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ON THE DIVERGENCE OF NÖRLUND LOGARITHMIC MEANS WITH RESPECT TO THE L^1 NORM ON SOME UNBOUNDED VILENKIN GROUPS

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Abstract. Using the results of the paper [1] we give a divergence result of Nörlund logarithmic means for some unbounded Vilenkin groups. We prove that the boundedness of the subsequence $(\|F_{M_n}\|_1)_n$ implies the divergence in the L^1 norm of the sequence $(t_n f)_n$ for a conveniently chosen integrable function f. We provide an example to illustrate a direct application of this result.

Keywords: Vilenkin groups; Integrable function; Sequence of integers; Fourier series.

1. Introduction

In their paper [1] the authors proved a convergence result of the subsequence $(t_{M_n} f)_n$ to the integrable function f in the L^1 norm for some unbounded Vilenkin groups. The main tool was the boundedness of the sequence $(\|F_{M_n}\|_1)_n$. Paradoxically, this is the reason of the divergence of the whole sequence $(t_n f)_n$.

Therefore, in order to construct unbounded groups on which the sequence $(t_n f)_n$ converges in the L^1 norm, the property of uniform boundedness needs to be avoided.

Other divergence results can also be found in [1] and [2]. Many known results and open problems are presented in the work of Gat [3].

Let $(m_0, m_1, \ldots, m_n, \ldots)$ be a sequence of integers not less than 2. The Vilenkin group G is defined by $G := \prod_{n=0}^{\infty} \mathbb{Z}_{m_n}$, where $\mathbb{Z}_{m_n}$ denotes a discrete group of order m_n, with addition *mod* m_n.

It is said that G is unbounded if the sequence $(m_0, m_1, \ldots, m_n, \ldots)$ is unbounded.

Each element from G can be represented as a sequence $(x_n)_n$, where $x_n \in \{0, 1, \ldots, m_n - 1\}$, for every integer $n \geq 0$. Addition in G is obtained coordinatewise.

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The topology on G is generated by the subgroups
$I_n := \{x = (x_i)_i \in G, x_i = 0 \text{ for } i < n\}$, and their translations
$I_n(y) := \{x = (x_i)_i \in G, x_i = y_i \text{ for } i < n\}$.

The basis $(e_n)_n$ is formed by elements $e_n = (\delta_{in})_i$.

Define the sequence $(M_n)_n$ as follows: $M_0 = 1$ and $M_{n+1} = m_n M_n$.

If $|I_n|$ denotes the normalized product measure of I_n then it can be easily seen that $|I_n| = M_n^{-1}$.

The generalized Rademacher functions are defined by

$$r_n(x) := e^{\frac{2\pi i x_n}{m_n}}, n \in \mathbb{N} \cup \{0\}, x \in G.$$

For every non-negative integer n, there exists a unique sequence $(n_i)_i$ so that

$$n = \sum_{i=0}^{\infty} n_i M_i.$$

and the system of Vilenkin functions (see [4]), by

$$\psi_n(x) := \prod_{i=0}^{\infty} r_i^{n_i}(x), n \in \mathbb{N} \cup \{0\}, x \in G.$$

The Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels and Fejér kernels are respectively defined as follows

$$\hat{f}(n) := \int f(x)\bar{\psi}_n(x)dx,$$

$$S_n f := \sum_{k=0}^{n-1} \hat{f}(k)\psi_k,$$

$$D_n := \sum_{k=0}^{n-1} \psi_k,$$

$$K_n := \frac{1}{n}\sum_{k=1}^{n} D_k.$$

It can be easily seen that

$$S_n f(y) = \int D_n(y - x) f(x)dx,$$

and

$$D_{M_n}(x) = M_n 1_{I_n}(x).$$

The notation C will be used for independent positive constant. Throughout this paper we write log for the function $\log_2$.

The Nörlund logarithmic means are defined by

$$t_n f := \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k f}{n-k}, \qquad l_n := \sum_{k=1}^{n-1}\frac{1}{k}.$$

The functions F_n, $n \in \mathbb{N}$ are defined by

$$F_n := \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{D_k}{n-k},$$

it is clear that

$$t_n f = F_n * f.$$

2. Results

Lemma 2.1. *The sequence of functions*

$$\frac{1}{l_{M_{n+1}}}\sum_{k=1}^{[\frac{m_n}{2}]M_n-1}\frac{D_k}{M_{n+1}-k}$$

is uniformly bounded in the L^1 *norm.*

Proof. Since (see [1, Lemma 1])

$$D_{M_k-j}(x) = D_{M_k}(x) - \overline{\psi_{M_k-1}(-x)}D_j(-x), \qquad 1 \le j < M_k,$$

we obtain that

$$\begin{aligned}
\sum_{k=1}^{[\frac{m_n}{2}]M_n-1}\frac{D_k(x)}{M_{n+1}-k} &= \sum_{k=M_{n+1}-[\frac{m_n}{2}]M_n+1}^{M_{n+1}-1}\frac{D_{M_{n+1}-k}(x)}{k}\\
&= \sum_{k=M_{n+1}-[\frac{m_n}{2}]M_n+1}^{M_{n+1}-1}\frac{1}{k}(D_{M_{n+1}}(x) - \bar{\psi}_{M_{n+1}-1}(-x)D_k(-x))\\
&= (l_{M_{n+1}} - l_{M_{n+1}-[\frac{m_n}{2}]M_n+1})D_{M_{n+1}}(x)\\
&\quad - \bar{\psi}_{M_{n+1}-1}(-x)\sum_{k=M_{n+1}-[\frac{m_n}{2}]M_n+1}^{M_{n+1}-1}\frac{1}{k}D_k(-x).
\end{aligned} \tag{2.1}$$

Now we have

$$
\begin{aligned}
&\sum_{k=M_{n+1}-[\frac{m_n}{2}]M_n+1}^{M_{n+1}-1} \frac{1}{k} D_k \\
&= \sum_{k=M_{n+1}-[\frac{m_n}{2}]M_n}^{M_{n+1}-1} \left(\frac{1}{k}-\frac{1}{k+1}\right) \sum_{j=1}^{k} D_j \\
&\quad - \frac{1}{M_{n+1}-[\frac{m_n}{2}]M_n} \sum_{j=1}^{M_{n+1}-[\frac{m_n}{2}]M_n} D_j + \frac{1}{M_{n+1}} \sum_{j=1}^{M_{n+1}-1} D_j \\
&= \sum_{k=M_{n+1}-[\frac{m_n}{2}]M_n}^{M_{n+1}-1} \frac{1}{k+1} K_k - K_{M_{n+1}-[\frac{m_n}{2}]M_n} + \frac{M_{n+1}-1}{M_{n+1}} K_{M_{n+1}-1}.
\end{aligned} \tag{2.2}
$$

From [1, Lemma 3] we get for every $k \in \{M_{n+1}-[\frac{m_n}{2}]M_n, \ldots, M_{n+1}-1\}$,

$$
\|K_k\|_1 \leq C \sum_{i=0}^{n+1} \frac{1}{2^i} \frac{1}{M_{n+1-i}} \sum_{t=0}^{n-i} M_{t+1} \log m_t \leq C \max_{t=0,\ldots,n} \log m_t.
$$

Using this fact and Formula (2.2) we get

$$
\begin{aligned}
\left\| \sum_{k=M_{n+1}-[\frac{m_n}{2}]M_n+1}^{M_{n+1}-1} \frac{1}{k} D_k \right\|_1 &\leq \sum_{k=M_{n+1}-[\frac{m_n}{2}]M_n}^{M_{n+1}-1} \frac{1}{k+1} \|K_k\|_1 \\
&+ \|K_{M_{n+1}-[\frac{m_n}{2}]M_n}\|_1 + \|K_{M_{n+1}-1}\|_1 \\
&\leq C \max_{t=0,\ldots,n} \log m_t.
\end{aligned}
$$

Using Formula (2.1) we get for every $n \in \mathbb{N}$,

$$
\frac{1}{l_{M_{n+1}}} \| \sum_{k=1}^{[\frac{m_n}{2}]M_n-1} \frac{D_k}{M_{n+1}-k} \|_1 \leq C \|D_{M_{n+1}}\|_1 + C \frac{\max_{t=0,\ldots,n} \log m_t}{\sum_{t=0}^{n} \log m_t} = O(1).
$$

Theorem 2.1. *If the sequence $(m_n)_n$ is unbounded and if the sequence $(F_{M_n})_n$ is bounded in L^1, then there exists a function $f \in L^1$ such that $t_n f \nrightarrow f$ in L^1.*

Proof. We first write

$$
\begin{aligned}
l_{M_{n+1}} F_{M_{n+1}} &= \sum_{k=1}^{M_{n+1}-1} \frac{D_k}{M_{n+1}-k} \\
&= \sum_{k=1}^{[\frac{m_n}{2}]M_n} \frac{D_k}{M_{n+1}-k} + \sum_{k=[\frac{m_n}{2}]M_n+1}^{M_{n+1}-1} \frac{D_k}{M_{n+1}-k} \\
&= \sum_{k=1}^{[\frac{m_n}{2}]M_n} \frac{D_k}{M_{n+1}-k} + \sum_{k=1}^{M_{n+1}-[\frac{m_n}{2}]M_n-1} \frac{D_{k+[\frac{m_n}{2}]M_n}}{M_{n+1}-[\frac{m_n}{2}]M_n-k} \\
&= I + II.
\end{aligned}
$$

Without loss of generality we may assume that m_n is even since the proof for odd numbers can be obtained in a similar way.

Since

$$
D_{sM_{n+1}+k} = D_{sM_{n+1}} + \psi_{sM_{n+1}} D_k, \qquad 1 \leq k < M_{n+1},
$$

we obtain that

$$
\begin{aligned}
II &= \sum_{k=1}^{\frac{M_{n+1}}{2}-1} \frac{D_{k+\frac{M_{n+1}}{2}}}{\frac{M_{n+1}}{2}-k} = \sum_{k=1}^{\frac{M_{n+1}}{2}-1} \frac{D_{\frac{M_{n+1}}{2}} + \psi_{\frac{M_{n+1}}{2}} D_k}{\frac{M_{n+1}}{2}-k} \\
&= D_{\frac{M_{n+1}}{2}} \sum_{k=1}^{\frac{M_{n+1}}{2}-1} \frac{1}{\frac{M_{n+1}}{2}-k} + \psi_{\frac{M_{n+1}}{2}} \sum_{k=1}^{\frac{M_{n+1}}{2}-1} \frac{D_k}{\frac{M_{n+1}}{2}-k} \\
&= l_{\frac{M_{n+1}}{2}} D_{\frac{M_{n+1}}{2}} + \psi_{\frac{M_{n+1}}{2}} l_{\frac{M_{n+1}}{2}} F_{\frac{M_{n+1}}{2}}.
\end{aligned}
$$

It follows that

$$
F_{M_{n+1}} = \frac{1}{l_{M_{n+1}}} \sum_{k=1}^{\frac{M_{n+1}}{2}} \frac{D_k}{M_{n+1}-k} + \frac{l_{\frac{M_{n+1}}{2}}}{l_{M_{n+1}}} D_{\frac{M_{n+1}}{2}} + \psi_{\frac{M_{n+1}}{2}} \frac{l_{\frac{M_{n+1}}{2}}}{l_{M_{n+1}}} F_{\frac{M_{n+1}}{2}},
$$

which leads from $\psi_{\frac{M_{n+1}}{2}} = \pm 1$,

$$
\psi_{\frac{M_{n+1}}{2}} F_{M_{n+1}} = \frac{\psi_{\frac{M_{n+1}}{2}}}{l_{M_{n+1}}} \sum_{k=1}^{\frac{M_{n+1}}{2}} \frac{D_k}{M_{n+1}-k} + \frac{l_{\frac{M_{n+1}}{2}}}{l_{M_{n+1}}} \psi_{\frac{M_{n+1}}{2}} D_{\frac{M_{n+1}}{2}} + \frac{l_{\frac{M_{n+1}}{2}}}{l_{M_{n+1}}} F_{\frac{M_{n+1}}{2}},
$$

and

$$
\begin{aligned}
\left\| F_{\frac{M_{n+1}}{2}} * f \right\|_1 \geq{}& \left\| \psi_{\frac{M_{n+1}}{2}} D_{\frac{M_{n+1}}{2}} * f \right\|_1 - C \left\| \psi_{\frac{M_{n+1}}{2}} F_{M_{n+1}} * f \right\|_1 \\
& - C \left\| \frac{\psi_{\frac{M_{n+1}}{2}}}{l_{M_{n+1}}} \sum_{k=1}^{\frac{M_{n+1}}{2}} \frac{D_k}{M_{n+1}-k} * f \right\|_1 .
\end{aligned}
$$

Under the boundedness assumption of $(F_{M_n})_n$ in L^1 we get

$$\|\psi_{\frac{M_{n+1}}{2}} F_{M_{n+1}} * f\|_1 \leq C\|f\|_1.$$

Applying Lemma 2.1 we get

$$\left\| \frac{\psi_{\frac{M_{n+1}}{2}}}{l_{M_{n+1}}} \sum_{k=1}^{\frac{M_{n+1}}{2}} \frac{D_k}{M_{n+1}-k} * f \right\|_1 \leq C\|f\|_1.$$

In order to prove the divergence of $\left(F_{\frac{M_{n+1}}{2}} * f\right)_n$ for some function $f \in L^1$ it suffices to prove that $\left(\psi_{\frac{M_{n+1}}{2}} D_{\frac{M_{n+1}}{2}} * f\right)_n$ diverges.

Let the subsequence of even numbers $(m_{n_k})_k$ be so that

$$\sum_{k=0}^{\infty} \frac{1}{\sqrt{\log m_{n_k}}} < +\infty.$$

We construct the integrable function

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{\sqrt{\log m_{n_k}}} D_{M_{n_k+1}}(x - e_{n_k}).$$

For arbitrary positive integers n, k and $y \in G$ we have

$$\begin{aligned} &\psi_{\frac{M_{n_k+1}}{2}} D_{\frac{M_{n_k+1}}{2}} * D_{M_{n_l+1}}(x - e_{n_l})(y) \\ &= M_{n_l+1} M_{n_k} \int\limits_{\{t: y-t-e_{n_l} \in I_{n_l+1}\} \cap I_{n_k}} \psi_{M_{n_k}}^{\frac{m_{n_k}}{2}}(t)(1 + \psi_{M_{n_k}}(t) + \ldots + \psi_{M_{n_k}}^{\frac{m_{n_k}}{2}-1}(t))dt. \end{aligned}$$

The last expression does not vanish only if

$$y \in e_{n_l} + I_{n_k} + I_{n_l+1}.$$

This is equivalent to

$$\begin{cases} y \in e_{n_l} + I_{n_l+1}, & k \geq l+1, \\ y \in I_{n_k}, & k < l+1. \end{cases}$$

Therefore, if $k \geq l+1$, we have for $y \in e_{n_l} + I_{n_l+1}$

$$\{t : y - t - e_{n_l} \in I_{n_l+1}\} \cap I_{n_k} = I_{n_l+1} \cap I_{n_k} = I_{n_k}.$$

In this case

$$\psi_{M_{n_k}}^{\frac{m_{n_k}}{2}} D_{\frac{M_{n_k+1}}{2}} * D_{M_{n_l+1}}(x - e_{n_l})(y) = 0.$$

For $l \geq k$ and $y \in I_{n_k}$, we have

$$\begin{aligned}
\{t : y - t - e_{n_l} \in I_{n_l+1}\} \cap I_{n_k} &= \{t : t - (y - e_{n_l}) \in I_{n_l+1}\} \cap I_{n_k} \\
&= y - e_{n_l} + I_{n_l+1} \cap I_{n_k} = y - e_{n_l} + I_{n_l+1}.
\end{aligned}$$

It follows that for $l \geq k$

$$\begin{aligned}
&\psi_{M_{n_k}}^{\frac{m_{n_k}}{2}} D_{\frac{M_{n_k+1}}{2}} * D_{M_{n_l+1}}(x - e_{n_l})(y) \\
&= M_{n_k} 1_{I_{n_k}}(y) \psi_{M_{n_k}}^{\frac{m_{n_k}}{2}}(y - e_{n_l}) \cdot (1 + \psi_{M_{n_k}}(y - e_{n_l}) + \ldots + \psi_{M_{n_k}}^{\frac{m_{n_k}}{2}-1}(y - e_{n_l})).
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\left\| \psi_{M_{n_k}}^{\frac{m_{n_k}}{2}} D_{\frac{M_{n_k+1}}{2}} * f \right\|_1 \\
&= M_{n_k} \int_{I_{n_k}} |(1 + \psi_{M_{n_k}}(y - e_{n_k}) + \ldots + \psi_{M_{n_k}}^{\frac{m_{n_k}}{2}-1}(y - e_{n_k})) \frac{\psi_{M_{n_k}}^{\frac{m_{n_k}}{2}}(y - e_{n_k})}{\sqrt{\log m_{n_k}}} \\
&\quad + \sum_{l=k+1}^{\infty} (1 + \psi_{M_{n_k}}(y) + \ldots + \psi_{M_{n_k}}^{\frac{m_{n_k}}{2}-1}(y)) \frac{\psi_{M_{n_k}}^{\frac{m_{n_k}}{2}}(y)}{\sqrt{\log m_{n_l}}} | dy.
\end{aligned} \tag{2.3}$$

We have

$$\begin{aligned}
1 + \psi_{M_{n_k}}(y) + \ldots + \psi_{M_{n_k}}^{\frac{m_{n_k}}{2}-1}(y) &= \frac{\sin \frac{\pi}{2} y_{n_k} \cos \frac{m_{n_k}-2}{2m_{n_k}} \pi y_{n_k}}{\sin \frac{\pi}{m_{n_k}} y_{n_k}} \\
&\quad + i \frac{\sin \frac{\pi}{2} y_{n_k} \sin \frac{m_{n_k}-2}{2m_{n_k}} \pi y_{n_k}}{\sin \frac{\pi}{m_{n_k}} y_{n_k}}.
\end{aligned}$$

Suppose that y_{n_k} is even, then we have

$$1 + \psi_{M_{n_k}}(y) + \cdots + \psi_{M_{n_k}}^{\frac{m_{n_k}}{2}-1}(y) = 0,$$

and

$$|1 + \psi_{M_{n_k}}(y - e_{n_k}) + \cdots + \psi_{M_{n_k}}^{\frac{m_{n_k}}{2}-1}(y - e_{n_k})| \sim |\cot \frac{\pi}{m_{n_k}} y_{n_k}|.$$

If in the right side of (2.3) we only integrate on even y_{n_k}, for

$$y_{n_k} \in \{1, \ldots, m_{n_k} - 1\}$$

we get

$$\left\|\psi_{M_{n_k}}^{\frac{m_{n_k}}{2}} D_{\frac{M_{n_k+1}}{2}} * f\right\|_1 \geq C\frac{1}{m_{n_k}\sqrt{\log m_{n_k}}} \sum_{y_{n_k}\in\{2,\dots,m_{n_k}-2\};y_{n_k}\ even} |\cot\frac{\pi}{m_{n_k}} y_{n_k}| \sim \sqrt{\log m_{n_k}}.$$

Since in [1, Theorem 2] the authors proved that under certain conditions $t_n f - f \nrightarrow 0$ in L^1, we may provide an example where $(t_n f)_n$ diverges and the condition of [1, Theorem 2] is not verified.

Example 2.1. There exists an unbounded Vilenkin group represented by the sequence $(m_n)_n$ such that

1. $\log m_{n_k} \sim \sqrt{n_k}$, for some subsequence $(m_{n_k})_k$ and
2. $t_n f \nrightarrow f$ in L^1.

Using Theorem 2.1 and [1, Lemma 4] it suffices to construct a sequence $(m_n)_n$ such that

$$\sup_n \frac{\sum_{k=0}^{n-1}(\log m_k)^2}{\sum_{k=0}^{n-1}\log m_k} < +\infty.$$

Let $m_k = 2$ if $k \neq 4^s$ for all positive integers s, and $\log m_k = 2^s = \sqrt{k}$ if $k = 4^s$. Hence we have

$$\sum_{k=0}^{n-1}(\log m_k)^2 = \sum_{s=[\log\sqrt{n-1}]+1}^{n-1}(\log 2)^2 + \sum_{s=0}^{[\log\sqrt{n-1}]} 4^s \leq n(\log 2)^2 + C4^{\log\sqrt{n}} \sim n.$$

On the other hand we have

$$\sum_{k=0}^{n-1}\log m_k \sim n\log 2 + 2^{\log\sqrt{n}} \sim n,$$

from which we easily obtain the result.

3. Conclusion

Example 2.1 is very similar to Example 1, given in [1], where the authors proved a divergence result for some sequence $(m_n)_n$ satisfying $\log m_n = O(n^{\frac{1}{4}})$. It is clear that in both cases divergence is a direct consequence of the boundedness of the subsequence $(\|F_{M_n}\|_1)_n$. This gives a better understanding on the behaviour of unbounded sequences $(m_n)_n$ that may define groups on which L^1-convergence of $(t_n f)_n$ is satisfied for all integrable functions.

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