FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 33, No 4 (2018), 505–516
https://doi.org/10.22190/FUMI1804505S

ITERATIONS FOR APPROXIMATING LIMIT REPRESENTATIONS OF GENERALIZED INVERSES

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Abstract. Our underlying motivation is the iterative method for the implementation of the limit representation of the Moore-Penrose inverse $\lim_{\alpha\to 0}(\alpha I + A^*A)^{-1}A^*$ from [Žukovski, Lipcer, On recurent computation of normal solutions of linear algebraic equations, Ž. Vicisl. Mat. i Mat. Fiz. 12 (1972), 843–857] and [Žukovski, Lipcer, On computation pseudoinverse matrices, Ž. Vicisl. Mat. i Mat. Fiz. 15 (1975), 489–492]. The iterative process for the implementation of the general limit formula $\lim_{\alpha\to 0}(\alpha I + R^*S)^{-1}R^*$ was defined in [P.S. Stanimirović, Limit representations of generalized inverses and related methods, Appl. Math. Comput. 103 (1999), 51–68]. In this paper we develop an improvement of this iterative process. The iterative method defined in such a way is able to produce the result in a predefined number of iterative steps. Convergence properties of defined iterations are further investigated.

Keywords. Generalized inverses; Moore-Penrose inverse; Drazin inverse; limit representation; Leverrier-Faddeev algorithm.

1. Introduction

We use the following notation. $\mathbb{C}^{m\times n}$: the set of $m \times n$ complex matrices; $\mathbb{C}_r^{m\times n}$ is the set of rank r: $\mathbb{C}_r^{m\times n} = \{X \in \mathbb{C}^{m\times n} : \text{rank}(X) = r\}$; $\mathcal{O}$ (resp. $\vec{0}$): the zero matrix of an appropriate order (resp. the zero vector); I_m: identity matrix of the order m; $\mathcal{R}(A)$ and $\mathcal{N}(A)$: the range and the null space of A; $\text{Tr}\,(A)$: the trace of A.

For any matrix $A \in \mathbb{C}^{m\times n}$ consider the following equations in X:

$$(1) \quad AXA = A, \quad (2) \quad XAX = X, \quad (3) \quad (AX)^* = AX, \quad (4) \quad (XA)^* = XA$$

and if $m = n$, also

$$(5) \quad AX = XA \quad (1^k) \quad A^{k+1}X = A^k.$$

Received December 20, 2018.; Accepted January 22, 2019.
2010 *Mathematics Subject Classification.* Primary 15A09; Secondary 15A23

For a sequence $\mathcal{S}$ of $\{1,2,3,4,5,1^k\}$ the set of matrices obeying the equations represented in $\mathcal{S}$ is denoted by $A\{\mathcal{S}\}$. A matrix from $A\{\mathcal{S}\}$ is called an $\mathcal{S}$-inverse of A and denoted by $A^{(\mathcal{S})}$. If X satisfies (1) and (2), it is said to be a reflexive g-inverse of A, whereas $X = A^\dagger$ is said to be the Moore-Penrose inverse of A if it satisfies (1)-(4). Also, A_L^{-1} (resp. A_R^{-1}) denote an arbitrary left (resp. right) inverse of A. The group inverse $A^\#$ is the unique $\{1,2,5\}$ inverse of A, and exists if and only if $\mathrm{ind}(A) = \min\{k:\ \mathrm{rank}(A^{k+1}) = \mathrm{rank}(A^k)\} = 1$. A matrix $G = A^D$ is said to be the Drazin inverse of A if (1^k) (for some positive integer k), (2) and (5) are satisfied.

Let there be given invertible matrices M and N of the order m and n, respectively. For any $m \times n$ matrix A, the weighted Moore-Penrose inverse of A is the unique solution $X = A^\dagger_{M,N}$ of the matrix equations (1), (2) and the following equations in X:

$$(3M) \qquad (MAX)^* = MAX \qquad\qquad (4N) \qquad (XA)^*N = NXA.$$

the next is valid for a rectangular matrix A [14]:

$$(1.1) \qquad A^\dagger = A^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)}, \quad A^\dagger_{M,N} = A^{(2)}_{\mathcal{R}(A^\sharp),\mathcal{N}(A^\sharp)}, \quad A^\sharp = N^{-1}A^*M,$$

where M, N are positive definite matrices. For a given square matrix A the next identities are satisfied:

$$(1.2) \qquad A^{\mathrm{D}} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^k)}, \ k \geq \mathrm{ind}(A), \quad A^\# = A^{(2)}_{\mathcal{R}(A),\mathcal{N}(A)}.$$

The core inverse of a complex matrix was originated by Baksalary and Trenkler in [1]. A matrix $A^{\oplus} \in \mathbb{C}^{n\times n}$ satisfying

$$AA^{\oplus} = P_{\mathcal{R}(A)} \text{ and } \mathcal{R}(A^{\oplus}) \subseteq \mathcal{R}(A)$$

is called the core inverse of A.

Manjunatha Prasad and Mohana in [10] discovered the core-EP inverse. A matrix X, denoted by $A^{\dagger}$, is called the core-EP inverse of $A \in \mathbb{C}^{n\times n}$ if it satisfies

$$XAX = X, \quad \mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k).$$

The following results can be derived using results from [9]:

$$A^{\dagger} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*)}, \qquad A^{\oplus} = A^{(2)}_{\mathcal{R}(A),\mathcal{N}(A^*)}.$$

The remainder of the manuscript is organized as follows. In order to complete the presentation and describe our motivation, limit representations of main generalized inverses are surveyed in Section 2.. Some additional results about the convergence of the iterations proposed in [11] are presented in Section 3.. An efficient method for the improved implementation of defined iterations is considered in Section 4.. An illustrative numerical example is presented in Section 5..

2. Survey of limit representations

Limit representations of main generalized inverses is restated in Proposition 2.1. The inverse of a nonsingular matrix A can be characterized in terms of the limiting process

$$A^{-1} = \lim_{\alpha\to 0}(\alpha I + A)^{-1}, \tag{2.1}$$

wherein it is assumed that $-\alpha \notin \sigma(A)$ and $\sigma(A)$ stands for the set of all eigenvalues of A.

Proposition 2.1. **(a)** [3] *Limit representation of the Moore-Penrose inverse of a matrix* $A \in \mathbb{C}^{m\times n}$ *is equal to*

$$A^{\dagger} = \lim_{\alpha\to 0} \left(\alpha I_n + A^* A\right)^{-1} A^*. \tag{2.2}$$

(b) [7] *Limit representation of the Drazin inverse of a matrix* $A \in \mathbb{C}^{n\times n}$ *whose index is* k *can be expressed as the limit*

$$A^{\mathrm{D}} = \lim_{\alpha\to 0} \left(\alpha I_n + A^{l+1}\right)^{-1} A^l, \quad l \geqslant k. \tag{2.3}$$

(c) [13] *Let* $A \in \mathbb{C}^{m\times n}$ *be of rank* r*, let* T *be a subspace of* $\mathbb{C}^n$ *of dimension* $s \le r$*, and let* S *be a subspace of* $\mathbb{C}^m$ *of dimension* $m - s$*. In addition, suppose that* $G \in \mathbb{C}^{n\times m}$ *satisfies* $\mathcal{R}(G) = T$ *and* $\mathcal{N}(G) = S$*. In the case of the existence,* $A^{(2)}_{T,S}$ *is defined by the limit representation*

$$A^{(2)}_{T,S} = \lim_{\alpha\to 0} \left(GA + \alpha I\right)^{-1} G. \tag{2.4}$$

(d)[16] *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\mathrm{ind}(A) = 1$*. Then*

$$A^{\#\!\!\!\!\bigcirc} = \lim_{\alpha\to 0} AA^*(A^2A^* + \alpha I)^{-1} = \lim_{\alpha\to 0}(AA^*A + \alpha I)^{-1}AA^*. \tag{2.5}$$

$$A^{\#\!\!\!\!\bigcirc} = \lim_{\alpha\to 0} A(A^2)^*(A^2(A^2)^* + \alpha I)^{-1} = \lim_{\alpha\to 0}(A(A^2)^*A + \alpha I)^{-1}A(A^2)^*. \tag{2.6}$$

(e) [16] *Let* $A \in \mathbb{C}^{n\times n}$ *and* $\mathrm{ind}(A) = k$*. Then*

$$A^{\dagger\!\!\!\!\bigcirc} = \lim_{\alpha\to 0} A^k(A^k)^*(A^{k+1}(A^k)^* + \alpha I)^{-1} = \lim_{\alpha\to 0}(A^k(A^k)^*A + \alpha I)^{-1}A^k(A^k)^*. \tag{2.7}$$

The limit representations of the outer inverse in Banach space were investigated in [6].

The following additional notation will be used in this section.

a_t, $\quad t = 1,\dots,m$: tth row of $A \in \mathbb{C}^{m\times n}$; $A_{\underline{t}} = \begin{bmatrix} a_1 \\ \dots \\ a_t \end{bmatrix}$, $t = 1,\dots,m$: the $t \times n$ submatrix which contains the first t rows of $A \in \mathbb{C}^{m\times n}$; $y_{\underline{t}} = A_{\underline{t}}x$, and specially $y_t = a_t x$, $\quad t = 1,\dots,m$;

Our idea in the present paper can be described in three steps. First step is an iterative method for the implementation of the limit representation of the Moore-Penrose inverse $\lim_{\alpha\to 0}(\alpha I + A^*A)^{-1}A^*$ from [17, 18]. Another step is the iterative process for the implementation of the general limit formula $\lim_{\alpha\to 0}(\alpha I + R^*S)^{-1}R^*$, originated in [11]. In this paper we develop an improvement of this iterative process. Detailed description is given in the rest of this section.

In Proposition 2.2 and Proposition 2.3 we restate known iterative methods from [17, 18].

Proposition 2.2. (Žukovski, Lipcer 1972) [17] *For a given $m\times n$ complex matrix A and given $m\times 1$ complex vector y, the solution of the iterative process*

$$
\begin{aligned}
&\gamma_{t+1}^{\alpha} = \gamma_t^{\alpha} - \frac{\gamma_t^{\alpha} a_{t+1}^* a_{t+1}\gamma_t^{\alpha}}{\alpha + a_{t+1}\gamma_t^{\alpha} a_{t+1}^*}, \quad \gamma_0^{\alpha} = I_n, \quad \alpha > 0, \\
&x_{t+1}^{\alpha} = x_t^{\alpha} + \frac{\gamma_t^{\alpha} a_{t+1}^*}{\alpha + a_{t+1}\gamma_t^{\alpha} a_{t+1}^*}\left(y_{t+1} - a_{t+1}x_t^{\alpha}\right), \quad x_0^{\alpha} = 0, \\
&t = 0,\dots,m-1
\end{aligned} \tag{2.8}
$$

is given by

$$
\begin{aligned}
x_t^{\alpha} &= \left(\alpha I_n + A_{\underline{t}}{}^* A_{\underline{t}}\right)^{-1} A_{\underline{t}}{}^* y_{\underline{t}} \\
\gamma_t^{\alpha} &= \left(\alpha I_n + A_{\underline{t}}{}^* A_{\underline{t}}\right)^{-1}\alpha, \quad t = 1,\dots,m.
\end{aligned}
$$

Proposition 2.3. (Žukovski, Lipcer 1975) [18] *Let $A \in \mathbb{C}^{m\times n}$. If the rows of the unit matrix I_m are denoted by i_t, $t = 1,\dots,m$, then the following iterative method*

$$
\begin{aligned}
&\gamma_{t+1}^{\alpha} = \gamma_t^{\alpha} - \frac{\gamma_t^{\alpha} a_{t+1}^* a_{t+1}\gamma_t^{\alpha}}{\alpha + a_{t+1}\gamma_t^{\alpha} a_{t+1}^*}, \quad \gamma_0^{\alpha} = I_n, \quad \alpha > 0, \\
&X_{t+1}^{\alpha} = X_t^{\alpha} + \frac{\gamma_t^{\alpha} a_{t+1}^*}{\alpha + a_{t+1}\gamma_t^{\alpha} a_{t+1}^*}\left(i_{t+1} - a_{t+1}X_t^{\alpha}\right), \quad X_0^{\alpha} = \mathcal{O} \in \mathbb{C}^{n\times m} \\
&t = 0,\dots,m-1
\end{aligned} \tag{2.9}
$$

produces the resulting matrices

$$
X_m^{\alpha} = \left(\alpha I_n + A^*A\right)^{-1}A^*, \quad \gamma_m^{\alpha} = \left(\alpha I_n + A^*A\right)^{-1}\alpha.
$$

Of special interest are the limits $\lim_{\alpha\to 0} X_m^{\alpha} = A^{\dagger}$ [2] and $\lim_{\alpha\to 0}\gamma_m^{\alpha} = I_n - A^{\dagger}A$ [17].

An interesting computational scheme was proposed in [17]. This scheme ensures indirect decreasing of the values for α: after computation of the values γ_i^{α} and X_i^{α}, $i = 1,\dots,m$ by means of (2.9), compute γ_i^{α} and X_i^{α}, $i > t$, by means of the rows $a_{m+1} = a_1,\dots,a_{2m} = a_m,\dots$ and the numbers $y_{m+1} = y_1,\dots,y_{2m} = y_m,\dots$ In this case is

$$
X_{mN}^{\alpha} = X_m^{\alpha/N}, \quad \gamma_{mN}^{\alpha} = \gamma_m^{\alpha/N}, \quad N = 1,2,\dots \tag{2.10}
$$

where X_{mN}^{α} and γ_{mN}^{α} are defined by

$$X_{mN}^{\alpha} = \left(\alpha I_n + A_{mN}^* A_{mN}\right)^{-1} A_{mN}^*, \quad \gamma_{mN}^{\alpha} = \left(\alpha I_n + A_{mN}^* A_{mN}\right)^{-1} \alpha$$

and A_{mN} is the block matrix which consists of N blocks of $A_m = A$:

$$A_{mN} = \begin{bmatrix} A_m \\ \dots \\ A_m \end{bmatrix} = \begin{bmatrix} A \\ \dots \\ A \end{bmatrix}.$$

The main result in [11] was an approximate method for computing generalized inverses and different matrix expressions involving generalized inverses which are determined by the limit expressions

$$\lim_{\alpha \to 0} \left(\alpha I_q + R^* S\right)^{-1} R^*, \tag{2.11}$$

$$\lim_{\alpha \to 0} \left(\alpha I_q + R^* S\right)^{-1} \alpha, \tag{2.12}$$

where R and S are two arbitrary $p \times q$ complex matrices.

For a given matrix $A \in \mathbb{C}_r^{m \times n}$, in the case $R = S = A$ we obtain the iterative method (2.9) for computing the Moore-Penrose inverse. In the case $m = n$, $R^* = A^l$, $l \geqslant \operatorname{ind}(A)$, $S = A$, we construct an iterative process for implementation of the limit representation (2.3) for computing the Drazin inverse.

The following result from [11] generalizes the iterative process (2.8).

Proposition 2.4. (Stanimirović 1999) [11] *Let given two arbitrary $p \times q$ complex matrices R and S and $p \times 1$ complex vector y. If the rows of the matrices R and S are denoted by r_u and s_u, respectively, $u = 1, \dots, p$, and r_u^* denotes conjugate and transpose of the vector r_u, then the following iterative sequences*

$$\begin{aligned} \gamma_{t+1}^{\alpha} &= \gamma_t^{\alpha} - \frac{\gamma_t^{\alpha} r_{t+1}^* s_{t+1} \gamma_t^{\alpha}}{\alpha + s_{t+1} \gamma_t^{\alpha} r_{t+1}^*}, \quad \gamma_0^{\alpha} = I_q, \quad \alpha > 0, \\ x_{t+1}^{\alpha} &= x_t^{\alpha} + \frac{\gamma_t^{\alpha} r_{t+1}^*}{\alpha + s_{t+1} \gamma_t^{\alpha} r_{t+1}^*} \left(y_{t+1} - s_{t+1} x_t^{\alpha}\right), \quad x_0^{\alpha} = \vec{0}, \\ t &= 0, \dots, p-1 \end{aligned} \tag{2.13}$$

exist if and only if

$$\alpha + s_{t+1} \gamma_t^{\alpha} r_{t+1}^* \neq \vec{0}, \quad t = 0, \dots, p-1. \tag{2.14}$$

In this case, (2.13) *produces the following values:*

$$\begin{aligned} \gamma_t^{\alpha} &= \left(\alpha I_q + R_{\underline{t}}^* S_{\underline{t}}\right)^{-1} \alpha \\ x_t^{\alpha} &= \left(\alpha I_q + R_{\underline{t}}^* S_{\underline{t}}\right)^{-1} R_{\underline{t}}^* y_{\underline{t}}, \quad t = 1, \dots, p, \end{aligned} \tag{2.15}$$

where $R_{\underline{t}}^*$ is $q \times t$ matrix, equal to the conjugate and transpose of the submatrix $R_{\underline{t}}$ of R.

An approximate method for the implementation of the limit formula (2.11) and its convergence properties were investigated in [11].

Proposition 2.5. (Stanimirović 1999) [11] *Consider $m \times n$ complex matrix A and two $p \times q$ complex matrices R and S, whose rows are denoted by r_t and s_t, $t = 1, \dots, p$, respectively. If the rows of the unit matrix I_p are denoted by i_t, $t = 1, \dots, p$, then the iterations*

$$\begin{aligned}
&\gamma_{t+1}^{\alpha} = \gamma_t^{\alpha} - \frac{\gamma_t^{\alpha} r_{t+1}^* s_{t+1} \gamma_t^{\alpha}}{\alpha + s_{t+1}\gamma_t^{\alpha} r_{t+1}^*}, \quad \gamma_0^{\alpha} = I_q, \quad \alpha > 0,\\
&X_{t+1}^{\alpha} = X_t^{\alpha} + \frac{\gamma_t^{\alpha} r_{t+1}^*}{\alpha + s_{t+1}\gamma_t^{\alpha} r_{t+1}^*}\left(i_{t+1} - s_{t+1}X_t^{\alpha}\right), \; X_0^{\alpha} = \mathcal{O} \in \mathbb{C}^{q\times p},\\
&t = 0, \dots, p-1
\end{aligned} \tag{2.16}$$

converge if and only if

$$\alpha + s_{t+1}\gamma_t^{\alpha} r_{t+1}^* \neq 0, \quad t = 0, \dots, p-1$$

and the limits $\lim\limits_{\alpha\to 0} X_p^{\alpha}$, $\lim\limits_{\alpha\to 0} \gamma_p^{\alpha}$ produce the following results:

$$\lim_{\alpha\to 0} X_p^{\alpha} = \lim_{\alpha\to 0}(\alpha I_q + R^*S)^{-1}R^*, \quad \lim_{\alpha\to 0} \gamma_p^{\alpha} = I_q - \lim_{\alpha\to 0}(\alpha I_q + R^*S)^{-1}R^*S. \tag{2.17}$$

(i) *In the case $p=m$, $q=n$, $R=S=A$ we get*

$$\lim_{\alpha\to 0} X_m^{\alpha} = A^{\dagger}, \quad \lim_{\alpha\to 0} \gamma_m^{\alpha} = I_n - A^{\dagger}A.$$

(ii) *If A is $n \times n$ matrix, selecting the values $p = q = n$, $R^* = A^l$, $l \geqslant \mathrm{ind}(A)$, $S = A$, we obtain*

$$\lim_{\alpha\to 0} X_n^{\alpha} = A^{\mathrm{D}}, \quad \lim_{\alpha\to 0} \gamma_n^{\alpha} = I_n - A^{\mathrm{D}}A.$$

(iii) *In the case $p > q = \mathrm{rank}(S)$, for arbitrary $R \in \mathbb{C}_q^{p\times q}$ such that R^*S is invertible, we get $\lim\limits_{\alpha\to 0} X_p^{\alpha} = S_L^{-1}$.*

(iv) *Consider the case $q > p = \mathrm{rank}(S)$ and an arbitrary matrix $R \in \mathbb{C}^{p\times q}$ such that SR^* is invertible. Then $\lim\limits_{\alpha\to 0} X_p^{\alpha} = S_R^{-1}$.*

(v) *Selection $S = R \in \mathbb{C}^{p\times q}$ in* (2.16) *implies*

$$\lim_{\alpha\to 0} X_p^{\alpha} = R^{\dagger}, \quad \lim_{\alpha\to 0} \gamma_p^{\alpha} = I_q - R^{\dagger}R.$$

(vi) *For $A \in \mathbb{C}^{n\times n}$, in the case $n=p=q$, $R^* = A^k$, $S = I_n$, the limit value $\lim\limits_{\alpha\to 0} X_n^{\alpha}$ exists if $\mathrm{ind}(A)=k$, in which case $\lim\limits_{\alpha\to 0} X_n^{\alpha} = AA^{\mathrm{D}}$.*

(vii) *If $A\in\mathbb{C}_r^{m\times n}$, $p=q=m=n$, $R^*=\alpha^k I_n$, $S=\alpha^{-k}I_n$, $k=\mathrm{ind}(A)>0$, then*

$$\lim_{\alpha\to 0} X_n^{\alpha} = (-1)^{k-1}(I - AA^{\mathrm{D}})A^{k-1}.$$

3. Further results on the convergence

Some further results about the convergence with respect to Proposition 3.1.

Theorem 3.1. *Let us observe $m \times n$ complex matrix A and two $p \times q$ complex matrices R and S with rows r_t and s_t, $t = 1, \dots, p$, respectively. If the rows of the unit matrix I_p are denoted by i_t, $t = 1, \dots, p$, then the iterations* (2.16) *converge if and only if*

$$\alpha + s_{t+1}\gamma_t^{\alpha} r_{t+1}^* \neq 0, \quad t = 0, \dots, p-1.$$

In this case, the limits $\lim\limits_{\alpha\to 0} X_p^{\alpha}$, $\lim\limits_{\alpha\to 0} \gamma_p^{\alpha}$ produce the following results:

$$\lim_{\alpha\to 0} X_p^{\alpha} = \lim_{\alpha\to 0}(\alpha I_q + R^* S)^{-1} R^*, \quad \lim_{\alpha\to 0} \gamma_p^{\alpha} = I_q - \lim_{\alpha\to 0}(\alpha I_q + R^* S)^{-1} R^* S.$$

(viii) *If A is of rank r, T is a subspace of $\mathbb{C}^n$ of dimension $s \le r$, and S be a subspace of $\mathbb{C}^m$ of dimension $m - s$. If $G \in \mathbb{C}^{n\times m}$ satisfies $\mathcal{R}(G) = T$ and $\mathcal{N}(G) = S$ and* $\operatorname{rank}(GA) = \operatorname{rank}(G)$, *then $A^{(2)}_{\mathcal{R}(G),\mathcal{N}(G)}$ then in the case $p = m$, $q = n$, $R = G^*$, $S = A$ we get*

$$\lim_{\alpha\to 0} X_m^{\alpha} = A^{(2)}_{\mathcal{R}(G),\mathcal{N}(G)}, \quad \lim_{\alpha\to 0} \gamma_m^{\alpha} = I_n - A^{(2)}_{\mathcal{R}(G),\mathcal{N}(G)} A.$$

(ix) *If $A \in \mathbb{C}^{n\times n}$ of index* $\operatorname{ind}(A) = 1$, *the selected values $p = q = n$, $R = A^* A$, $S = A$ initiate*

$$\lim_{\alpha\to 0} X_n^{\alpha} = A^{\oplus}, \quad \lim_{\alpha\to 0} \gamma_n^{\alpha} = I_n - A^{\oplus} A.$$

(x) *If $A \in \mathbb{C}^{n\times n}$ of index* $\operatorname{ind}(A) = k$, *the selected values $p = q = n$, $R = (A^k)^* A^k$, $S = A$ initiate*

$$\lim_{\alpha\to 0} X_n^{\alpha} = A^{\textcircled{\dagger}}, \quad \lim_{\alpha\to 0} \gamma_n^{\alpha} = I_n - A^{\textcircled{\dagger}} A.$$

Proof. The proof can be verified using (2.17) in conjunction with (2.4), (2.5) and (2.6). □

4. An improved implementation

In this paper we propose an improvement of the iterative processes (2.13), (2.15) and (2.16). According to the improvement, these iterations can converge in an arbitrary prescribed number of iterations. If b is the required number of iterations , and integers c, d are defined as $c = Quotient[b, p]$, $d = Mod[b, p]$, then the iterations (2.13), (2.15) and (2.16) terminate in $p - 1 + d$ steps, where $c = Quotient[b, p]$, $d = Mod[b, p]$.

Also, an implementation of the introduced approximate methods in the programming package *Mathematica* is developed.

Theorem 4.1. *Let given two arbitrary $p \times q$ complex matrices R and S and $p \times 1$ complex vector y. Let the rows of the matrices R and S be denoted by r_u and s_u, respectively, for each $u = 1, \dots, p$. Also, assume that the rows of the unit matrix I_p are denoted by i_t, $t = 1, \dots, p$. If b is an arbitrary prescribed number of iterations, and integers c, d are defined as $c = Quotient[b, p]$, $d = Mod[b, p]$, then the following iterative sequences:*

$$
\begin{aligned}
\gamma_{t+1}^{\alpha/c} &= \gamma_t^{\alpha/c} - \frac{\gamma_t^{\alpha/c} r_{t+1}^* s_{t+1} \gamma_t^{\alpha/c}}{\alpha/c + s_{t+1}\gamma_t^{\alpha/c} r_{t+1}^*}, \quad \gamma_0^{\alpha/c} = I_q, \quad \alpha > 0,\\
x_{t+1}^{\alpha/c} &= x_t^{\alpha/c} + \frac{\gamma_t^{\alpha/c} r_{t+1}^*}{\alpha/c + s_{t+1}\gamma_t^{\alpha/c} r_{t+1}^*}\left(y_{t+1} - s_{t+1} x_t^{\alpha/c}\right), \quad x_0^{\alpha/c} = \vec{0},\\
X_{t+1}^{\alpha/c} &= X_t^{\alpha/c} + \frac{\gamma_t^{\alpha/c} r_{t+1}^*}{\alpha/c + s_{t+1}\gamma_t^{\alpha/c} r_{t+1}^*}\left(i_{t+1} - s_{t+1} X_t^{\alpha/c}\right), \ X_0^{\alpha/c} = \mathcal{O} \in \mathbb{C}^{q\times p}\\
& t = 0, \dots, p-1+d
\end{aligned} \tag{4.1}
$$

exist if and only if

$$
\alpha/c + s_{t+1}\gamma_t^{\alpha/c} r_{t+1}^* \neq 0, \quad t = 0, \dots, p-1. \tag{4.2}
$$

In the case when (4.2) *holds, the iterations* (4.1) *produce the following values:*

$$
\begin{aligned}
\gamma_{p+d-1}^{\alpha/c} &= \gamma_b^{\alpha} = \left(\frac{\alpha}{c} I_q + R_{\underline{d}}{}^* S_{\underline{d}}\right)^{-1} \frac{\alpha}{c}\\
x_{p+d-1}^{\alpha/c} &= x_b^{\alpha} = \left(\frac{\alpha}{c} I_q + R_{\underline{d}}{}^* S_{\underline{d}}\right)^{-1} R_{\underline{d}}^* y_{\underline{d}},\\
X_{p+d-1}^{\alpha/c} &= X_b^{\alpha} = \left(\frac{\alpha}{c} I_q + R_{\underline{d}}{}^* S_{\underline{d}}\right)^{-1} R_{\underline{d}}^*,
\end{aligned} \tag{4.3}
$$

where $R_{\underline{d}}{}^*$ is $q \times t$ matrix, equal to the conjugate and transpose of the submatrix $R_{\underline{d}}$ of R.

Proof. Utilizing a result from [11], for an arbitrary integer $N \geqslant 1$, we get the following statements for the iterative process:

$$
\gamma_{pN}^{\alpha} = \gamma_p^{\alpha/N}, \quad X_{pN}^{\alpha} = X_p^{\alpha/N}, \quad x_{pN}^{\alpha} = x_p^{\alpha/N}, \quad N = 1, 2, \dots. \tag{4.4}
$$

Consequently, after the first $p-1$ iterations we obtain

$$
\gamma_p^{\alpha/c} = \gamma_{pc}^{\alpha}, \quad X_p^{\alpha/c} = X_{pc}^{\alpha}, \quad x_p^{\alpha/c} = x_{pc}^{\alpha}.
$$

Finally, applying another d iterations we obtain

$$
\begin{aligned}
\gamma_{p+d}^{\alpha/c} &= \gamma_{pc+d}^{\alpha} = \gamma_b^{\alpha},\\
X_{p+d}^{\alpha/c} &= X_{pc+d}^{\alpha} = X_b^{\alpha},\\
x_{p+d}^{\alpha/c} &= x_{pc+d}^{\alpha} = x_b^{\alpha}.
\end{aligned}
$$

This completes the proof. □

Now we describe implementation of the iterative methods presented in (4.1). Input parameters in the algorithm are:

*r_*, *s_*: input matrices R and S;

*it_*: a prescribed number of iterations;

*alpha_*: a small real number representing the initial value of the parameter α.

STEP 1. Initial values of used local variables:

```
{m,n}=Dimensions[a];
in=IdentityMatrix[n]; im=IdentityMatrix[m];
g0=in; x0=ConstantArray[0, {n, m}];
```

STEP 2. Implementation of the iterative step. A major problem arising in the implementation of the limit $\lim_{\alpha\to 0} X_p^\alpha$ by means of (4.1) is the increase of dimensions. Namely, according to the property (4.4), decrease of the value α to α/N, $N \geq 1$ requires usage of block matrices γ_{pN}^α, X_{pN}^α, x_{pN}^α. This fact initiates a significant increase of number of arithmetic operations during the iterations. In order to avoid this problem, we use the standard function `Mod` of the programming language *Mathematica*. Further improvement is achieved using the iterations (4.1). Detailed implementation of the iterative rule (4.1) is presented as follows.

```
c=Quotient[it,m]; d=Mod[it,m];
alpha=alpha/c;
i=1;
While[i<=p-1+d,
 j=i;
 If[i>m,j=Mod[i,m];If[j==0,j=m];
 g1=g0-(g0.Transpose[{r[[j]]}].{s[[j]]}.g0)/
      (alpha+({s[[j]]}.g0.Transpose[{r[[j]]}])[[1,1]]);
 x1=x0+g0.Transpose[{r[[j]]}].({in[[j]]}-{s[[j]]}.x0)/
                           (alpha+({s[[j]]}.g0.Transpose[{r[[j]]}])[[1,1]]);
 g0=g1; x0=x1; i=i+1
];
```

STEP 3. Generate the output: `Return[{x1,g1}];`

5. Numerical example

In this section we present a few numerical comparisons between the implementation given in [11] and the implementation introduced in this paper. Assume that R, S are $p \times q$ matrices. Let us denote by b an arbitrary prescribed number of iterations, $c = Quotient[b,p]$ and $d = Mod[b,p]$. Implementation presented in [11] terminates after $b = pc + d$ iterations. On the other hand, modification defined in (4.1) terminates after $p + d - 1$ iterations. It is clear that the improved method requires $(p-1)c - 1$ iterations less than the original one.

Let us choose the matrices `r=s={{1,2,3},{3,2,1}}`. Using the modified implementation with $\alpha = 0.01$, and the maximal number of steps equal to 20, we perform 301 usual iterations from [11] in only two steps

```
c = 150 d = 1
alpha= 0.0000666667
it=1
x1={{0.07142823129413669,0}, {0.1428564625882733, 0},{0.21428469388241,0}}

it=2
x1={{-0.166663, 0.3333289352552714}, {0.0833331, 0.0833331018524948},
    {0.333329, -0.1666627315502817}}

it=3
x1={{-0.1666627315502817, 0.3333289352552714},
    {0.0833331018524949, 0.0833331018524948},
    {0.3333289352552714, -0.1666627315502817}}
```

Implementation described in [11] gives the following result after 301 iterations:

```
x1={{-0.1666627422808011, 0.3333289429198914},
    {0.0833331072178662,0.083333098020105},
    {0.333328956716534, -0.166662746879682}}
```

6. Conclusion and possible further research

Starting point in our research was the iterative method for the implementation of the limit representation of the Moore-Penrose inverse $A^{\dagger} = \lim_{\alpha \to 0} (\alpha I + A^* A)^{-1} A^*$ and the matrix expression $I - A^{\dagger} A$ from [17] and [18]. Further, we used the iterative process for the implementation of the general limit formula $\lim_{\alpha \to 0} (\alpha I + R^* S)^{-1} R^*$ from [11]. In this paper we further investigate convergence of these iterations. Moreover, an improvement of the iterations from from [11] is proposed and investigated. The efficacy of the proposed method is confirmed by its ability to produce the result in a predefined number of iterative steps. Convergence properties of defined iterations and iterations from [11] are further investigated.

An efficient algorithm for the implementation of the iterative processes (2.13), (2.15) and (2.16) from [11] is proposed and described. Firstly, an useful rule for avoiding usage of increasing block matrices during the iterations is proposed. Instead of growing block matrices we propose usage of the function `Mod` on the indices of the input matrices. In addition, according to certain rules, the introduced algorithm can converge in an arbitrary prescribed number of iterations.

Also, an implementation of the introduced approximate methods in the programming package *Mathematica* is developed.

An alternative limit expression of the Drazin inverse of the form

$$A^{\mathrm{D}} = \lim_{\alpha \to 0} (\alpha I_n + A)^{-(l+1)} A^l, \quad l \geqslant k$$

was presented in [5]. One possibility for further research could be development of iterations for the implementation of this alternative limiting formula.

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