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INSERTION OF A CONTRA-CONTINUOUS FUNCTION BETWEEN TWO COMPARABLE CONTRA-$\alpha-$CONTINUOUS (CONTRA-$C-$CONTINUOUS) FUNCTIONS *

Majid Mirmiran and Binesh Naderi

Abstract. Necessary and sufficient conditions in terms of lower cut sets are given for the insertion of a contra-continuous function between two comparable real-valued functions on topological spaces on which the kernel of sets is open.
Keywords: Insertion, Strong binary relation, $C-$open set, Semi-preopen set, $\alpha-$open set, Contra-continuous function, Lower cut set.

1. Introduction

The concept of a $C-$open set in a topological space was introduced by E. Hatir, T. Noiri and S. Yksel in [12]. The authors define a set S to be a $C-$*open* set if $S = U \cap A$, where U is open and A is semi-preclosed. A set S is a $C-$*closed* set if its complement (denoted by S^c) is a $C-$open set or equivalently if $S = U \cup A$, where U is closed and A is semi-preopen. The authors show that a subset of a topological space is open if and only if it is an $\alpha-$open set and a $C-$open set or equivalently a subset of a topological space is closed if and only if it is an $\alpha-$closed set and a $C-$closed set. This enables them to provide the following decomposition of continuity: a function is continuous if and only if it is $\alpha-$continuous and $C-$continuous or equivalently a function is contra-continuous if and only if it is contra-$\alpha-$continuous and contra-$C-$continuous.
Recall that a subset A of a topological space (X, τ) is called $\alpha-$*open* if A is the difference of an open and a nowhere dense subset of X. A set A is called $\alpha-$*closed* if its complement is $\alpha-$open or equivalently if A is the union of a closed and a nowhere dense set. Sets which are dense in some regular closed subspace are called *semi-preopen or* $\beta-$*open.* A set is *semi-preclosed or* $\beta-$*closed* if its complement is

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semi-preopen or $\beta-$open.

In [7] it was shown that a set A is $\beta-$open if and only if $A \subseteq Cl(Int(Cl(A)))$. A generalized class of closed sets was considered by Maki in [19]. He investigated the sets that can be represented as union of closed sets and called them $V-$sets. Complements of $V-$sets, i.e., sets that are intersection of open sets are called $\Lambda-$sets [19].

Recall that a real-valued function f defined on a topological space X is called $A-$continuous [23] if the preimage of every open subset of $\mathbb{R}$ belongs to A, where A is a collection of subsets of X. Most of the definitions of function used throughout this paper are consequences of the definition of $A-$continuity. However, for unknown concepts the reader may refer to [4, 11]. In the recent literature many topologists have focused their research in the direction of investigating different types of generalized continuity.

J. Dontchev in [5] introduced a new class of mappings called contra-continuity.S. Jafari and T. Noiri in [13, 14] exhibited and studied among others a new weaker form of this class of mappings called contra-$\alpha-$continuous. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 3, 8, 9, 10, 22].

Hence, a real-valued function f defined on a topological space X is called *contra-continuous* (resp. *contra-$C-$continuous* , *contra-$\alpha-$continuous*) if the preimage of every open subset of $\mathbb{R}$ is closed (resp. $C-$closed , $\alpha-$closed) in X[5].

Results of Katětov [15, 16] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient conditions for the insertion of a contra-continuous function between two comparable real-valued functions on such topological spaces that $\Lambda-$sets or kernel of sets are open [19].

If g and f are real-valued functions defined on a space X, we write $g \leq f$ (resp. $g < f$) in case $g(x) \leq f(x)$ (resp. $g(x) < f(x)$) for all x in X.

The following definitions are modifications of conditions considered in [17].

A property P, defined relative to a real-valued function on a topological space, is a *$cc-$property* provided that any constant function has property P and provided that the sum of a function with property P and any contra-continuous function also has property P. If P_1 and P_2 are $cc-$properties, the following terminology is used:(i) A space X has the *weak $cc-$insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f, g$ has property P_1 and f has property P_2, then there exists a contra-continuous function h such that $g \leq h \leq f$.(ii) A space X has the *$cc-$insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g < f, g$ has property P_1 and f has property P_2, then there exists a contra-continuous function h such that $g < h < f$.(iii) A space X has the *strong $cc-$insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f, g$ has property P_1 and f has property P_2, then there exists a contra-continuous function h such that $g \leq h \leq f$ and if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$.(iv) A space X has the *weakly $cc-$insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g < f, g$ has property P_1 , f has property P_2 and $f - g$ has property P_2, then there exists a

contra-continuous function h such that $g < h < f$.
In this paper, for a topological space whose $\Lambda-$sets or kernel of sets are open, is given a sufficient condition for the weak $cc-$insertion property. Also for a space with the weak $cc-$insertion property, we give a necessary and sufficient condition for the space to have the $cc-$insertion property. Several insertion theorems are obtained as corollaries of these results.

2. The Main Result

Before giving a sufficient condition for the insertability of a contra-continuous function, the necessary definitions and terminology are stated.
Definition 2.1. Let A be a subset of a topological space (X, τ). We define the subsets A^Λ and A^V as follows:
$A^\Lambda = \cap\{O : O \supseteq A, O \in (X, \tau)\}$ and $A^V = \cup\{F : F \subseteq A, F^c \in (X, \tau)\}$.
In [6, 18, 21], A^Λ is called the *kernel* of A.
The family of all $\alpha-$open, $\alpha-$closed, $C-$open and $C-$closed will be denoted by $\alpha O(X, \tau)$, $\alpha C(X, \tau)$, $CO(X, \tau)$ and $CC(X, \tau)$, respectively.
We define the subsets $\alpha(A^\Lambda), \alpha(A^V), C(A^\Lambda)$ and $C(A^V)$ as follows:
$\alpha(A^\Lambda) = \cap\{O : O \supseteq A, O \in \alpha O(X, \tau)\}$,
$\alpha(A^V) = \cup\{F : F \subseteq A, F \in \alpha C(X, \tau)\}$,
$C(A^\Lambda) = \cap\{O : O \supseteq A, O \in CO(X, \tau)\}$ and
$C(A^V) = \cup\{F : F \subseteq A, F \in CC(X, \tau)\}$.
$\alpha(A^\Lambda)$ (resp. $C(A^\Lambda)$) is called the $\alpha - kernel$ (resp. $C - kernel$) of A.
The following first two definitions are modifications of conditions considered in [15, 16].
Definition 2.2. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x \ \bar{\rho} \ y$ if and only if $y \ \rho \ v$ implies $x \ \rho \ v$ and $u \ \rho \ x$ implies $u \ \rho \ y$ for any u and v in S.
Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:
1) If $A_i \ \rho \ B_j$ for any $i \in \{1, \ldots, m\}$ and for any $j \in \{1, \ldots, n\}$, then there exists a set C in $P(X)$ such that $A_i \ \rho \ C$ and $C \ \rho \ B_j$ for any $i \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, n\}$.
2) If $A \subseteq B$, then $A \ \bar{\rho} \ B$.
3) If $A \ \rho \ B$, then $A^\Lambda \subseteq B$ and $A \subseteq B^V$.
The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:
Definition 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number ℓ, then $A(f, \ell)$ is called a *lower indefinite cut set* in the domain of f at the level ℓ.
We now give the following main result:
Theorem 2.1. Let g and f be real-valued functions on the topological space X, in which kernel sets are open, with $g \leq f$. If there exists a strong binary relation ρ on

the power set of X and if there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f,t_1)\ \rho\ A(g,t_2)$, then there exists a contra-continuous function h defined on X such that $g \leq h \leq f$.

Proof. Let g and f be real-valued functions defined on the X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f,t_1)\ \rho\ A(g,t_2)$.

Define functions F and G mapping the rational numbers $\mathbb{Q}$ into the power set of X by $F(t) = A(f,t)$ and $G(t) = A(g,t)$. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1)\ \bar{\rho}\ F(t_2), G(t_1)\ \bar{\rho}\ G(t_2)$, and $F(t_1)\ \rho\ G(t_2)$. By Lemmas 1 and 2 of [16] it follows that there exists a function H mapping $\mathbb{Q}$ into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1)\ \rho\ H(t_2), H(t_1)\ \rho\ H(t_2)$ and $H(t_1)\ \rho\ G(t_2)$.

For any x in X, let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If x is in $H(t)$ then x is in $G(t')$ for any $t' > t$; since x is in $G(t') = A(g,t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f,t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1,t_2) = H(t_2)^V \setminus H(t_1)^\Lambda$. Hence $h^{-1}(t_1,t_2)$ is closed in X, i.e., h is a contra-continuous function on X. ■

The above proof used the technique of theorem 1 in [15].

Theorem 2.2. Let P_1 and P_2 be $cc-$property and X be a space that satisfies the weak $cc-$insertion property for (P_1,P_2). Also assume that g and f are functions on X such that $g < f, g$ has property P_1 and f has property P_2.The space X has the $cc-$insertion property for (P_1,P_2) if and only if there exist lower cut sets $A(f-g, 3^{-n+1})$ and there exists a decreasing sequence $\{D_n\}$ of subsets of X with empty intersection and such that for each $n, X \setminus D_n$ and $A(f-g, 3^{-n+1})$ are completely separated by contra-continuous functions.

Proof. Theorem 2.1 of [20].■

3. Applications

The abbreviations $c\alpha c$ and cCc are used for contra-$\alpha-$continuous and contra-$C-$continuous, respectively.

Before stating the consequences of theorems 2.1, 2.2, we suppose that X is a topological space whose kernel sets are open.

Corollary 3.1. If for each pair of disjoint $\alpha-$open (resp. $C-$open) sets G_1, G_2 of X , there exist closed sets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then X has the weak $cc-$insertion property for $(c\alpha c, c\alpha c)$ (resp. (cCc, cCc)).

Proof. Let g and f be real-valued functions defined on X, such that f and g are $c\alpha c$ (resp. cCc), and $g \leq f$.If a binary relation ρ is defined by $A\ \rho\ B$ in case

$\alpha(A^\Lambda) \subseteq \alpha(B^V)$ (resp. $C(A^\Lambda) \subseteq C(B^V)$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f,t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g,t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is an $\alpha-$open (resp. $C-$open) set and since $\{x \in X : g(x) < t_2\}$ is an $\alpha-$closed (resp. $C-$closed) set, it follows that $\alpha(A(f,t_1)^\Lambda) \subseteq \alpha(A(g,t_2)^V)$ (resp. $C(A(f,t_1)^\Lambda) \subseteq C(A(g,t_2)^V)$). Hence $t_1 < t_2$ implies that $A(f,t_1)\ \rho\ A(g,t_2)$. The proof follows from Theorem 2.1. ■

Corollary 3.2. If for each pair of disjoint $\alpha-$open (resp. $C-$open) sets G_1, G_2, there exist closed sets F_1 and F_2 such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then every contra-$\alpha-$continuous (resp. contra-$C-$continuous) function is contra-continuous.

Proof. Let f be a real-valued contra-$\alpha-$continuous (resp. contra-$C-$continuous) function defined on X. Set $g = f$, then by Corollary 3.1, there exists a contra-continuous function h such that $g = h = f$.■

Corollary 3.3. If for each pair of disjoint $\alpha-$open (resp. $C-$open) sets G_1, G_2 of X , there exist closed sets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then X has the strong $cc-$insertion property for $(c\alpha c, c\alpha c)$ (resp. (cCc, cCc)).

Proof. Let g and f be real-valued functions defined on the X, such that f and g are $c\alpha c$ (resp. cCc), and $g \leq f$. Set $h = (f+g)/2$, thus $g \leq h \leq f$ and if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$. Also, by Corollary 3.2, since g and f are contra-continuous functions hence h is a contra-continuous function.■

Corollary 3.4. If for each pair of disjoint subsets G_1, G_2 of X , such that G_1 is $\alpha-$open and G_2 is $C-$open, there exist closed subsets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then X have the weak $cc-$insertion property for $(c\alpha c, cCc)$ and $(cCc, c\alpha c)$.

Proof. Let g and f be real-valued functions defined on X, such that g is $c\alpha c$ (resp. cCc) and f is cCc (resp. $c\alpha c$), with $g \leq f$.If a binary relation ρ is defined by $A\ \rho\ B$ in case $C(A^\Lambda) \subseteq \alpha(B^V)$ (resp. $\alpha(A^\Lambda) \subseteq C(B^V)$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f,t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g,t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a $C-$open (resp. $\alpha-$open) set and since $\{x \in X : g(x) < t_2\}$ is an $\alpha-$closed (resp. $C-$closed) set, it follows that $C(A(f,t_1)^\Lambda) \subseteq \alpha(A(g,t_2)^V)$ (resp. $\alpha(A(f,t_1)^\Lambda) \subseteq C(A(g,t_2)^V)$). Hence $t_1 < t_2$ implies that $A(f,t_1)\ \rho\ A(g,t_2)$. The proof follows from Theorem 2.1. ■

Before stating consequences of Theorem 2.2, we state and prove the necessary lemmas.

Lemma 3.1. The following conditions on the space X are equivalent:

(i) For each pair of disjoint subsets G_1, G_2 of X, such that G_1 is $\alpha-$open and G_2 is $C-$open, there exist closed subsets F_1, F_2 of X such that $G_1 \subseteq F_1, G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$.

(ii) If G is a $C-$open (resp. $\alpha-$open) subset of X which is contained in an $\alpha-$closed (resp. $C-$closed) subset F of X, then there exists a closed subset H of X such that $G \subseteq H \subseteq H^{\Lambda} \subseteq F$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $G \subseteq F$, where G and F are $C-$open (resp. $\alpha-$open) and $\alpha-$closed (resp. $C-$closed) subsets of X, respectively. Hence, F^c is an $\alpha-$open (resp. $C-$open) and $G \cap F^c = \varnothing$.

By (i) there exists two disjoint closed subsets F_1, F_2 such that $G \subseteq F_1$ and $F^c \subseteq F_2$. But

$$F^c \subseteq F_2 \Rightarrow F_2^c \subseteq F,$$

and

$$F_1 \cap F_2 = \varnothing \Rightarrow F_1 \subseteq F_2^c$$

hence

$$G \subseteq F_1 \subseteq F_2^c \subseteq F$$

and since F_2^c is an open subset containing F_1, we conclude that $F_1^{\Lambda} \subseteq F_2^c$, i.e.,

$$G \subseteq F_1 \subseteq F_1^{\Lambda} \subseteq F.$$

By setting $H = F_1$, condition (ii) holds.

(ii) $\Rightarrow$ (i) Suppose that G_1, G_2 are two disjoint subsets of X, such that G_1 is $\alpha-$open and G_2 is $C-$open.

This implies that $G_2 \subseteq G_1^c$ and G_1^c is an $\alpha-$closed subset of X. Hence by (ii) there exists a closed set H such that $G_2 \subseteq H \subseteq H^{\Lambda} \subseteq G_1^c$.

But

$$H \subseteq H^{\Lambda} \Rightarrow H \cap (H^{\Lambda})^c = \varnothing$$

and

$$H^{\Lambda} \subseteq G_1^c \Rightarrow G_1 \subseteq (H^{\Lambda})^c.$$

Furthermore, $(H^{\Lambda})^c$ is a closed subset of X. Hence $G_2 \subseteq H, G_1 \subseteq (H^{\Lambda})^c$ and $H \cap (H^{\Lambda})^c = \varnothing$. This means that condition (i) holds.$\blacksquare$

Lemma 3.2. Suppose that X is a topological space. If each pair of disjoint subsets G_1, G_2 of X, where G_1 is $\alpha-$open and G_2 is $C-$open, can be separated by closed subsets of X then there exists a contra-continuous function $h : X \to [0, 1]$ such that $h(G_2) = \{0\}$ and $h(G_1) = \{1\}$.

Proof. Suppose G_1 and G_2 are two disjoint subsets of X, where G_1 is $\alpha-$open and G_2 is $C-$open. Since $G_1 \cap G_2 = \varnothing$, hence $G_2 \subseteq G_1^c$. In particular, since G_1^c is an $\alpha-$closed subset of X containing the $C-$open subset G_2 of X,by Lemma 3.1, there exists a closed subset $H_{1/2}$ such that

$$G_2 \subseteq H_{1/2} \subseteq H_{1/2}^{\Lambda} \subseteq G_1^c.$$

Note that $H_{1/2}$ is also an $\alpha-$closed subset of X and contains G_2, and G_1^c is an $\alpha-$closed subset of X and contains the $C-$open subset $H_{1/2}^{\Lambda}$ of X. Hence, by Lemma 3.1, there exists closed subsets $H_{1/4}$ and $H_{3/4}$ such that

$$G_2 \subseteq H_{1/4} \subseteq H_{1/4}^{\Lambda} \subseteq H_{1/2} \subseteq H_{1/2}^{\Lambda} \subseteq H_{3/4} \subseteq H_{3/4}^{\Lambda} \subseteq G_1^c.$$

By continuing this method for every $t \in D$, where $D \subseteq [0,1]$ is the set of rational numbers that their denominators are exponents of 2, we obtain closed subsets H_t with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function h on X by $h(x) = \inf\{t : x \in H_t\}$ for $x \notin G_1$ and $h(x) = 1$ for $x \in G_1$.
Note that for every $x \in X, 0 \leq h(x) \leq 1$, i.e., h maps X into [0,1]. Also, we note that for any $t \in D, G_2 \subseteq H_t$; hence $h(G_2) = \{0\}$. Furthermore, by definition, $h(G_1) = \{1\}$. It remains only to prove that h is a contra-continuous function on X. For every $\alpha \in \mathbb{R}$, we have if $\alpha \leq 0$ then $\{x \in X : h(x) < \alpha\} = \varnothing$ and if $0 < \alpha$ then $\{x \in X : h(x) < \alpha\} = \cup\{H_t : t < \alpha\}$, hence, they are closed subsets of X. Similarly, if $\alpha < 0$ then $\{x \in X : h(x) > \alpha\} = X$ and if $0 \leq \alpha$ then $\{x \in X : h(x) > \alpha\} = \cup\{(H_t^\Lambda)^c : t > \alpha\}$ hence, every of them is a closed subset. Consequently h is a contra-continuous function. ■

Lemma 3.3. Suppose that X is a topological space such that every two disjoint $C-$open and $\alpha-$open subsets of X can be separated by closed subsets of X. The following conditions are equivalent:
(i) Every countable convering of $C-$closed (resp. $\alpha-$closed) subsets of X has a refinement consisting of $\alpha-$closed (resp. $C-$closed) subsets of X such that for every $x \in X$, there exists a closed subset of X containing x such that it intersects only finitely many members of the refinement.
(ii) Corresponding to every decreasing sequence $\{G_n\}$ of $C-$open (resp. $\alpha-$open) subsets of X with empty intersection there exists a decreasing sequence $\{F_n\}$ of $\alpha-$closed (resp. $C-$closed) subsets of X such that $\bigcap_{n=1}^{\infty} F_n = \varnothing$ and for every $n \in \mathbb{N}, G_n \subseteq F_n$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $\{G_n\}$ is a decreasing sequence of $C-$open (resp. $\alpha-$open) subsets of X with empty intersection. Then $\{G_n^c : n \in \mathbb{N}\}$ is a countable covering of $C-$closed (resp. $\alpha-$closed) subsets of X . By hypothesis (i) and Lemma 3.1, this covering has a refinement $\{V_n : n \in \mathbb{N}\}$ such that every V_n is a closed subset of X and $V_n^\Lambda \subseteq G_n^c$. By setting $F_n = (V_n^\Lambda)^c$, we obtain a decreasing sequence of closed subsets of X with the required properties.
(ii) $\Rightarrow$ (i) Now if $\{H_n : n \in \mathbb{N}\}$ is a countable covering of $C-$closed (resp. $\alpha-$closed) subsets of X, we set for $n \in \mathbb{N}, G_n = (\bigcup_{i=1}^{n} H_i)^c$. Then $\{G_n\}$ is a decreasing sequence of $C-$open (resp. $\alpha-$open) subsets of X with empty intersection. By (ii) there exists a decreasing sequence $\{F_n\}$ consisting of $\alpha-$closed (resp. $C-$closed) subsets of X such that $\bigcap_{n=1}^{\infty} F_n = \varnothing$ and for every $n \in \mathbb{N}, G_n \subseteq F_n$.Now we define the subsets W_n of X in the following manner:
W_1 is a closed subset of X such that $F_1^c \subseteq W_1$ and $W_1^\Lambda \cap G_1 = \varnothing$.
W_2 is a closed subset of X such that $W_1^\Lambda \cup F_2^c \subseteq W_2$ and $W_2^\Lambda \cap G_2 = \varnothing$, and so on. (By Lemma 3.1, W_n exists).
Then since $\{F_n^c : n \in \mathbb{N}\}$ is a covering for X, hence $\{W_n : n \in \mathbb{N}\}$ is a covering for X consisting of closed sets. Moreover, we have
(i) $W_n^\Lambda \subseteq W_{n+1}$
(ii) $F_n^c \subseteq W_n$
(iii) $W_n \subseteq \bigcup_{i=1}^{n} H_i$.
Now setting $S_1 = W_1$ and for $n \geq 2$, we set $S_n = W_{n+1} \setminus W_{n-1}^\Lambda$.
Then since $W_{n-1}^\Lambda \subseteq W_n$ and $S_n \supseteq W_{n+1} \setminus W_n$, it follows that $\{S_n : n \in \mathbb{N}\}$ consists

of closed sets and covers X. Furthermore, $S_i \cap S_j \neq \varnothing$ if and only if $|i-j| \leq 1$. Finally, consider the following sets:

$$\begin{array}{llllll}
S_1 \cap H_1, & S_1 \cap H_2 & & & & \\
S_2 \cap H_1, & S_2 \cap H_2, & S_2 \cap H_3 & & & \\
S_3 \cap H_1, & S_3 \cap H_2, & S_3 \cap H_3, & S_3 \cap H_4 & & \\
\vdots & & & & & \\
S_i \cap H_1, & S_i \cap H_2, & S_i \cap H_3, & S_i \cap H_4, & \cdots, & S_i \cap H_{i+1} \\
\vdots & & & & &
\end{array}$$

These sets are closed sets, cover X and refine $\{H_n : n \in \mathbb{N}\}$. In addition, $S_i \cap H_j$ can intersect at most the sets in its row, immediately above, or immediately below row.

Hence if $x \in X$ and $x \in S_n \cap H_m$, then $S_n \cap H_m$ is a closed set containing x that intersects at most finitely many of sets $S_i \cap H_j$. Consequently, $\{S_i \cap H_j : i \in \mathbb{N}, j = 1, \ldots, i+1\}$ refines $\{H_n : n \in \mathbb{N}\}$ such that its elements are closed sets, and for every point in X we can find a closed set containing the point that intersects only finitely many elements of that refinement.■

Corollary 3.5. If every two disjoint $C-$open and $\alpha-$open subsets of X can be separated by closed subsets of X and, in addition, every countable covering of $C-$closed (resp. $\alpha-$closed) subsets of X has a refinement that consists of $\alpha-$closed (resp. $C-$closed) subsets of X such that for every point of X we can find a closed subset containing that point such that it intersects only a finite number of refining members then X has the weakly $cc-$insertion property for $(c\alpha c, cCc)$ (resp. $(cCc, c\alpha c)$).

Proof. Since every two disjoint $C-$open and $\alpha-$open sets can be separated by closed subsets of X, therefore by Corollary 3.4, X has the weak $cc-$insertion property for $(c\alpha c, cCc)$ and $(cCc, c\alpha c)$. Now suppose that f and g are real-valued functions on X with $g < f$, such that g is $c\alpha c$ (resp. cCc), f is cCc (resp. $c\alpha c$) and $f-g$ is cCc (resp. $c\alpha c$). For every $n \in \mathbb{N}$, set

$$A(f-g, 3^{-n+1}) = \{x \in X : (f-g)(x) \leq 3^{-n+1}\}.$$

Since $f-g$ is cCc (resp. $c\alpha c$), hence $A(f-g, 3^{-n+1})$ is a $C-$open (resp. $\alpha-$open) subset of X. Consequently, $\{A(f-g, 3^{-n+1})\}$ is a decreasing sequence of $C-$open (resp. $\alpha-$open) subsets of X and furthermore since $0 < f-g$, it follows that $\bigcap_{n=1}^{\infty} A(f-g, 3^{-n+1}) = \varnothing$. Now by Lemma 3.3, there exists a decreasing sequence $\{D_n\}$ of $\alpha-$closed (resp. $C-$closed) subsets of X such that $A(f-g, 3^{-n+1}) \subseteq D_n$ and $\bigcap_{n=1}^{\infty} D_n = \varnothing$. But by Lemma 3.2, the pair $A(f-g, 3^{-n+1})$ and $X \setminus D_n$ of $C-$open (resp. $\alpha-$open) and $\alpha-$open (resp. $C-$open) subsets of X can be completely separated by contra-continuous functions. Hence by Theorem 2.2, there exists a contra-continuous function h defined on X such that $g < h < f$, i.e., X has the weakly $cc-$insertion property for $(c\alpha c, cCc)$ (resp. $(cCc, c\alpha c)$).■.

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Majid Mirmiran
Department of Mathematics
University of Isfahan
Isfahan 81746-73441, Iran
mirmir@sci.ui.ac.ir

Binesh Naderi
School of Management and Medical Information
Medical University of Isfahan, Iran
naderi@mng.mui.ac.ir