x) + \frac{1}{4} d(\frac{y}{5}, x) + \frac{1}{4} d(-\frac{x}{5}, -\frac{y}{5}) \\ &\leq \frac{1}{4} d(x, y) + \frac{1}{2} d(x, T(y)) + \frac{1}{4} d(T(x), T(y)). \end{aligned}$$

This implies that $d(T(x), T(y)) \leq \frac{1}{3} d(x, y) + \frac{2}{3} d(x, T(y))$.

Case 2: $x \in [-1, \frac{1}{2}), y \in [\frac{1}{2}, 1]$. In this case, we have

$$\begin{aligned} d(T(x), T(y)) &= d(\frac{x}{5}, \frac{y}{5}) \\ &= d(\frac{x}{5} + \frac{T(y)}{5} - \frac{T(y)}{5}, \frac{y}{5} + \frac{T(x)}{5} - \frac{T(x)}{5}) \\ &\leq \frac{1}{5} d(x, T(x)) + \frac{1}{5} d(T(x), T(y)) + \frac{1}{5} d(y, T(y)) \\ &\leq \frac{1}{5} d(x, T(y)) + \frac{1}{5} d(T(x), T(y)) + \frac{1}{5} d(T(x), T(y)) \\ &\quad + \frac{1}{5} d(x, y) + \frac{1}{5} d(x, T(y)) \\ &= \frac{2}{5} d(x, T(y)) + \frac{2}{5} d(T(x), T(y)) + \frac{1}{5} d(x, y). \end{aligned}$$

This implies that $d(T(x), T(y)) \leq \frac{1}{3} d(x, y) + \frac{2}{3} d(x, T(y))$.

Case 3: $y \in [-1, \frac{1}{2}), x \in [\frac{1}{2}, 1]$. The argument is similar to the one in Case 2.

Case 4: $x, y \in [\frac{1}{2}, 1]$. The proof is the same as in Case 1.

Hence, T is mean nonexpansive by taking $a = \frac{1}{3}, b = \frac{2}{3}$.

Clearly, 0.5 is the only fixed point of the mapping T. Put $\alpha_n = \beta_n = \gamma_n = \dfrac{1}{n + 100}$. By using MATHEMATICA, we computed the iterates of the algorithm for two different initial points $x_1 = -0.9 \in [-1, 1]$ and $x_1 = 0.9 \in [-1, 1]$. Finally, using numerical experiments we compared the Zhou and Cui iteration process with our algorithm (see Table 4.1). Moreover, the convergence behavior of these algorithms is shown in Figure 4.1. We conclude that x_n converges to 0.5.

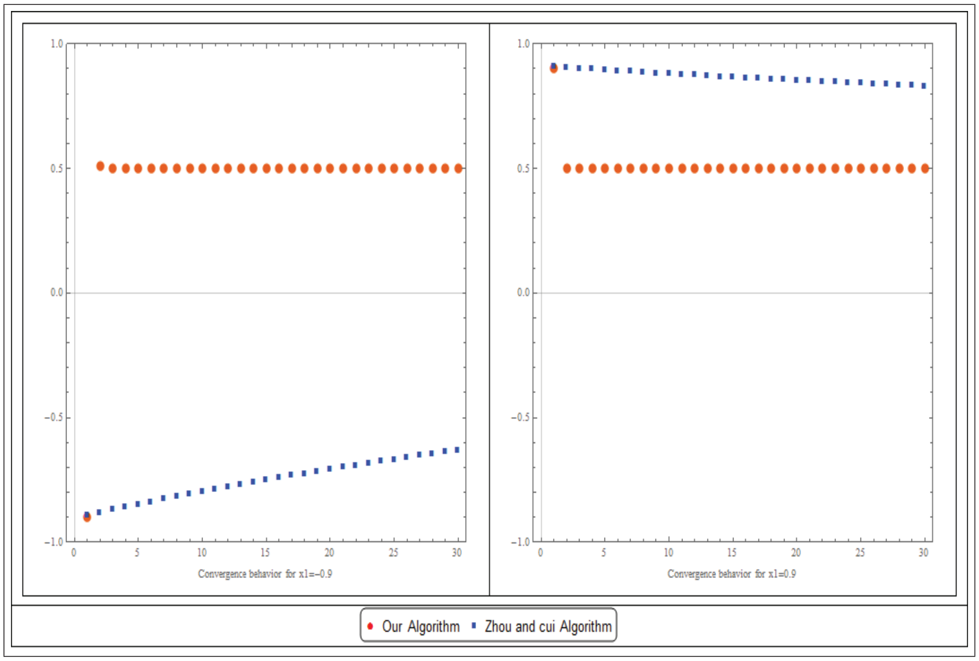

Figure 4.1: Convergence behaviors corresponding to $x_1 = -0.9$ and $x_1 = 0.9$ for 30 steps.

Example 4.2. Consider $X = \mathbb{R}^2$ equipped with the Euclidean norm. Let $x = (x_1, x_2) \in \mathbb{R}^2$, then the squared distance of x from the origin, O, is

$$\|x\|^2 = x_1^2 + x_2^2.$$

Consider $\mathcal{C} = [-1,1] \times [-1,1]$ which is a bounded, closed, and convex subset of X. We define the mapping $\mathrm{K} : \mathcal{C} \longrightarrow \mathcal{C}$ by

$$\mathrm{K}(x_1, x_2) := (\frac{1}{3}x_1, \frac{1}{3}x_2)$$

K is a nonexpansive mapping. This means that K is a mean nonexpansive mapping with $a = 1$ and $b = 0$. Clearly, zero is the only fixed point of the mapping K. In this case, our algorithm is the following:

$$(4.1)\qquad \begin{cases} x_{(1)} = (x_{(1)_1}, x_{(1)_2}) \in \mathcal{C}, \\ (z_{(n)_1}, z_{(n)_2}) = K((1-\alpha_n)(x_{(n)_1}, x_{(n)_2}) + \alpha_n K(x_{(n)_1}, x_{(n)_2})), \\ (y_{(n)_1}, y_{(n)_2}) = K((1-\beta_n)(z_{(n)_1}, z_{(n)_2}) + \beta_n K(z_{(n)_1}, z_{(n)_2})), \\ (x_{(n+1)_1}, x_{(n+1)_2}) = K((1-\gamma_n)K(z_{(n)_1}, z_{(n)_2}) + \gamma_n K(y_{(n)_2})). \end{cases}$$

Put $\alpha_n = \beta_n = \gamma_n = \dfrac{1}{n+100}$. By using MATHEMATICA, we computed the iterates of the algorithm (4.1) for $x_{(1)} = (\frac{1}{2}, \frac{1}{2}) \in \mathcal{C}$ for 500 steps. Finally, using numerical experiments we compared the Zhou and Cui iteration process with our algorithm (4.1). The convergence behavior of these algorithms is shown in Figure 4.2. The conclusion is that x_n converges to $(0,0)$.

Table 4.1: Numerical results corresponding to $x_1 = -0.9$ and $x_1 = 0.9$ for 30 steps.

Step	Our Algorithm for $x_1 = -0.9$	Zhou and Cui Algorithm for $x_1 = -0.9$	Our Algorithm for $x_1 = 0.9$	Zhou and Cui Algorithm for $x_1 = 0.9$
1	−0.9	−0.9	0.9	0.9
2	0.509646	−0.888724	0.501821	0.896694
3	0.500044	−0.877647	0.500008	0.893448
4	0.5	−0.866763	0.5	0.89026
5	0.5	−0.856069	0.5	0.887127
6	0.5	−0.845558	0.5	0.88405
7	0.5	−0.835227	0.5	0.881026
8	0.5	−0.483408	0.5	0.878054
9	0.5	−0.815081	0.5	0.875132
10	0.5	−0.805258	0.5	0.87226
11	0.5	−0.795596	0.5	0.869436
12	0.5	−0.786091	0.5	0.866658
13	0.5	−0.776739	0.5	0.863926
14	0.5	−0.767537	0.5	0.861238
15	0.5	−0.75848	0.5	0.858594
16	0.5	−0.749565	0.5	0.855991
17	0.5	−0.740788	0.5	0.85343
18	0.5	−0.732147	0.5	0.850909
19	0.5	−0.723638	0.5	0.848428
20	0.5	−0.715258	0.5	0.845984
21	0.5	−0.707003	0.5	0.843578
22	0.5	−0.698872	0.5	0.841209
23	0.5	−0.690861	0.5	0.838875
24	0.5	−0.682967	0.5	0.836576
25	0.5	−0.675188	0.5	0.834311
26	0.5	−0.667521	0.5	0.832079
27	0.5	−0.659964	0.5	0.82988
28	0.5	−0.652514	0.5	0.827713
29	0.5	−0.645169	0.5	0.825576
30	0.5	−0.637927	0.5	0.82347

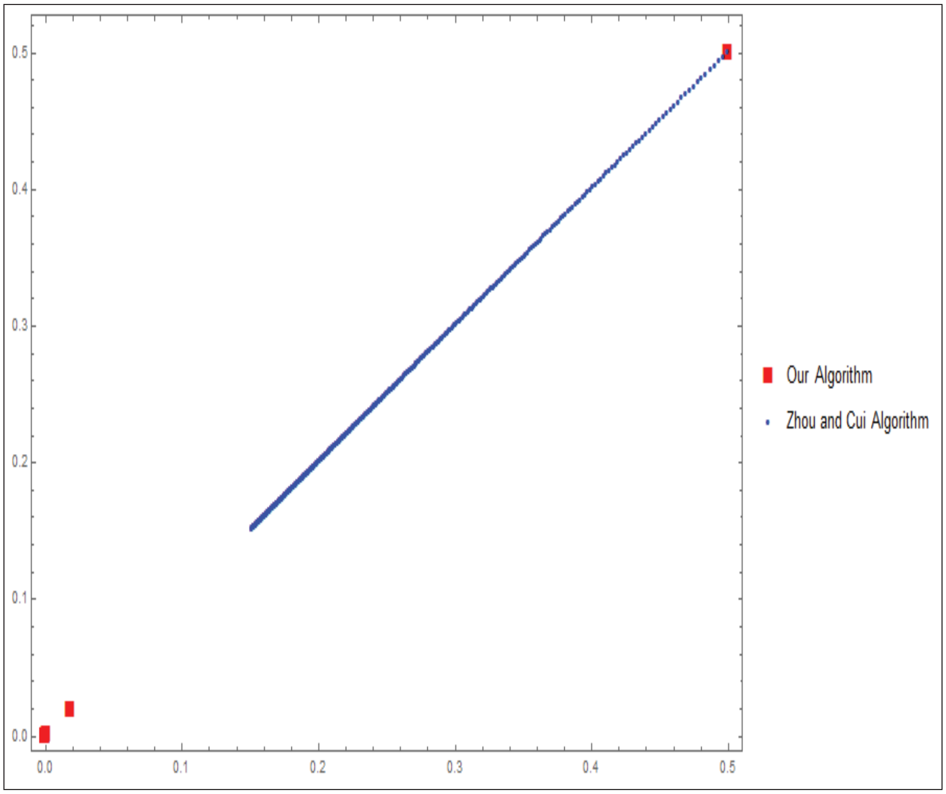

Figure 4.2: Convergence behaviors corresponding to $x_1 = (\frac{1}{2}, \frac{1}{2})$ for 500 steps.

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