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ENERGY DECAY OF SOLUTION TO PLATE EQUATION WITH MEMORY IN $\mathbb{R}^n$

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Abstract. The viscoelastic equation with fading memory in bounded spaces has been extensively studied by several authors. Here, the energy decay results are established for weak-viscoelastic plate equation in $\mathbb{R}^n$, which depends on the behavior of both α and g. The key to the proof is to construct an appropriate Lyapunov function of the system obtained after taking the Fourier transform.

1. Introduction

The models considered here are well known and refer to materials with memory as they are termed in the wide array of literature concerned with their physical and mechanical behavior, as well as many interesting analytical problems. The characteristic physical property of such materials is that their behavior depends on time not only through the present but also through their past history. Let us consider the weak-viscoelastic case in the following problem:

$$(1.1)\quad \begin{cases} u'' + \Delta^2 u + \alpha(t)\int_0^t g(t-s)\Delta u(s,x)ds - \Delta u' = 0, x \in \mathbb{R}^n, t \in \mathbb{R}^+_* \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}^n), u'(0,x) = u_1(x) \in L^2(\mathbb{R}^n), x \in \mathbb{R}^n, \end{cases}$$

where $n \geq 2$. It is well known that the presence of a viscoelastic term with and without the weighted function α does not preclude the question of existence, but its effects are only on the stability of the existing solution. For the existence, we refer the reader to the studies in [4], [5], [6], [10], [13], [14], [16]. This type of problem is usually encountered in viscoelasticity in various areas of mathematical physics. It was first considered by Dafermos in [3], where the general decay was discussed. The problems related to (1.1) have attracted a great deal of attention in the last decades and numerous results have appeared on the existence and long-time behavior of solutions. The results are by now rather developed, especially in any space dimension when it comes to nonlinear problems.
As for literature data, in $\mathbb{R}^n$ we quote [1], [7], [8], [9], [11], [12], [15]. In [8], the

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authors showed that for compactly supported initial data and for an exponentially decaying relaxation function, the decay of the energy of solution to a linear Cauchy problem related to (1.1) is polynomial. The finite-speed propagation is used to compensate for the lack of Poincare inequality. In [7], the author looked into a linear Cauchy viscoelastic equation with density. His study included the exponential and polynomial rates, where he used the spaces weighted by density to compensate for the lack of Poincare's inequality. The problem treated in [7] was dealt with in [9], where they consider a Cauchy problem for a viscoelastic wave equation. Under suitable conditions on the initial data and the relaxation function, they prove a polynomial decay result of solutions. The conditions used on the relaxation function g and its derivative g' are different from the usual ones.
Ikehata in [4] considered in one-dimensional half space the mixed problem of the equation

$$v_{tt} - v_{xx} + v_t = 0 \tag{1.2}$$

with weighted initial data and presented new decay estimates of solutions which can also be derived for the Cauchy problem in $\mathbb{R}^n$. Let us mention a pioneer question of the long time asymptotic of strongly damped wave equations in [6], where the authors studied the Cauchy the problem for abstract dissipative equations in Hilbert spaces generalizing wave equations with strong damping terms in $\mathbb{R}^n$ or exterior domains

$$u_{tt}(t) + Au(t) + Au'(t) = 0, \qquad t \in (0, \infty). \tag{1.3}$$

$$u(0) = u_0, \qquad u_t(0) = u_1, \tag{1.4}$$

where $A : D(A) \subset H \to H$ is a nonnegative self-adjoint operator in $(H, \|.\|)$ with a dense domain $D(A)$. Using the energy method in the Fourier space and its generalization based on the spectral theorem for self-adjoint operators, their main result was a combination of solutions of diffusion and wave equations.

2. Statement

We omit the space variable x of $u(x,t), u'(x,t)$ and for simplicity reason denote $u(x,t) = u$ and $u'(x,t) = u'$, when there is no confusion. The constants c used throughout this paper are positive generic constants which may be different in various settings, here $u' = du(t)/dt$ and $u'' = d^2u(t)/dt^2$.
The following notation will be used throughout this paper

$$(g \circ \Psi) = \int_0^t g(t-\tau)\,|\Psi(t) - \Psi(\tau)|^2\,d\tau, \text{ for any } \Psi \in L^\infty(0,T; L^2(\mathbb{R}^n)) \tag{2.1}$$

In order to investigate the decay structure based on the weak- memory and the damping terms, we also consider the following assumptions:

$g, \alpha : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are non-increasing differentiable functions of class C^1 satisfying:

$$|\xi|^2 - \alpha(t) \int_0^t g(t)dt \geq k > 0, \quad g(0) = g_0 > 0 \tag{2.2}$$

$$\infty > \int_0^\infty g(t)dt, \quad \alpha(t) > 0, \tag{2.3}$$

Here ξ is the variable associated with the Fourier transform.
In addition, there exists a non-increasing differentiable function $\beta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying

$$\beta(t) > 0, \quad g'(t) + \beta(t)g(t) \leq 0, \quad \forall t \geq 0, \quad \lim_{t\to\infty} = \frac{\alpha'(t)}{\beta(t)\alpha(t)} = 0. \tag{2.4}$$

We give some notations to be used below. Let F denote the Fourier transform in $L^2(\mathbb{R}^n)$ defined as follows:

$$F[f](\xi) = \widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp(-ix.\xi)f(x)dx. \tag{2.5}$$

where $i = \sqrt{-1}, \quad x.\xi = \sum_{i=1}^n x_i\xi_i$ and denote its inverse transform by F^{-1}. The operator $-\Delta$ is defined by

$$-\Delta v(x) = F^{-1}\left(|\xi|^2 F(v)(\xi)\right)(x), v \in H^2(\mathbb{R}^n), x \in \mathbb{R}^n.$$

For $1 \leq p \leq \infty$, we denote by $L^p(\mathbb{R}^n)$ the usual Lebesgue space on $\mathbb{R}^n$ with the norm $\|.\|_{L^p}$. For a nonnegative integer $m, H^m(\mathbb{R}^n)$ denotes the Sobolev space of $L^2(\mathbb{R}^n)$ functions on $\mathbb{R}^n$, equipped with the norm $\|.\|_{H^m}$. By direct calculations, we have the following technical Lemma which will play an important role in the sequel.

Lemma 2.1. *([16], Lemma 2.1) For any two functions $g \in C^1(\mathbb{R}), v \in W^{1,2}(0,T)$, it holds that*

$$\begin{aligned} Re\left\{\alpha(t)\int_0^t g(t-s)v(s)ds\overline{v}'(t)\right\} &= -\frac{1}{2}\alpha(t)g(t)|v(t)|^2 + \frac{1}{2}\alpha(t)(g'\circ v)(t) \\ &- \frac{1}{2}\frac{d}{dt}\alpha(t)(g\circ v)(t) + \frac{1}{2}\frac{d}{dt}\alpha(t)\int_0^t g(s)ds|v(t)|^2 \\ &+ \frac{1}{2}\alpha'(t)(g\circ v)(t) - \frac{1}{2}\alpha'(t)\int_0^t g(s)ds|v|^2 \end{aligned} \tag{2.6}$$

and

$$|\int_0^t g(t-s)(v(t)-v(s))ds|^2 \leq \int_0^t |g(s)|ds \int_0^t |g|(t-s)|v(t)-v(s)|^2 ds$$

We can now state and prove the asymptotic behavior of the solution of (1.1). Throughout this paper, let us set $\widehat{u}(t,\xi) = F(u(t,.))(\xi)$.

3. Main result

We show that our solution decays time asymptotically to zero and the rate of decay for the solution is similar to both α and g.

Theorem 3.1. *Assume u is the solution of (1.1), then the next general exponential estimate satisfies in the Fourier space*

$$E(t) \leq W \exp\left(-\omega \int_0^t \alpha(s)\beta(s)ds\right), \qquad \forall t \geq 0. \tag{3.1}$$

for some positive constants W, ω.

Proof. We take the Fourier transform of both sides of (1.1). Then one has the reduced equation for $\xi \in \mathbb{R}^n, t \in \mathbb{R}^+_*$:

$$\begin{cases} \widehat{u}''(t,\xi) + |\xi|^4\widehat{u}(t,\xi) - |\xi|^2\alpha(t)\int_0^t g(t-s)\widehat{u}(s,\xi)ds + |\xi|^2\widehat{u}'(t,\xi) = 0 \\ \widehat{u}(0,\xi) = \widehat{u}_0(\xi) \in H^1(\mathbb{R}^n), \widehat{u}'(0,\xi) = \widehat{u}_1(\xi) \in L^2(\mathbb{R}^n). \end{cases} \tag{3.2}$$

We apply the multiplier techniques in Fourier space in order to obtain useful estimates and prepare some functionals associated with the nature of our problem to introduce appropriate Lyapunov functions.
First, to derive the equality for the physical energy, we multiply both sides of (3.2) by $\overline{\widehat{u}'}$. We denote

$$E_1(t) = \frac{1}{2}|\widehat{u}'|^2 + \frac{1}{2}|\xi|^2(|\xi|^2 - \alpha(t)\int_0^t g(s)ds)|\widehat{u}|^2 + \frac{1}{2}|\xi|^2\alpha(t)(g \circ \widehat{u})(t)$$

and

$$\begin{aligned} e_1(t) &= \frac{1}{2}|\xi|^2\left(\alpha(t)g(t)|\widehat{u}|^2 - \alpha(t)(g' \circ \widehat{u})(t) + 2|\widehat{u}'|^2\right) \\ &+ \frac{1}{2}|\xi|^2\left(\alpha'(t)(g \circ \widehat{u})(t) - \alpha'(t)\int_0^t g(s)ds|\widehat{u}|^2\right). \end{aligned}$$

Then, taking the real part of the resulting identities and by Lemma 2.1, we obtain

$$\frac{d}{dt}E_1(t) + e_1(t) = 0. \tag{3.3}$$

Second, the existence of the memory term forces us to make the first modification of the energy by multiplying (3.2) by $\Big(-\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\Big)$ and taking the real part, we have that

$$
\begin{aligned}
0 = & -Re\left\{\widehat{u}''\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\} \\
& -Re\left\{|\xi|^4\widehat{u}\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\} \\
& +\frac{1}{2}|\xi|^2\frac{d}{dt}\left(\Big|\alpha(t)\int_0^t g(t-s)\widehat{u}(s)ds\Big|^2\right) \\
& -Re\left\{|\xi|^2\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}
\end{aligned}
\tag{3.4}
$$

Since

$$
\begin{aligned}
& \frac{d}{dt}\left(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right) \\
& = \alpha'(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds + \alpha(t)\frac{d}{dt}\left(\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right) \\
& = \alpha'(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds + \alpha(t)g_0\overline{\widehat{u}} + \alpha(t)\int_0^t g'(t-s)\overline{\widehat{u}}(s)ds.
\end{aligned}
$$

The first term in Eq.(3.4) takes the form

$$
\begin{aligned}
- \quad & Re\left\{\widehat{u}''\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\} \\
= \quad & -Re\left\{\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}' \\
+ \quad & Re\left\{\widehat{u}'\frac{d^2}{dt^2}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\} \\
= \quad & -Re\left\{\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}' + \alpha(t)g_0|\widehat{u}'|^2 \\
+ \quad & Re\left\{\widehat{u}'\Big(\alpha(t)\frac{d}{dt}\Big(\int_0^t g'(t-s)\overline{\widehat{u}}(s)ds\Big) + \alpha'(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}.
\end{aligned}
$$

Denote by

$$
E_2(t) = \frac{1}{2}\left(|\xi|^2\Big|\alpha(t)\int_0^t g(t-s)\widehat{u}(s)ds\Big|^2 - Re\left\{\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\right),
$$

and

$$
\begin{aligned}
e_2(t) \quad = \quad & \alpha(t)g_0|\widehat{u}'|^2 - Re\left\{|\xi|^2\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\} \\
+ \quad & Re\left\{\alpha'(t)\widehat{u}'\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right\}
\end{aligned}
$$

$$R_2(t) = -Re\left\{|\xi|^4\widehat{u}\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\} + Re\left\{\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g'(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}.$$

Then,

$$\frac{d}{dt}E_2(t) + e_2(t) + R_2(t) = 0. \tag{3.5}$$

Next, to make the second modification of the energy which corresponds to the strong damping, we multiply (3.2) by $\overline{\widehat{u}}$ and taking the real part, we have

$$\begin{aligned} 0 &= (Re\{\widehat{u}'\overline{\widehat{u}}\})' - |\widehat{u}'|^2 + |\xi|^4|\widehat{u}|^2 \\ &- |\xi|^2 Re\left\{\alpha(t)\int_0^t g(t-s)\widehat{u}(s)\overline{\widehat{u}}(t)ds\right\} + \frac{1}{2}|\xi|^2(|\widehat{u}|^2)', \end{aligned}$$

using results in Lemma 2.1, we get

$$\begin{aligned} 0 &= (Re\{\widehat{u}'\overline{\widehat{u}}\})' - |\widehat{u}'|^2 + |\xi|^4|\widehat{u}|^2 + \frac{1}{2}|\xi|^2(|\widehat{u}|^2)' \\ &- |\xi|^2\left(\alpha(t)\int_0^t g(s)ds|\widehat{u}|^2 - Re\left\{\alpha(t)\int_0^t g(t-s)(\widehat{u}(s)-\widehat{u}(t))\overline{\widehat{u}}(s)ds\right\}\right). \end{aligned}$$

Denote

$$E_3(t) = Re\{\widehat{u}'\overline{\widehat{u}}\} + \frac{1}{2}|\xi|^2|\widehat{u}|^2,$$

and

$$\begin{aligned} e_3(t) &= |\xi|^2\Big(|\xi|^2 - \alpha(t)\int_0^t g(s)ds\Big)|\widehat{u}|^2. \\ R_3(t) &= -|\widehat{u}'|^2 - Re\left\{\alpha(t)\int_0^t g(t-s)(\widehat{u}(s)-\widehat{u}(t))\overline{\widehat{u}}(s)ds\right\} \end{aligned}$$

Then,

$$\frac{d}{dt}E_3(t) + e_3(t) + R_3(t) = 0. \tag{3.6}$$

Let us define for some constants $\varepsilon_1, \varepsilon_2 > 0$ to be chosen later

$$\begin{aligned} E_4(t) &= E_1(t) + \varepsilon_1\alpha(t)E_2(t) + \varepsilon_2\alpha(t)E_3(t) \\ &= \frac{1}{2}\left\{\frac{2(1)}{2}|\widehat{u}'|^2 + |\xi|^2\Big(|\xi|^2 - \alpha(t)\int_0^t g(s)ds\Big)|\widehat{u}|^2 + |\xi|^2\alpha(t)(g\circ\widehat{u})(t)\right\} \\ &+ \frac{\varepsilon_1\alpha(t)}{2}\left(|\xi|^2\Big|\alpha(t)\int_0^t g(t-s)\widehat{u}(s)ds\Big|^2 - Re\left\{\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\right) \\ &+ \varepsilon_2\alpha(t)\left(Re\left\{\widehat{u}'\overline{\widehat{u}}\right\} + \frac{1}{2}|\xi|^2|\widehat{u}|^2\right) \end{aligned}$$

and

$$\begin{aligned}
e_4(t) &= e_1(t) + \varepsilon_1\alpha(t)e_2(t) + \varepsilon_2\alpha(t)e_3(t) \\
&= \frac{|\xi|^2}{2}\alpha(t)\left(g(t)|\widehat{u}|^2 - (g'\circ\widehat{u})(t) + 2\alpha^{-1}(t)|\widehat{u}'|^2\right) \\
&+ \frac{|\xi|^2}{2}\alpha'(t)\left((g\circ\widehat{u})(t) - \widehat{u}'\int\limits_0^t g(s)ds|\widehat{u}|^2\right) \\
&+ \varepsilon_1\alpha(t)\left(\alpha(t)g_0|\widehat{u}'|^2 - Re\left\{|\xi|^2\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\right) \\
&+ \varepsilon_1\alpha(t)\left(Re\left\{\alpha'(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right\}\right) \\
&+ \varepsilon_2|\xi|^2\alpha(t)\left(|\xi|^2 - \alpha(t)\int_0^t g(s)ds\right)|\widehat{u}|^2
\end{aligned}$$

and

$$\begin{aligned}
R_4(t) &= \varepsilon_1\alpha(t)R_2(t) + \varepsilon_2\alpha(t)R_3(t) \\
&= \varepsilon_1\alpha(t)\left(-Re\left\{|\xi|^4\widehat{u}\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\right) \\
&+ \varepsilon_1\alpha(t)\left(Re\left\{\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g'(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\right) \\
&+ \varepsilon_2\alpha(t)\left(-|\widehat{u}'|^2 - Re\left\{\alpha(t)\int_0^t g(t-s)(\widehat{u}(s)-\widehat{u}(t))\overline{\widehat{u}}(s)ds\right\}\right)
\end{aligned}$$

At this point, we introduce as in [12], the Lyapunov functions as

$$L_1(t) = \left\{|\widehat{u}'|^2 + k|\xi|^2|\widehat{u}|^2 + |\xi|^2\alpha(t)(g\circ\widehat{u})(t)\right\} \tag{3.7}$$

and

$$L_2(t) = \alpha(t)g(t)|\widehat{u}|^2 + \alpha(t)\beta(t)(g\circ\widehat{u})(t). \tag{3.8}$$

It is easy to verify that there exists positive constants $c_1(g_0), c_2(g_0)$ such that

$$c_1L_1(t) \le E_1(t) \le c_2L_1(t), \forall t > 0. \tag{3.9}$$

Thanks to Holder, Young's inequalities, one gets for some constant $c_3 > 0$

$$|\varepsilon_1E_2(t) + \varepsilon_2E_3(t)| \le c_3L_1(t),$$

which means that $L_1(t) \sim E(t)$. Using again (2.4), Holder and Young's inequalities and assumptions on g to obtain

$$|R_4(t)| = \varepsilon_1R_2(t) + \varepsilon_2R_3(t)$$

$$\begin{aligned}
&\leq \varepsilon_1 Re\left\{|\xi|^4\widehat{u}\alpha(t)\frac{d}{dt}\Big(\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\\
&+ \varepsilon_1 Re\left\{\widehat{u}'\alpha(t)\frac{d}{dt}\Big(\int_0^t g'(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\\
&+ \varepsilon_2\left(|\widehat{u}'|^2 + Re\left\{\alpha(t)\int_0^t g(t-s)(\widehat{u}(s)-\widehat{u}(t))\overline{\widehat{u}}(s)ds\right\}\right)\\
&\leq \varepsilon_1|\widehat{u}'|^2 + c_4\varepsilon_1|\xi|^4|\widehat{u}|^2 + c_5\varepsilon_1|\xi|^2 L_2(t)\\
&+ \varepsilon_2\left[|\widehat{u}'|^2 + c_6|\xi|^2\left(\lambda|\widehat{u}|^2 + c_\lambda\alpha(t)(g\circ\widehat{u})(t)\right)\right]\\
&\leq (\varepsilon_1+\varepsilon_2)|\widehat{u}'|^2 + (c_4\varepsilon_1|\xi|^2 + \varepsilon_2 c_6\lambda)|\xi|^2|\widehat{u}|^2 + (c_5\varepsilon_1 + c_\lambda\varepsilon_2)|\xi|^2 L_2(t).
\end{aligned}$$

Since $L_2(t) \leq c_3\ e_1(t)$, one can easily check that there exists positive constants $\varepsilon_1, \varepsilon_2, \lambda, c_4, c_5, c_6$ such that

$$|R_4(t)| \leq ce_4(t), c > 0. \tag{3.10}$$

By (3.3), (3.5) and (3.6), we get

$$\frac{d}{dt}E_4(t) = \frac{d}{dt}E_1(t) + \varepsilon_1\alpha(t)\frac{d}{dt}E_2(t) + \varepsilon_2\alpha(t)\frac{d}{dt}E_3(t) + \varepsilon_1\alpha'(t)E_2(t) + \varepsilon_2\alpha'(t)E_3(t).$$

We use $\lim\limits_{t\to\infty}\frac{\alpha'(t)}{\alpha(t)} = 0$ by (2.2)-(2.4) to choose $t_1 > t_0$ and since $e_4(t) \geq cE_4(t)$, then (3.10) gives for some positive constant N

$$\frac{d}{dt}E_4(t) \leq -N\alpha(t)E_4(t) + c\alpha(t)(g\circ\widehat{u})(t). \tag{3.11}$$

Multiplying (3.11) by $\beta(t)$ and using (2.4), (3.8), we obtain

$$\begin{aligned}
\beta(t)\frac{d}{dt}E_4(t) &\leq -N\beta(t)\alpha(t)E_4(t) + c\beta(t)\alpha(t)(g\circ\widehat{u})(t)\\
&\leq -N\beta(t)\alpha(t)E_4(t) - c\alpha(t)(g'\circ\widehat{u})(t)\\
&\leq -N\beta(t)\alpha(t)E_4(t) - c|\xi|^2\alpha'(t)\int_0^t g(s)ds|\widehat{u}|^2 - 2c\frac{d}{dt}E_4(t), \quad \forall t > t_1. \qquad (3.12)
\end{aligned}$$

Since $\beta'(t) \leq 0$, we set $L(s) = (\beta(s) + 2c)E_4(s)$ which is equivalent to $E_4(t)$, then

$$\begin{aligned}
\frac{d}{dt}L(t) &\leq -N\beta(t)\alpha(t)E_4(t) - c|\xi|^2\alpha'(t)\int_0^t g(s)ds|\widehat{u}|^2\\
&\leq -\beta(t)\alpha(t)\left[N - \frac{2\alpha'(t)}{k\beta(t)\alpha(t)}\int_0^t g(s)ds\right]E_4(t), \quad \forall t > t_1. \qquad (3.13)
\end{aligned}$$

By (2.4), we can choose $t_2 > t_1$ such that

$$\begin{aligned}
\frac{d}{dt}L(t) &\leq -c\beta(t)\alpha(t)E_4(t)\\
&\leq -c\beta(t)\alpha(t)L(t), \quad \forall t > t_2. \qquad (3.14)
\end{aligned}$$

Integrating (3.14) over $[t_2, t]$ using equivalence between Lyapunov function and the energy function, it yields that

$$E(t) \leq W \exp(-\omega \int_0^t \alpha(s)\beta(s)ds), W, \omega > 0.$$

□

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