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GENERAL DECAY OF SOLUTION FOR COUPLED SYSTEM OF VISCOELASTIC WAVE EQUATIONS OF KIRCHHOFF TYPE WITH DENSITY IN $\mathbb{R}^n$

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Abstract. A system of viscoelastic wave equations of Kirchhoff type is considered. For a wider class of relaxation functions, we use spaces weighted by the density function to establish a very general decay rate of the solution.

Keywords: Lyapunov function; Viscoelastic; Kirchhoff type; Density; Decay rate; Weighted spaces; Coupled system

1. Introduction

We consider the following system

$$
\begin{cases} \left(|u'|^{q-2}u'\right)' - \phi(x)\left(M(\|\nabla_x u\|_2^2)\Delta_x u - \int_0^t g_1(t-s)\Delta_x u(s)ds\right) + \alpha v = 0 \\ \left(|v'|^{q-2}v'\right)' - \phi(x)\left(M(\|\nabla_x v\|_2^2)\Delta_x v - \int_0^t g_2(t-s)\Delta_x v(s)ds\right) + \alpha u = 0 \end{cases} \tag{1.1}
$$

where $x \in \mathbb{R}^n, \alpha, t > 0, q, n \geq 2$ and M is a positive C^1 function satisfying for $s \geq 0, m_0 > 0, m_1 \geq 0, \gamma \geq 1$, $M(s) = m_0 + m_1 s^\gamma$ and the scalar functions $g_i(s), i = 1,2$ (the so-called relaxation kernel) are assumed to satisfy (A1).

The problem (1.1) is equipped by the following initial data.

$$
u(0,x) = u_0(x) \in \mathcal{H}(\mathbb{R}^n), \quad u'(0,x) = u_1(x) \in L^q_\rho(\mathbb{R}^n), \tag{1.2}
$$

$$
v(0,x) = v_0(x) \in \mathcal{H}(\mathbb{R}^n), \quad v'(0,x) = v_1(x) \in L^q_\rho(\mathbb{R}^n), \tag{1.3}
$$

where the weighted spaces $\mathcal{H}$ is given in Definition 2.1 and the density function $(\phi(x))^{-1} = \rho(x)$ satisfies

$$
\rho : \mathbb{R}^n \to \mathbb{R}^*_+, \quad \rho(x) \in C^{0,\tilde{\gamma}}(\mathbb{R}^n) \tag{1.4}
$$

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with $\tilde{\gamma} \in (0,1)$ and $\rho \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where $s = \frac{2n}{2n-qn+2q}$.

In this framework, (see [8], [9], [15], [16], [19], [21]), it is well known that for any initial data $u_0, v_0 \in \mathcal{H}(\mathbb{R}^n)$ and $u_1, v_1 \in L^q_\rho(\mathbb{R}^n)$, the problem (1.1)-(1.3) has a unique solution $(u,v) \in \left(C([0,T),\mathcal{H}(\mathbb{R}^n))\right)^2, (u',v') \in \left(C([0,T),L^q_\rho(\mathbb{R}^n))\right)^2$, under the hypothesis $(A1)-(A2)$. The problem (1.1) is usually encountered in viscoelasticity in various areas of mathematical physics, it was first considered by Dafermos in [7], where the general decay was discussed. The problems related to (1.1) have attracted a great deal of attention in the last decades and numerous results appeared on the existence and long time behavior of solutions but their results are by now rather developed, especially in any space dimension.
This kind of system appears in the models of the nonlinear Kirchhoff-type. It is a generalization of a model introduced by Kirchhoff [13] as the equation in the case $n = 1$ describes this type of problem as a small amplitude vibration of an elastic string. The original equation is:

$$\rho h u_{tt} + \tau u_t = \left(P_0 + \frac{Eh}{2L}\int_0^L |u_x(x,t)|^2 ds\right) u_{xx} + f, \tag{1.5}$$

where $0 \leq x \leq L$ and $t > 0, u(x,t)$ is the lateral displacement at the space coordinate x and the time t, ρ the mass density, h the cross-section area, L the length, P_0 the initial axial tension, τ the resistance modulus, E the Young modulus and f the external force (for example, the action of gravity).

The motivation for our study is due to some results regarding viscoelastic wave equations of the Kirchhoff type in a bounded domain. The wave equation of the form

$$u'' - M(\|\nabla_x u\|_2^2)\Delta_x u + \int_0^t g(t-s)\Delta_x u(s)ds + h(u') = f(u), \quad x \in \Omega, t > 0 \tag{1.6}$$

is a model to describe the motion of deformable solids as the hereditary effect is incorporated. Eq.(1.6) was studied by [20] and they proved the existence of weak solution for a large datum. Later, for a small datum and under some restrictions, global solutions and exponential decay to zero is shown in [17].

The work with weighted spaces was studied by many authors (see in this direction [5], [12], [18] and [22]). For the decay rate of solution for equations in $\mathbb{R}^n$, we quote the results by [2], [9], [10], [11], [16]. In [10], the authors showed that for compactly supported initial data and for an exponentially decaying relaxation function the decay of the energy of the solution to the linear Cauchy problem $(1.1),(1.3)$ in one equation with $\alpha = 0, q = 2, \rho(x) = 1, M \equiv 1$ is polynomial. The finite-speed propagation is used to compensate for the lack of Poincare's inequality. In case $\alpha = 0, q = 2, M \equiv 1$, in [9], the author looked into a linear Cauchy viscoelastic problem with density. His study included the exponential and polynomial rates, where he used the spaces weighted by density to compensate for the lack of Poincare's inequality. The same problem, treated in [9], was considered in [11], where they considered the Cauchy problem for the viscoelastic wave equation. Under suitable conditions on the initial data and the relaxation function, they prove a polynomial decay result of solutions. The conditions used on the relaxation function g and its derivative g' are different from the usual ones.

The problem (1.1),(1.3) in case $\alpha = 0, x \in \mathbb{R}^n$ with the relaxation function g is a positive nonincreasing function, was considered as one equation in [21], where the author established a general decay rate result for relaxation functions satisfying the assumptions $(A1) - (A2)$. The main purpose of the present paper is to extend this result to a coupled system of linear equations.
We omit the space variable x of $u(x,t), u'(x,t)$ and, for simplicity reasons, denote $u(x,t) = u$ and $u'(x,t) = u'$, when no confusion arises. The constants c used throughout this paper are positive generic constants which may be different, here $u' = du(t)/dt$ and $u'' = d^2u(t)/dt^2$.

2. Material, spaces and Assumptions

First, we recall and make use the following assumptions on the functions $g_i, i = 1,2$ as:

(A1) We assume that the functions $g_i : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is of class C^1 satisfying:

$$m_0 - \overline{g_i} = l_i > 0, \quad g_i(0) = g_{0i} > 0, \tag{2.1}$$

where $\overline{g_i} = \int_0^\infty g_i(t)dt$.

(A2) There exists a positive function $H \in C^1(\mathbb{R}^+)$ such that

$$g_i'(t) + H(g_i(t)) \leq 0, t \geq 0, \quad H(0) = 0 \tag{2.2}$$

and H is linear or strictly increasing and strictly convex C^2 function on $(0,r], 1 > r$.

Remark 2.1. [16]

A) We can deduce that there exists $t_1 > 0$ large enough such that for $i = 1,2$:

<u>1) $\forall t \geq t_1$:</u> We have $\lim\limits_{s\to+\infty} g_i(s) = 0$, which implies that $\lim\limits_{s\to+\infty} -g_i'(s)$ cannot be positive, so $\lim\limits_{s\to+\infty} -g_i'(s) = 0$. Then $g_i(t_1) > 0$ and

$$\max\{g_1(s), g_2(s), -g_1'(s), -g_2'(s)\} < \min\{r, H(r), H_0(r)\}, \tag{2.3}$$

where $H_0(t) = H(D(t))$ provided that D is a positive C^1 function, with $D(0) = 0$, for which H_0 is a strictly increasing and strictly convex C^2 function on $(0,r]$ and

$$\int_0^{+\infty} g_i(s)H_0(-g_i'(s))ds < +\infty.$$

<u>2) $\forall t \in [0,t_1]$:</u> As g_i are nonincreasing, $g_i(0) > 0$ and $g_i(t_1) > 0$ then $g_i(t) > 0$ and

$$g_i(0) \geq g_i(t) \geq g_i(t_1) > 0.$$

Therefore, since H is a positive continuous function, then

$$a \leq H(g_1(t)) \leq b$$

$$c \leq H(g_2(t)) \leq d$$

for some positive constants a, b, c and d. Consequently,

$$g_i'(t) \leq -H(g_i(t)) \leq -kg_i(t), \quad k > 0$$

which gives

$$g_i'(t) \leq -kg_i(t), k > 0 \tag{2.4}$$

B) Let H_0^* be the convex conjugate of H_0 in the sense of Young (see [3], pages 61-64), then

$$H_0^*(s) = s(H_0')^{-1}(s) - H_0[(H_0')^{-1}(s)], \quad s \in (0, H_0'(r))$$

and satisfies the following Young's inequality

$$AB \leq H_0^*(A) + H_0(B), \quad A \in (0, H_0'(r)), B \in (0, r]. \tag{2.5}$$

Definition 2.1. [[9], [18]] We define the function spaces of our problem and its norm as follows:

$$\mathcal{H}(\mathbb{R}^n) = \left\{ f \in L^{2n/(n-2)}(\mathbb{R}^n) : \nabla_x f \in (L^2(\mathbb{R}^n))^n \right\} \tag{2.6}$$

and the space $L^2_\rho(\mathbb{R}^n)$ to be the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to the inner product

$$(f, h)_{L^2_\rho(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho f h dx.$$

For $1 < q < \infty$, if f is a measurable function on $\mathbb{R}^n$, we define

$$\|f\|_{L^q_\rho(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \rho |f|^q dx \right)^{1/q}. \tag{2.7}$$

The space $L^2_\rho(\mathbb{R}^n)$ is a separable Hilbert space.

The following technical lemma will play an important role in the sequel.

Lemma 2.1. [6] *(Lemma 1.1) For any two functions h, $v \in C^1(\mathbb{R})$ and $\theta \in [0,1]$ we have*

$$v'(t)\int_0^t h(t-s)v(s)ds = -\frac{1}{2}\frac{d}{dt}\int_0^t h(t-s)|v(t)-v(s)|^2 ds + \frac{1}{2}\frac{d}{dt}\left(\int_0^t h(s)ds\right)|v(t)|^2$$

$$+\frac{1}{2}\int_0^t h'(t-s)|v(t)-v(s)|^2 ds - \frac{1}{2}h(t)|v(t)|^2. \tag{2.8}$$

and

$$\left|\int_0^t h(t-s)(v(t)-v(s))ds\right|^2 \le \left(\int_0^t |h(s)|^{2(1-\theta)}ds\right)\left(\int_0^t |h(t-s)|^{2\theta}|v(t)-v(s)|^2 ds\right) \tag{2.9}$$

The energy of (u,v) at time t is defined by

$$\begin{aligned} E(t) &= \frac{(q-1)}{q}\left[\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + \|v'\|^q_{L^q_\rho(\mathbb{R}^n)}\right] \\ &+ \frac{m_1}{2(\gamma+1)}\left[\|\nabla_x u\|_2^{2(\gamma+1)} + \|\nabla_x v\|_2^{2(\gamma+1)}\right] \\ &+ \frac{1}{2}\Big(m_0 - \int_0^t g_1(s)ds\Big)\|\nabla_x u\|_2^2 + \frac{1}{2}\Big(m_0 - \int_0^t g_2(s)ds\Big)\|\nabla_x v\|_2^2 \\ &+ \frac{1}{2}(g_1 \circ \nabla_x u) + \frac{1}{2}(g_2 \circ \nabla_x v) + \alpha \int_{\mathbb{R}^n} \rho u v dx \end{aligned} \tag{2.10}$$

For α small enough we use Lemma 3.1 to deduce for $c>0$ that:

$$\begin{aligned} E(t) &\ge (1 - c|\alpha|\|\rho\|^{-1}_{L^{n/2}})\frac{(q-1)}{q}\left[\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + \|v'\|^q_{L^q_\rho(\mathbb{R}^n)}\right] \\ &+ \frac{m_1}{2(\gamma+1)}\left[\|\nabla_x u\|_2^{2(\gamma+1)} + \|\nabla_x v\|_2^{2(\gamma+1)}\right] \\ &+ \frac{1}{2}\Big(m_0 - \int_0^t g_1(s)ds\Big)\|\nabla_x u\|_2^2 + \frac{1}{2}\Big(m_0 - \int_0^t g_2(s)ds\Big)\|\nabla_x v\|_2^2 \\ &+ \frac{1}{2}(g_1 \circ \nabla_x u) + \frac{1}{2}(g_2 \circ \nabla_x v) \end{aligned} \tag{2.11}$$

and the following energy functional law holds:

$$E'(t) \le \frac{1}{2}(g_1' \circ \nabla_x u)(t) + \frac{1}{2}(g_2' \circ \nabla_x v)(t), \quad \text{for all} \quad t \ge 0. \tag{2.12}$$

which means that our energy is uniformly bounded and decreasing along the trajectories. The following notation will be used throughout this paper

$$(g_i \circ \nabla_x \psi)(t) = \int_0^t g_i(t-\tau)\,\|\nabla_x \psi(t) - \nabla_x \psi(\tau)\|_2^2\, d\tau, i = 1, 2, \tag{2.13}$$

for $\psi(t) \in \mathcal{H}(\mathbb{R}^n), t \ge 0$.
We are now ready to state and prove our main results.

3. Main results

The next Lemma can be easily shown (see [12], Lemma 2.1).

Lemma 3.1. *Let ρ satisfies (1.4), then for any $u \in \mathcal{H}(\mathbb{R}^n)$*

$$\|u\|_{L^q_\rho(\mathbb{R}^n)} \leq \|\rho\|_{L^s(\mathbb{R}^n)} \|\nabla_x u\|_{L^2(\mathbb{R}^n)} \tag{3.1}$$

with $s = \frac{2n}{2n-qn+2q}, 2 \leq q \leq \frac{2n}{n-2}$

Our main result reads as follows.

Theorem 3.1. *Let $(u_0, v_0) \in (\mathcal{H}(\mathbb{R}^n))^2, (u_1, v_1) \in (L^q_\rho(\mathbb{R}^n))^2$ and suppose that $(A1)-(A2)$ hold. Then there exist positive constants a, b, c, d such that the energy of the solution given by* (1.1),(1.3) *satisfies,*

$$E(t) \leq dH_1^{-1}(bt + c), \quad \textit{for all} \quad t \geq 0,$$

where

$$H_1(t) = \int_t^1 \frac{1}{sH_0'(as)} ds \tag{3.2}$$

To prove Theorem 3.1, let us define

$$L(t) = \xi_1 E(t) + \psi_1(t) + \xi_2 \psi_2(t) \tag{3.3}$$

for $\xi_1, \xi_2 > 1$. In order to obtain useful estimates, we prepare some functionals associated with the nature of our problem introduced in Lyapunov function L as

$$\psi_1(t) = \int_{\mathbb{R}^n} \rho(x) \left[u|u'|^{q-2}u' + v|v'|^{q-2}v'\right] dx, \tag{3.4}$$

and the existence of the memory terms force us to introduce the next functional

$$\begin{aligned} \psi_2(t) &= -\int_{\mathbb{R}^n} \rho(x)|u'|^{q-2}u' \int_0^t g_1(t-s)(u(t)-u(s)) ds dx \\ &\quad - \int_{\mathbb{R}^n} \rho(x)|v'|^{q-2}v' \int_0^t g_2(t-s)(v(t)-v(s)) ds dx. \end{aligned} \tag{3.5}$$

Lemma 3.2. *Under the assumptions (A1) and (A2), the functional ψ_1 satisfies, along the solution to* (1.1),(1.3)

$$\begin{aligned} \psi_1'(t) \leq & (1 - d|\alpha| \|\rho\|^{-1}_{L^{n/2}}) \left[\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + \|v'\|^q_{L^q_\rho(\mathbb{R}^n)}\right] \\ & + c_1 m_1 \left(\|\nabla_x u\|_2^{2(\gamma+1)} + \|\nabla_x v\|_2^{2(\gamma+1)}\right) \\ & + (\sigma - l)\left(\|\nabla_x u\|_2^2 + \|\nabla_x v\|_2^2\right) + \frac{(1-l)}{4\sigma}\left((g_1 \circ \nabla_x u) + (g_2 \circ \nabla_x v)\right), \end{aligned}$$

where $l = \min\{l_1, l_2\}$.

Proof. From (3.4), integrate over $\mathbb{R}^n$, we have

$$\begin{aligned}
\psi_1'(t) &= \int_{\mathbb{R}^n} \rho(x)|u'|^q dx + \int_{\mathbb{R}^n} \rho(x)u\left(|u'|^{q-2}u'\right)' dx \\
&+ \int_{\mathbb{R}^n} \rho(x)|v'|^q dx + \int_{\mathbb{R}^n} \rho(x)v\left(|v'|^{q-2}v'\right)' dx \\
&= \int_{\mathbb{R}^n} \left(\rho(x)|u'|^q + M(\|\nabla_x u\|_2^2)u\Delta_x u - u\int_0^t g_1(t-s)\Delta_x u(s,x)ds - \alpha uv\right) dx \\
&+ \int_{\mathbb{R}^n} \left(\rho(x)|v'|^q + M(\|\nabla_x v\|_2^2)v\Delta_x v - v\int_0^t g_2(t-s)\Delta_x v(s,x)ds - \alpha uv\right) dx \\
&\leq (1 - c|\alpha|\|\rho\|_{L^{n/2}}^{-1})\left[\|u'\|_{L^q_\rho(\mathbb{R}^n)}^q + \|v'\|_{L^q_\rho(\mathbb{R}^n)}^q\right] + m_1\left(\|\nabla_x u\|_2^{2(\gamma+1)} + \|\nabla_x v\|_2^{2(\gamma+1)}\right) \\
&+ \int_{\mathbb{R}^n} \nabla_x u \int_0^t g_1(t-s)(\nabla_x u(s) - \nabla_x u(t))dsdx - l_1\|\nabla_x u\|_2^2 \\
&+ \int_{\mathbb{R}^n} \nabla_x v \int_0^t g_2(t-s)(\nabla_x v(s) - \nabla_x v(t))dsdx - l_2\|\nabla_x v\|_2^2.
\end{aligned}$$

Using Young's inequality and Lemma 2.1 for $\theta = 1/2$, we obtain for $l = \min\{l_1, l_2\}$

$$\begin{aligned}
\psi_1'(t) &\leq (1 - c|\alpha|\|\rho\|_{L^{n/2}}^{-1})\left[\|u'\|_{L^q_\rho(\mathbb{R}^n)}^q + \|v'\|_{L^q_\rho(\mathbb{R}^n)}^q\right] + m_1\left(\|\nabla_x u\|_2^{2(\gamma+1)} + \|\nabla_x v\|_2^{2(\gamma+1)}\right) \\
&- l_1\|\nabla_x u\|_2^2 - l_2\|\nabla_x v\|_2^2 \\
&+ \sigma\|\nabla_x u\|_2^2 + \frac{1}{4\sigma}\int_{\mathbb{R}^n}\left(\int_0^t g_1(t-s)|\nabla_x u(s) - \nabla_x u(t)|ds\right)^2 dx \qquad (3.6) \\
&+ \sigma\|\nabla_x v\|_2^2 + \frac{1}{4\sigma}\int_{\mathbb{R}^n}\left(\int_0^t g_2(t-s)|\nabla_x v(s) - \nabla_x v(t)|ds\right)^2 dx \\
&\leq (1 - c|\alpha|\|\rho\|_{L^{n/2}}^{-1})\left[\|u'\|_{L^q_\rho(\mathbb{R}^n)}^q + \|v'\|_{L^q_\rho(\mathbb{R}^n)}^q\right] + m_1\left(\|\nabla_x u\|_2^{2(\gamma+1)} + \|\nabla_x v\|_2^{2(\gamma+1)}\right) \\
&+ (\sigma - l)\left(\|\nabla_x u\|_2^2 + \|\nabla_x v\|_2^2\right) + \frac{(1-l)}{4\sigma}\left((g_1 \circ \nabla_x u) + (g_2 \circ \nabla_x v)\right).
\end{aligned}$$

□

Lemma 3.3. *Under the assumptions (A1) and (A2), the functional ψ_2 satisfies, along the solution of (1.1),(1.3), for any $\sigma \in (0, m_0)$*

$$\begin{aligned}
\psi_2'(t) &\leq \sigma\left(1 + \alpha\|\rho\|_{L^{n/2}(\mathbb{R}^n)}^2\right)\left(\|\nabla_x u\|_2^2 + \|\nabla_x v\|_2^2\right) \\
&+ c_\sigma\left(1 + \alpha\|\rho\|_{L^{n/2}(\mathbb{R}^n)}^2\right)\left((g_1 \circ \nabla_x u) + (g_2 \circ \nabla_x v)\right) \\
&- c_\sigma\|\rho\|_{L^s(\mathbb{R}^n)}^q\left((g_1' \circ \nabla_x u)^{q/2} + (g_2' \circ \nabla_x v)^{q/2}\right) \\
&+ \left(\sigma - \int_0^t g(s)ds\right)\left(\|u'\|_{L^q_\rho(\mathbb{R}^n)}^{q/2} + \|v'\|_{L^q_\rho(\mathbb{R}^n)}^{q/2}\right) \\
&+ m_1 c_3\left(\|\nabla_x u\|_2^{2(\gamma+1)} + \|\nabla_x v\|_2^{2(\gamma+1)}\right).
\end{aligned}$$

where

$$\int_0^t g(s)ds \leq \min\left\{\int_0^t g_1(s)ds, \int_0^t g_2(s)ds\right\} \qquad (3.7)$$

Proof. Exploiting Eq. (1.1), (3.5) to get

$$
\begin{aligned}
\psi_2'(t) &= -\int_{\mathbb{R}^n} \rho(x) \left(|u'|^{q-2}u'\right)' \int_0^t g_1(t-s)(u(t)-u(s))dsdx \\
&- \int_{\mathbb{R}^n} \rho(x)|u'|^{q-2}u' \int_0^t g_1'(t-s)(u(t)-u(s))dsdx - \int_0^t g_1(s)ds\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} \\
&- \int_{\mathbb{R}^n} \rho(x) \left(|v'|^{q-2}v'\right)' \int_0^t g_2(t-s)(v(t)-v(s))dsdx \\
&- \int_{\mathbb{R}^n} \rho(x)|v'|^{q-2}v' \int_0^t g_2'(t-s)(v(t)-v(s))dsdx - \int_0^t g_2(s)ds\|v'\|^q_{L^q_\rho(\mathbb{R}^n)} \\
&= \int_{\mathbb{R}^n} M(\|\nabla u\|_2^2)\nabla_x u \int_0^t g_1(t-s)(\nabla_x u(t) - \nabla_x u(s))dsdx \\
&- \int_{\mathbb{R}^n} \left(\int_0^t g_1(t-s)\nabla_x u(s,x)ds\right)\left(\int_0^t g_1(t-s)(\nabla_x u(t) - \nabla_x u(s))ds\right)dx \\
&- \int_{\mathbb{R}^n} \rho(x)|u'|^{q-2}u' \int_0^t g_1'(t-s)(u(t)-u(s))dsdx \\
&- \int_0^t g_1(s)ds\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + \alpha\int_{\mathbb{R}^n} \rho(x)v \int_0^t g_1(t-s)(u(t)-u(s))dsdx \\
&+ \int_{\mathbb{R}^n} M(\|\nabla v\|_2^2)\nabla_x v \int_0^t g_2(t-s)(\nabla_x v(t) - \nabla_x v(s))dsdx \\
&- \int_{\mathbb{R}^n} \left(\int_0^t g_2(t-s)\nabla_x v(s,x)ds\right)\left(\int_0^t g_2(t-s)(\nabla_x v(t) - \nabla_x v(s))ds\right)dx \\
&- \int_{\mathbb{R}^n} \rho(x)|v'|^{q-2}v' \int_0^t g_2'(t-s)(v(t)-v(s))dsdx \\
&- \int_0^t g_2(s)ds\|v'\|^q_{L^q_\rho(\mathbb{R}^n)} + \alpha\int_{\mathbb{R}^n} \rho(x)u \int_0^t g_2(t-s)(v(t)-v(s))dsdx,
\end{aligned}
$$

then

$$
\begin{aligned}
\psi_2'(t) &= \left(m_0 - \int_0^t g_1(s)ds\right)\int_{\mathbb{R}^n} \nabla_x u \int_0^t g_1(t-s)(\nabla_x u(t) - \nabla_x u(s))dsdx \\
&+ \int_{\mathbb{R}^n} \left(\int_0^t g_1(t-s)(\nabla_x u(t) - \nabla_x u(s))ds\right)^2 dx + c_1 m_1\|\nabla_x u\|_2^{2(\gamma+1)} \\
&- \int_{\mathbb{R}^n} \rho(x)|u'|^{q-2}u' \int_0^t g_1'(t-s)(u(t)-u(s))dsdx \\
&- \int_0^t g_1(s)ds\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + c(g_1 \circ \nabla_x u)(t) \\
&+ \left(m_0 - \int_0^t g_2(s)ds\right)\int_{\mathbb{R}^n} \nabla_x v \int_0^t g_2(t-s)(\nabla_x v(t) - \nabla_x v(s))dsdx \\
&+ \int_{\mathbb{R}^n} \left(\int_0^t g_2(t-s)(\nabla_x v(t) - \nabla_x v(s))ds\right)^2 dx + c_2 m_1\|\nabla_x v\|_2^{2(\gamma+1)}
\end{aligned}
$$

$$
\begin{aligned}
&- \int_{\mathbb{R}^n} \rho(x)|v'|^{q-2}v' \int_0^t g_2'(t-s)(v(t)-v(s))dsdx \\
&- \int_0^t g_2(s)ds\|v'\|^q_{L^q_\rho(\mathbb{R}^n)} + c(g_2 \circ \nabla_x v)(t) \\
&+ \alpha \int_{\mathbb{R}^n} \rho(x) \left(v \int_0^t g_1(t-s)(u(t)-u(s))ds + u \int_0^t g_2(t-s)(v(t)-v(s))ds \right) dx
\end{aligned}
$$

By Holder's and Young's inequalities and Lemma 3.1, we estimate the last term as

$$
\begin{aligned}
&\alpha \int_{\mathbb{R}^n} \rho(x) v \int_0^t g_1(t-s)(u(t)-u(s))dsdx \\
&\leq \alpha \left(\int_{\mathbb{R}^n} \rho(x)|v|^2 dx \right)^{1/2} \times \\
&\left(\int_{\mathbb{R}^n} \rho(x) \Big| \int_0^t g_1(t-s)(u(t)-u(s))ds \Big|^2 \right)^{1/2} \\
&\leq \alpha\sigma \|v\|^2_{L^2_\rho(\mathbb{R}^n)} + \alpha c_\sigma \Big\| \int_0^t g_1(t-s)(u(t)-u(s))ds \Big\|^2_{L^2_\rho(\mathbb{R}^n)} \\
&\leq \alpha\sigma \|\rho\|^2_{L^{n/2}(\mathbb{R}^n)} \|\nabla_x v\|_2^2 + \alpha c_\sigma \|\rho\|^2_{L^{n/2}(\mathbb{R}^n)} (g_1 \circ \nabla_x u)(t).
\end{aligned}
$$

and

$$
\begin{aligned}
&\alpha \int_{\mathbb{R}^n} \rho(x) u \int_0^t g_2(t-s)(v(t)-v(s))dsdx \\
&\leq \alpha \left(\int_{\mathbb{R}^n} \rho(x)|u|^2 dx \right)^{1/2} \times \\
&\left(\int_{\mathbb{R}^n} \rho(x) \Big| \int_0^t g_2(t-s)(v(t)-v(s))ds \Big|^2 \right)^{1/2} \\
&\leq \alpha\sigma \|u\|^2_{L^2_\rho(\mathbb{R}^n)} + \alpha c_\sigma \Big\| \int_0^t g_2(t-s)(v(t)-v(s))ds \Big\|^2_{L^2_\rho(\mathbb{R}^n)} \\
&\leq \alpha\sigma \|\rho\|^2_{L^{n/2}(\mathbb{R}^n)} \|\nabla_x u\|_2^2 + \alpha c_\sigma \|\rho\|^2_{L^{n/2}(\mathbb{R}^n)} (g_2 \circ \nabla_x v)(t).
\end{aligned}
$$

and for the exponents $\frac{q}{q-1}, q$

$$
\begin{aligned}
&-\int_{\mathbb{R}^n} \rho(x)|u'|^{q-2}u' \int_0^t g_1'(t-s)(u(t)-u(s))dsdx \\
&\leq \left(\int_{\mathbb{R}^n} \rho(x)|u'|^q dx\right)^{(q-1)/q} \times \\
&\left(\int_{\mathbb{R}^n} \rho(x)\Big|\int_0^t g_1'(t-s)(u(t)-u(s))ds\Big|^q\right)^{1/q} \\
&\leq \sigma\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + c_\sigma \Big\|\int_0^t -g_1'(t-s)(u(t)-u(s))ds\Big\|^q_{L^q_\rho(\mathbb{R}^n)} \\
&\leq \sigma\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} - c_\sigma\|\rho\|^q_{L^s(\mathbb{R}^n)}(g_1' \circ \nabla_x u)^{q/2}(t).
\end{aligned}
$$

and

$$
\begin{aligned}
&-\int_{\mathbb{R}^n} \rho(x)|v'|^{q-2}v' \int_0^t g_2'(t-s)(v(t)-v(s))dsdx \\
&\leq \left(\int_{\mathbb{R}^n} \rho(x)|v'|^q dx\right)^{(q-1)/q} \times \\
&\left(\int_{\mathbb{R}^n} \rho(x)\Big|\int_0^t g_2'(t-s)(v(t)-v(s))ds\Big|^q\right)^{1/q} \\
&\leq \sigma\|v'\|^q_{L^q_\rho(\mathbb{R}^n)} + c_\sigma \Big\|\int_0^t -g_2'(t-s)(v(t)-v(s))ds\Big\|^q_{L^q_\rho(\mathbb{R}^n)} \\
&\leq \sigma\|v'\|^q_{L^q_\rho(\mathbb{R}^n)} - c_\sigma\|\rho\|^q_{L^s(\mathbb{R}^n)}({g_2}' \circ \nabla_x v)^{q/2}(t).
\end{aligned}
$$

Using Young's and Poincare's inequalities and Lemma 2.1 for $\theta = 1/2$, we obtain

$$
\begin{aligned}
\psi_2'(t) \;\leq\;& \sigma\left(1+\alpha\|\rho\|^2_{L^{n/2}(\mathbb{R}^n)}\right)\left(\|\nabla_x u\|_2^2 + \|\nabla_x v\|_2^2\right) \\
&+ c_\sigma\left(1+\alpha\|\rho\|^2_{L^{n/2}(\mathbb{R}^n)}\right)\left((g_1\circ\nabla_x u)+(g_2\circ\nabla_x v)\right) \\
&- c_\sigma\|\rho\|^q_{L^s(\mathbb{R}^n)}\left((g_1'\circ\nabla_x u)^{q/2}+(g_2'\circ\nabla_x v)^{q/2}\right) \\
&+ \left(\sigma-\int_0^t g_1(s)ds\right)\|u'\|^{q/2}_{L^q_\rho(\mathbb{R}^n)} + \left(\sigma-\int_0^t g_2(s)ds\right)\|v'\|^{q/2}_{L^q_\rho(\mathbb{R}^n)} \\
&+ m_1\left(c_1\|\nabla_x u\|_2^{2(\gamma+1)} + c_2\|\nabla_x v\|_2^{2(\gamma+1)}\right).
\end{aligned}
$$

□

We need the next Lemma, which means that there is equivalent between the Lyapunov and energy functions, that is for $\xi_1, \xi_2 > 1$, we have

$$\beta_1 L(t) \leq E(t) \leq \beta_2 L(t) \tag{3.8}$$

holds for two positive constants β_1 and β_2.

Lemma 3.4. *For $\xi_1, \xi_2 > 1$, we have*

$$L(t) \sim E(t). \tag{3.9}$$

Proof. By (3.3) we have

$$\begin{aligned}
|L(t) - \xi_1 E(t)| &\leq |\psi_1(t)| + \xi_2 |\psi_2(t)| \\
&\leq \int_{\mathbb{R}^n} \left|\rho(x) u |u'|^{q-2} u'\right| dx + \int_{\mathbb{R}^n} \left|\rho(x) v |v'|^{q-2} v'\right| dx \\
&+ \xi_2 \int_{\mathbb{R}^n} \left|\rho(x) |u'|^{q-2} u' \int_0^t g_1(t-s)(u(t)-u(s)) ds\right| dx \\
&+ \xi_2 \int_{\mathbb{R}^n} \left|\rho(x) |v'|^{q-2} v' \int_0^t g_2(t-s)(v(t)-v(s)) ds\right| dx.
\end{aligned}$$

Thanks to Holder and Young's inequalities with exponents $\frac{q}{q-1}, q$, since $\frac{2n}{n+2} \geq q \geq 2$, we have by using Lemma 3.1

$$\begin{aligned}
\int_{\mathbb{R}^n} \left|\rho(x) u |u'|^{q-2} u'\right| dx &\leq \left(\int_{\mathbb{R}^n} \rho(x) |u|^q dx\right)^{1/q} \left(\int_{\mathbb{R}^n} \rho(x) |u'|^q dx\right)^{(q-1)/q} \\
&\leq \frac{1}{q} \left(\int_{\mathbb{R}^n} \rho(x) |u|^q dx\right) + \frac{q-1}{q} \left(\int_{\mathbb{R}^n} \rho(x) |u'|^q dx\right) \\
&\leq c \|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + c \|\rho\|^q_{L^s(\mathbb{R}^n)} \|\nabla_x u\|_2^q
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^n} \left|\rho(x) v |v'|^{q-2} v'\right| dx &\leq \left(\int_{\mathbb{R}^n} \rho(x) |v|^q dx\right)^{1/q} \left(\int_{\mathbb{R}^n} \rho(x) |v'|^q dx\right)^{(q-1)/q} \\
&\leq \frac{1}{q} \left(\int_{\mathbb{R}^n} \rho(x) |v|^q dx\right) + \frac{q-1}{q} \left(\int_{\mathbb{R}^n} \rho(x) |v'|^q dx\right) \\
&\leq c \|v'\|^q_{L^q_\rho(\mathbb{R}^n)} + c \|\rho\|^q_{L^s(\mathbb{R}^n)} \|\nabla_x v\|_2^q
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left| \left(\rho(x)^{\frac{q-1}{q}} |u'|^{q-2} u'\right) \left(\rho(x)^{\frac{1}{q}} \int_0^t g_1(t-s)(u(t)-u(s)) ds\right) \right| dx \\
&\leq \left(\int_{\mathbb{R}^n} \rho(x) |u'|^q dx\right)^{(q-1)/q} \times \\
&\left(\int_{\mathbb{R}^n} \rho(x) \left| \int_0^t g_1(t-s)(u(t)-u(s)) ds\right|^q dx\right)^{1/q} \\
&\leq \frac{q-1}{q} \|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + \frac{1}{q} \left\| \int_0^t g_1(t-s)(u(t)-u(s)) ds \right\|^q_{L^q_\rho(\mathbb{R}^n)} \\
&\leq \frac{q-1}{q} \|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + \frac{1}{q} \|\rho\|^q_{L^s(\mathbb{R}^n)} (g_1 \circ \nabla_x u)^{q/2}(t).
\end{aligned}$$

and

$$\int_{\mathbb{R}^n} \left| \left(\rho(x)^{\frac{q-1}{q}} |v'|^{q-2} v'\right) \left(\rho(x)^{\frac{1}{q}} \int_0^t g_2(t-s)(v(t)-v(s))ds\right) \right| dx$$
$$\leq \left(\int_{\mathbb{R}^n} \rho(x)|v'|^q dx \right)^{(q-1)/q} \times$$
$$\left(\int_{\mathbb{R}^n} \rho(x) \Big| \int_0^t g_2(t-s)(v(t)-v(s))ds \Big|^q dx \right)^{1/q}$$
$$\leq \frac{q-1}{q} \|v'\|^q_{L^q_\rho(\mathbb{R}^n)} + \frac{1}{q} \Big\| \int_0^t g_2(t-s)(v(t)-v(s))ds \Big\|^q_{L^q_\rho(\mathbb{R}^n)}$$
$$\leq \frac{q-1}{q} \|v'\|^q_{L^q_\rho(\mathbb{R}^n)} + \frac{1}{q} \|\rho\|^q_{L^s(\mathbb{R}^n)} (g_2 \circ \nabla_x v)^{q/2}(t).$$

Then, since $q \geq 2$, we have

$$\begin{aligned} |L(t) - \xi_1 E(t)| &\leq c(E(t) + E^{q/2}(t)) \\ &\leq c(E(t) + E(t)E^{(q/2)-1}(t)) \\ &\leq c(E(t) + E(t)E^{(q/2)-1}(0)) \\ &\leq cE(t). \end{aligned}$$

Therefore, we can choose ξ_1 so that

$$L(t) \sim E(t). \tag{3.12}$$

□

Proof of Theorem 3.1 From (2.12), results of Lemma 3.2 and Lemma 3.3, we have

$$\begin{aligned} L'(t) &= \xi_1 E'(t) + \psi_1'(t) + \xi_2 \psi_2'(t) \\ &\leq (\frac{\xi_1}{2} - c_\sigma \xi_2 \|\rho\|^q_{L^s(\mathbb{R}^n)}) \left[(g_1' \circ \nabla_x u)^{q/2} + (g_2' \circ \nabla_x v)^{q/2} \right] \\ &+ M_0 \left[(g_2 \circ \nabla_x u) + (g_2 \circ \nabla_x v) \right] - M_1 \left[\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + \|v'\|^q_{L^q_\rho(\mathbb{R}^n)} \right] \\ &+ cm_1(1+\xi_2) \left[\|\nabla_x u\|_2^{2(\gamma+1)} + \|\nabla_x v\|_2^{2(\gamma+1)} \right] - M_2 \left[\|\nabla_x u\|_2^2 + \|\nabla_x v\|_2^2 \right] \end{aligned}$$

where

$$M_0 = (\frac{4\xi_2 c \left(1 + \alpha \|\rho\|^2_{L^{n/2}(\mathbb{R}^n)}\right) + (1-l)}{4\sigma}),$$
$$M_1 = \left(\xi_2 \left(\int_0^{t_1} g(s)ds - \sigma \right) + c|\alpha| \|\rho\|^{-1}_{L^{n/2}(\mathbb{R}^n)} - 1 \right),$$
$$M_2 = \left(-\xi_2 \sigma \left(1 + \alpha \|\rho\|^2_{L^{n/2}(\mathbb{R}^n)}\right) + (l - \sigma) \right),$$

and t_1 was introduced in Remark 2.1.

We choose σ so small that $\xi_1 > 2c_\sigma \|\rho\|^q_{L^s(\mathbb{R}^n)}\xi_2$. Whence σ is fixed, we can choose ξ_1, ξ_2 large enough so that $M_1, M_2 > 0$, which yields

$$L'(t) \leq M_0\left[(g_1 \circ \nabla_x u) + (g_2 \circ \nabla_x v)\right] + cm_1\left[\|\nabla_x u\|_2^{2(\gamma+1)} + \|\nabla_x v\|_2^{2(\gamma+1)}\right] - cE(t), \quad \text{for all} \quad t \geq t_1. \tag{3.13}$$

Now we set $F(t) = L(t) + cE(t)$, which is equivalent to $E(t)$. Then by (3.13), we get for a positive constant c

$$\begin{aligned} F'(t) &= L'(t) + cE'(t) \\ &\leq -cE(t) + c\int_{\mathbb{R}^n}\int_{t_1}^t g_1(t-s)|\nabla_x u(t) - \nabla_x u(s)|^2 dsdx \\ &+ c\int_{\mathbb{R}^n}\int_{t_1}^t g_2(t-s)|\nabla_x v(t) - \nabla_x v(s)|^2 dsdx,, \quad \text{for all} \quad t \geq t_1. \end{aligned} \tag{3.14}$$

By (2.4) and (2.12), we have for all $t \geq t_1$

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_0^{t_1} g_1(t-s)|\nabla_x u(t) - \nabla_x u(s)|^2 dsdx + \int_{\mathbb{R}^n}\int_0^{t_1} g_2(t-s)|\nabla_x v(t) - \nabla_x v(s)|^2 dsdx \\ &\leq -\frac{1}{k}\left(\int_{\mathbb{R}^n}\int_0^{t_1} g_1'(t-s)|\nabla_x u(t) - \nabla_x u(s)|^2 dsdx + \int_{\mathbb{R}^n}\int_0^{t_1} g_2'(t-s)|\nabla_x v(t) - \nabla_x v(s)|^2 dsdx\right) \\ &\leq -cE'(t). \end{aligned}$$

At this point, we define

$$\begin{aligned} I(t) &= \int_{t_1}^t H_0(-g_1'(s))(g_1 \circ \nabla_x u)(t)ds \\ &+ \int_{t_1}^t H_0(-g_2'(s))(g_2 \circ \nabla_x v)(t)ds. \end{aligned} \tag{3.15}$$

Since $\int_0^{+\infty} H_0(-g_i'(s))g(s)ds < +\infty, i = 1, 2$, from (2.12) we have

$$\begin{aligned} I(t) &= \int_{t_1}^t H_0(-g_1'(s))\int_{\mathbb{R}^n} g_1(s)|\nabla_x u(t) - \nabla_x u(t-s)|^2 dxds \\ &+ \int_{t_1}^t H_0(-g_2'(s))\int_{\mathbb{R}^n} g_2(s)|\nabla_x v(t) - \nabla_x v(t-s)|^2 dxds \\ &\leq 2\int_{t_1}^t H_0(-g_1'(s))g_1(s)\int_{\mathbb{R}^n} |\nabla_x u(t)|^2 + |\nabla_x u(t-s)|^2 dxds \\ &+ 2\int_{t_1}^t H_0(-g_2'(s))g_2(s)\int_{\mathbb{R}^n} |\nabla_x v(t)|^2 + |\nabla_x v(t-s)|^2 dxds \\ &\leq cE(0)\left[\int_{t_1}^t H_0(-g_1'(s))g_1(s)ds + \int_{t_1}^t H_0(-g_2'(s))g_2(s)ds\right]. \end{aligned} \tag{3.16}$$

As in ([16], Eq. (3.11)), we have $I(t) < 1$. Now, we define again a new functional $\lambda(t)$ related to $I(t)$ as

$$\begin{aligned}\lambda(t) &= -\int_{t_1}^{t} H_0(-g_1'(s))g_1'(s)\int_{\mathbb{R}^n} g_1(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2 dxds\\ &- \int_{t_1}^{t} H_0(-g_2'(s))g_2'(s)\int_{\mathbb{R}^n} g_2(s)|\nabla_x v(t)-\nabla_x v(t-s)|^2 dxds. \qquad (3.17)\end{aligned}$$

From (A1)-(A2) and Remark 2.1 we get

$$H_0(-g_i'(s))g_i(s) \le H_0(H(g_i(s)))g_i(s) = D(g_i(s))g_i(s) \le k_0.$$

for some positive constant k_0. Then, for all $t \ge t_1$

$$\begin{aligned}\lambda(t) &\le -k_0\int_{t_1}^{t} g_1'(s)\int_{\mathbb{R}^n}|\nabla_x u(t)-\nabla_x u(t-s)|^2 dxds\\ &- k_0\int_{t_1}^{t} g_2'(s)\int_{\mathbb{R}^n}|\nabla_x v(t)-\nabla_x v(t-s)|^2 dxds\\ &\le -k_0\int_{t_1}^{t} g_1'(s)\int_{\mathbb{R}^n}|\nabla_x u(t)|^2+|\nabla_x u(t-s)|^2 dxds\\ &- k_0\int_{t_1}^{t} g_2'(s)\int_{\mathbb{R}^n}|\nabla_x v(t)|^2+|\nabla_x v(t-s)|^2 dxds\\ &\le -cE(0)\left[\int_{t_1}^{t} g_1'(s)ds+\int_{t_1}^{t} g_2'(s)ds\right]\\ &\le cE(0)\max\{g_1(t_1),g_2(t_1)\}\\ &< \min\{r,H(r),H_0(r)\}. \qquad (3.18)\end{aligned}$$

Using the properties of H_0 (strictly convex in $(0,r], H_0(0)=0$), then for $x \in (0,r], \theta\in[0,1]$

$$H_0(\theta x) \le \theta H_0(x).$$

Using Remark 2.1, (3.16), (3.18) and Jensen's inequality leads to

$$\begin{aligned}\lambda(t) = \ & I^{-1}(t)\{\int_{t_1}^{t} I(t)H_0[H_0^{-1}(-g_1'(s))]H_0(-g_1'(s))g_1'(s)\int_{\mathbb{R}^n} g_1(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2 dxds \\ & + \int_{t_1}^{t} I(t)H_0[H_0^{-1}(-g_2'(s))]H_0(-g_2'(s))g_2'(s)\int_{\mathbb{R}^n} g_2(s)|\nabla_x v(t)-\nabla_x v(t-s)|^2 dxds\} \\ & \geq I^{-1}(t)\{\int_{t_1}^{t} H_0[I(t)H_0^{-1}(-g_1'(s))]H_0(-g_1'(s))g_1'(s)\int_{\mathbb{R}^n} g_1(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2 dxds \\ & + \int_{t_1}^{t} H_0[I(t)H_0^{-1}(-g_2'(s))]H_0(-g_2'(s))g_2'(s)\int_{\mathbb{R}^n} g_2(s)|\nabla_x v(t)-\nabla_x v(t-s)|^2 dxds\} \\ & \geq H_0(I^{-1}(t)\int_{t_1}^{t} I(t)H_0^{-1}(-g_1'(s))H_0(-g_1'(s))g_1'(s)\int_{\mathbb{R}^n} g_1(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2 dxds \\ & + I^{-1}(t)\int_{t_1}^{t} I(t)H_0^{-1}(-g_2'(s))H_0(-g_2'(s))g_2'(s)\int_{\mathbb{R}^n} g_2(s)|\nabla_x v(t)-\nabla_x v(t-s)|^2 dxds) \\ & \geq H_0\left(\int_{t_1}^{t}\int_{\mathbb{R}^n} g_1(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2 dxds + \int_{t_1}^{t}\int_{\mathbb{R}^n} g_2(s)|\nabla_x v(t)-\nabla_x v(t-s)|^2 dxds\right)\end{aligned}$$

which implies

$$\int_{t_1}^{t}\int_{\mathbb{R}^n} g_1(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2 dxds + \int_{t_1}^{t}\int_{\mathbb{R}^n} g_2(s)|\nabla_x v(t)-\nabla_x v(t-s)|^2 dxds \leq H_0^{-1}(\lambda(t)).$$

Then

$$F'(t) \quad \leq \quad -cE(t) + cH_0^{-1}(\lambda(t)), \quad \text{for all} \quad t \geq t_1.$$

Now, we will following the steps in ([16]) and using the fact that $E' \leq 0, 0 < H_0', 0 < H_0''$ on $(0, r]$ to define the functional

$$F_1(t) = H_0'\left(a\frac{E(t)}{E(0)}\right)F(t) + cE(t), \quad a < r, 0 < c,$$

where $F_1(t) \sim E(t)$ and

$$\begin{aligned}F_1'(t) \quad &= \quad a\frac{E'(t)}{E(0)}H_0''\left(a\frac{E(t)}{E(0)}\right)F(t) + H_0'\left(a\frac{E(t)}{E(0)}\right)F'(t) + cE'(t) \\ &\leq \quad -cE(t)H_0'\left(a\frac{E(t)}{E(0)}\right) + cH_0'\left(a\frac{E(t)}{E(0)}\right)H_0^{-1}(\lambda(t)) + cE'(t).\end{aligned}$$

Let H_0^* given in Remark 2.1 and using Young's inequality (2.5) with $A = H_0'\left(a\frac{E(t)}{E(0)}\right)$, $B = H_0^{-1}(\lambda(t))$, to get

$$\begin{aligned}F_1'(t) \quad &\leq \quad -cE(t)H_0'\left(a\frac{E(t)}{E(0)}\right) + cH_0^*\left(H_0'\left(a\frac{E(t)}{E(0)}\right)\right) + c\lambda(t) + cE'(t) \\ &\leq \quad -cE(t)H_0'\left(a\frac{E(t)}{E(0)}\right) + ca\frac{E(t)}{E(0)}H_0'\left(a\frac{E(t)}{E(0)}\right) - c'E'(t) + cE'(t).\end{aligned}$$

Choosing a, c, c', such that for all $t \geq t_1$ we have

$$\begin{aligned} F_1'(t) &\leq -k\frac{E(t)}{E(0)}H_0'\left(a\frac{E(t)}{E(0)}\right) \\ &= -kH_2\left(\frac{E(t)}{E(0)}\right), \end{aligned}$$

where $H_2(t) = tH_0'(\alpha_0 t)$. Using the strict convexity of H_0 on $(0, r]$, to find that H_2', H_2 are strict positives on $(0, 1]$, then

$$R(t) = \tau\frac{k_1 F_1(t)}{E(0)} \sim E(t), \quad \tau \in (0, 1) \tag{3.19}$$

and

$$R'(t) \leq -\tau k_0 H_2(R(t)), \quad k_0 \in (0, +\infty), t \geq t_1.$$

Then, a simple integration and a suitable choice of τ yield,

$$R(t) \leq H_1^{-1}(bt + c), \quad b, c \in (0, +\infty), t \geq t_1.$$

here $H_1(t) = \int_t^1 H_2^{-1}(s)ds$. From (3.19), for a positive constant α_3, we have

$$E(t) \leq dH_1^{-1}(bt + c).$$

The fact that H_1 is strictly decreasing function on $(0, 1]$ and due to properties of H_2, we have

$$\lim_{t\to 0} H_1(t) = +\infty.$$

Then

$$E(t) \leq dH_1^{-1}(bt + c), \quad \text{for all} \quad t \geq 0.$$

This completes the proof of Theorem 3.1.

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