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## ON THE HERMITE-HADAMARD INEQUALITIES FOR CONVEX FUNCTIONS VIA HADAMARD FRACTIONAL INTEGRALS\*

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**Abstract.** In this paper, we establish new Hermite-Hadamard inequalities involving Hadamard fractional integrals, which are described by series. To achieve our aim, we use established fractional integral identities and elementary inequalities based on convex functions and monotonicity introduced in our previous works. Finally, some applications to special means of real numbers are given.

### 1. Introduction

Fractional calculus has been recognized as one of the best tools to describe long-memory processes. The corresponding mathematical models of these processes are fractional systems. Due to the extensive applications of fractional systems in engineering and science, research in this area has grown significantly all around the world [1, 2, 3, 4, 5, 6, 7].

Recently, fractional integral inequalities have attracted many researchers. For example, Anastassiou [8] presented the first fractional differentiation inequalities of Opial type involving so-called balanced fractional derivatives and continued with right and mixed fractional differentiation Ostrowski inequalities in the univariate and multivariate cases; Set [9] studied fractional Ostrowski inequalities involving Riemann-Liouville fractional integrals; Sarikaya et al. [10] studied Hermite-Hadamard type inequalities involving Riemann-Liouville fractional integrals. Strongly motivated by the contributions of the above works, our researcher group established many fractional Hermite-Hadamard inequalities involving Riemann-Liouville and Hadamard fractional integrals for different convex functions [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25].

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However, there are few papers [12, 16] on Hermite-Hadamard inequalities for Hadamard fractional integrals, even though Hadamard fractional integrals were presented many years ago as Riemann-Liouville fractional integrals.

For  $f \in L[a, b]$ , Hadamard fractional integrals [3]  $J_{a+}^{\alpha} f$  and  $J_{b-}^{\alpha} f$  of order  $\alpha \in \mathbb{R}^+$  with  $a \geq 0$  are defined by

$$({}_H J_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left( \ln \frac{x}{t} \right)^{\alpha-1} f(t) \frac{dt}{t}, \quad (0 < a < x \leq b),$$

and

$$({}_H J_{b-}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left( \ln \frac{t}{x} \right)^{\alpha-1} f(t) \frac{dt}{t}, \quad (0 < a \leq x < b),$$

where  $\Gamma(\cdot)$  is Gamma function.

It seems that Wang et al. [12, 16] firstly established the following two powerful fractional integral identities involving Hadamard fractional integrals for once differential functions.

**Lemma 1.1.** (see Lemma 3.1, [12]) Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  with  $0 < a < b$ . If  $f' \in L[a, b]$ , then the following equality for fractional integrals holds:

$$\begin{aligned} & \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(\ln b - \ln a)^{\alpha}} [{}_H J_{a+}^{\alpha} f(b) + {}_H J_{b-}^{\alpha} f(a)] \\ &= \frac{\ln b - \ln a}{2} \int_0^1 [(1-t)^{\alpha} - t^{\alpha}] e^{t \ln a + (1-t) \ln b} f'(e^{t \ln a + (1-t) \ln b}) dt \\ &= \frac{\ln b - \ln a}{2} \int_0^1 [(1-t)^{\alpha} - t^{\alpha}] a^t b^{1-t} f'(a^t b^{1-t}) dt. \end{aligned}$$

**Lemma 1.2.** (see Lemma 2.1, [16]) Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  with  $0 < a < b$ . If  $f' \in L[a, b]$ , then the following equality for fractional integrals holds:

$$\begin{aligned} & \frac{(\ln x - \ln a)^{\alpha} + (\ln b - \ln x)^{\alpha}}{\ln b - \ln a} f(x) - \frac{\Gamma(\alpha + 1)}{\ln b - \ln a} [{}_H J_{x-}^{\alpha} f(a) + {}_H J_{x+}^{\alpha} f(b)] \\ &= \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a} \int_0^1 t^{\alpha} e^{t \ln x + (1-t) \ln a} f'(e^{t \ln x + (1-t) \ln a}) dt \\ & \quad - \frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a} \int_0^1 t^{\alpha} e^{t \ln x + (1-t) \ln b} f'(e^{t \ln x + (1-t) \ln b}) dt \\ &= \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a} \int_0^1 t^{\alpha} x^t a^{1-t} f'(x^t a^{1-t}) dt \\ & \quad - \frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a} \int_0^1 t^{\alpha} x^t b^{1-t} f'(x^t b^{1-t}) dt, \end{aligned}$$

for any  $x \in (a, b)$ .

In [16], the authors explored a new concept named by *s-e*-condition to overcome some essential difficulties from the singular kernels in Hadamard fractional integrals. Some new Ostrowski type inequalities for Hadamard fractional integrals are obtained. In the present paper, we will use Lemma 1.1 and Lemma 1.2 via elementary equalities and inequalities via convex functions to derive new type Hermite-Hadamard inequalities involving Hadamard fractional integrals. Here, we do not need the *s-e*-condition [16] by using Lemma and monotonicity. Finally, we give some applications to special means of real numbers.

## 2. Preliminaries

In this section, we recall the definition of convex functions and collect some elementary equalities and inequalities which will be used in the sequel.

**Definition 2.1.** Let  $f : I \subseteq \mathbb{R}^+ \rightarrow \mathbb{R}^+$  and  $I$  is a convex set. A function  $f$  is said to be convex on  $I$  if for every  $x, y \in I$  and  $\lambda \in [0, 1]$ , we have

$$(2.1) \quad f[\lambda x + (1 - \lambda)y] \leq \lambda f(x) + (1 - \lambda)f(y).$$

**Lemma 2.1.** (see Lemma 2.1, [13]) For  $\alpha > 0$  and  $k > 0$ , we have

$$J(\alpha) := \int_0^1 t^{\alpha-1} k^t dt = k \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln k)^{i-1}}{(\alpha)_i} < +\infty$$

where

$$(\alpha)_i = \alpha(\alpha + 1)(\alpha + 2) \cdots (\alpha + i - 1).$$

**Lemma 2.2.** (see Lemma 2.2, [13]) For  $\alpha > 0$  and  $k > 0, z > 0$ , we have

$$\int_0^z t^{\alpha-1} k^t dt = z^\alpha k^z \sum_{i=0}^{\infty} \frac{(-z \ln k)^{i-1}}{(\alpha)_i} < +\infty.$$

where

$$(\alpha)_i = \alpha(\alpha + 1)(\alpha + 2) \cdots (\alpha + i - 1).$$

**Lemma 2.3.** (see [15]) For  $A > B > 0$ , it holds

$$\begin{aligned} (A - B)^\theta &\leq A^\theta - B^\theta && \text{when } \theta \geq 1, \\ (A - B)^\theta &\geq A^\theta - B^\theta && \text{when } 0 < \theta \leq 1. \end{aligned}$$

**Lemma 2.4.** (see Lemma 2.4, [13]) For  $A \geq 0, B \geq 0$ , it holds

$$\begin{aligned} (A + B)^\theta &\leq 2^{\theta-1}(A^\theta + B^\theta) && \text{when } \theta \geq 1, \\ (A + B)^\theta &\leq A^\theta + B^\theta && \text{when } 0 < \theta \leq 1. \end{aligned}$$

**Lemma 2.5.** (see Lemma 2.5, [18]) For  $t \in [0, 1]$ , we have

$$\begin{aligned} (1-t)^n &\leq 2^{1-n} - t^n \quad \text{when } n \in [0, 1], \\ (1-t)^n &\geq 2^{1-n} - t^n \quad \text{when } n \in [1, \infty). \end{aligned}$$

**Lemma 2.6.** (see Lemma 2.5, [19]) For  $t \in [0, 1]$ ,  $x, y > 0$ , we have

$$y^{1-t}x^t \leq tx + (1-t)y.$$

### 3. Main results

In this section, we will use convex functions via the results in Section 2 to derive our main results in this paper.

#### 3.1. Main results via Lemma 1.1

**Theorem 3.1.** Let  $f : (0, b) \rightarrow \mathbb{R}$  be a differentiable mapping. If  $|f'|$  is measurable and  $|f'|$  is convex and increasing function on  $(0, b)$  for some fixed  $\alpha \in (0, \infty)$ ,  $0 < a < b$ , then the following inequality for fractional integrals holds:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(\ln b - \ln a)^\alpha} [{}_H J_{a^+}^\alpha f(b) + {}_H J_{b^-}^\alpha f(a)] \right| \\ & \leq \max \left\{ \frac{\ln b - \ln a}{2} b \left[ (2^{1-\alpha} + 1) \left( (|f'(a)| - |f'(b)|) \frac{1}{4} (ab^{-1})^{\frac{1}{2}} \sum_{i=0}^{\infty} \frac{(-\frac{1}{2} \ln(ab^{-1}))^{i-1}}{(2)_i} \right. \right. \right. \\ & \quad + |f'(b)| \frac{(ab^{-1})^{\frac{1}{2}} - 1}{\ln(ab^{-1})} \Big) - 4 \left( (|f'(a)| - |f'(b)|) \left( \frac{1}{2} \right)^{\alpha+2} (ab^{-1})^{\frac{1}{2}} \sum_{i=0}^{\infty} \frac{(-\frac{1}{2} \ln(ab^{-1}))^{i-1}}{(\alpha+2)_i} \right. \right. \\ & \quad + |f'(b)| \left( \frac{1}{2} \right)^{\alpha+1} (ab^{-1})^{\frac{1}{2}} \sum_{i=0}^{\infty} \frac{(-\frac{1}{2} \ln(ab^{-1}))^{i-1}}{(\alpha+1)_i} \Big) \\ & \quad + 2 \left( (|f'(a)| - |f'(b)|) (ab^{-1}) \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(ab^{-1}))^{i-1}}{(\alpha+2)_i} \right. \\ & \quad + |f'(b)| (ab^{-1}) \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(ab^{-1}))^{i-1}}{(\alpha+1)_i} \Big) \\ & \quad \left. \left. - \left( (|f'(a)| - |f'(b)|) (ab^{-1}) \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(ab^{-1}))^{i-1}}{(2)_i} + |f'(b)| \frac{(ab^{-1}) - 1}{\ln(ab^{-1})} \right) \right] \right\}, \\ & \frac{\ln b - \ln a}{2} b \left[ (2^{1-\alpha} + 1) \left( (|f'(a)| - |f'(b)|) \frac{1}{4} (ab^{-1})^{\frac{1}{2}} \right. \right. \\ & \quad \left. \left. + \sum_{i=0}^{\infty} \frac{(-\frac{1}{2} \ln(ab^{-1}))^{i-1}}{(2)_i} + |f'(b)| \frac{(ab^{-1})^{\frac{1}{2}} - 1}{\ln(ab^{-1})} \right) \right] \end{aligned}$$

$$\begin{aligned}
 & -4 \left( (|f'(a)| - |f'(b)|) \left( \frac{1}{2} \right)^{\alpha+2} (ab^{-1})^{\frac{1}{2}} \sum_{i=0}^{\infty} \frac{(-\frac{1}{2} \ln(ab^{-1}))^{i-1}}{(\alpha+2)_i} \right. \\
 & + |f'(b)| \left( \frac{1}{2} \right)^{\alpha+1} (ab^{-1})^{\frac{1}{2}} \sum_{i=0}^{\infty} \frac{(-\frac{1}{2} \ln(ab^{-1}))^{i-1}}{(\alpha+1)_i} \left. \right) \\
 & + 2 \left( (|f'(a)| - |f'(b)|)(ab^{-1}) \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(ab^{-1}))^{i-1}}{(\alpha+2)_i} \right. \\
 & + |f'(b)|(ab^{-1}) \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(ab^{-1}))^{i-1}}{(\alpha+1)_i} \left. \right) \\
 & - 2^{1-\alpha} \left( (|f'(a)| - |f'(b)|)(ab^{-1}) \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(ab^{-1}))^{i-1}}{(2)_i} + |f'(b)| \frac{(ab^{-1}) - 1}{\ln(ab^{-1})} \right) \Bigg\},
 \end{aligned}$$

where  $(\alpha+1)_i = (\alpha+1)(\alpha+2) \cdots (\alpha+i)$ .

**Proof.** To achieve our aim, we divide our proof into two cases.

Case 1:  $\alpha \in (0, 1)$ . By using Definition 2.1, Lemma 1.1, Lemma 2.1, Lemma 2.2, Lemma 2.3, Lemma 2.5, Lemma 2.6 and Hölder inequality, we have

$$\begin{aligned}
 & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha} [{}_H J_{a^+}^\alpha f(b) + {}_H J_{b^-}^\alpha f(a)] \right| \\
 & = \left| \frac{\ln b - \ln a}{2} \int_0^1 [(1-t)^\alpha - t^\alpha] a^t b^{1-t} f'(a^t b^{1-t}) dt \right| \\
 & \leq \frac{\ln b - \ln a}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| a^t b^{1-t} |f'(a^t b^{1-t})| dt \\
 & \leq \frac{\ln b - \ln a}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| a^t b^{1-t} |f'(at + (1-t)b)| dt \\
 & \leq \frac{\ln b - \ln a}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| a^t b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \\
 & = \frac{\ln b - \ln a}{2} \left[ \int_0^{\frac{1}{2}} ((1-t)^\alpha - t^\alpha) a^t b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \right. \\
 & \quad \left. + \int_{\frac{1}{2}}^1 (-(1-t)^\alpha + t^\alpha) a^t b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \right] \\
 & = \frac{\ln b - \ln a}{2} \left[ \int_0^{\frac{1}{2}} (1-t)^\alpha a^t b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \right. \\
 & \quad \left. - \int_0^{\frac{1}{2}} t^\alpha a^t b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \right. \\
 & \quad \left. - \int_{\frac{1}{2}}^1 (1-t)^\alpha a^t b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \right]
 \end{aligned}$$

$$\begin{aligned}
& + \int_{\frac{1}{2}}^1 t^\alpha a^t b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \Big] \\
\leq & \frac{\ln b - \ln a}{2} \left[ \int_0^{\frac{1}{2}} (2^{1-\alpha} - t^\alpha) a^t b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \right. \\
& - \int_0^{\frac{1}{2}} t^\alpha a^t b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \\
& - \int_{\frac{1}{2}}^1 (1 - t^\alpha) a^t b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \\
& \left. + \int_{\frac{1}{2}}^1 t^\alpha a^t b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \right] \\
= & \frac{\ln b - \ln a}{2} \left[ \int_0^{\frac{1}{2}} 2^{1-\alpha} a^t b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \right. \\
& - 2 \int_0^{\frac{1}{2}} t^\alpha a^t b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \\
& - \int_{\frac{1}{2}}^1 a^t b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt + 2 \int_{\frac{1}{2}}^1 t^\alpha a^t b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \Big] \\
= & \frac{\ln b - \ln a}{2} \left[ (2^{1-\alpha} + 1) \int_0^{\frac{1}{2}} a^t b^{1-t} \left( t(|f'(a)| - |f'(b)|) + |f'(b)| \right) dt \right. \\
& - 4 \int_0^{\frac{1}{2}} t^\alpha a^t b^{1-t} \left( t(|f'(a)| - |f'(b)|) + |f'(b)| \right) dt \\
& + 2 \int_0^1 t^\alpha a^t b^{1-t} \left( t(|f'(a)| - |f'(b)|) + |f'(b)| \right) dt \\
& \left. - \int_0^1 a^t b^{1-t} \left( t(|f'(a)| - |f'(b)|) + |f'(b)| \right) dt \right] \\
= & \frac{\ln b - \ln a}{2} b \left[ (2^{1-\alpha} + 1) \left( \int_0^{\frac{1}{2}} \left(\frac{a}{b}\right)^t t(|f'(a)| - |f'(b)|) dt + \int_0^{\frac{1}{2}} \left(\frac{a}{b}\right)^t |f'(b)| dt \right) \right. \\
& - 4 \left( \int_0^{\frac{1}{2}} t^{\alpha+1} \left(\frac{a}{b}\right)^t (|f'(a)| - |f'(b)|) dt + \int_0^{\frac{1}{2}} t^\alpha \left(\frac{a}{b}\right)^t |f'(b)| dt \right) \\
& + 2 \left( \int_0^1 t^{\alpha+1} \left(\frac{a}{b}\right)^t (|f'(a)| - |f'(b)|) dt + \int_0^1 t^\alpha \left(\frac{a}{b}\right)^t |f'(b)| dt \right) \\
& \left. - \left( \int_0^1 \left(\frac{a}{b}\right)^t t(|f'(a)| - |f'(b)|) dt + \int_0^1 \left(\frac{a}{b}\right)^t |f'(b)| dt \right) \right] \\
= & \frac{\ln b - \ln a}{2} b \left[ (2^{1-\alpha} + 1) \left( (|f'(a)| - |f'(b)|) \frac{1}{4} (ab^{-1})^{\frac{1}{2}} \sum_{i=0}^{\infty} \frac{(-\frac{1}{2} \ln(ab^{-1}))^{i-1}}{(2)_i} \right. \right.
\end{aligned}$$

$$\begin{aligned}
 & +|f'(b)|\frac{(ab^{-1})^{\frac{1}{2}}-1}{\ln(ab^{-1})} - 4\left(|f'(a)|-|f'(b)|\right)\left(\frac{1}{2}\right)^{\alpha+2}(ab^{-1})^{\frac{1}{2}}\sum_{i=0}^{\infty}\frac{(-\frac{1}{2}\ln(ab^{-1}))^{i-1}}{(\alpha+2)_i} \\
 & +|f'(b)|\left(\frac{1}{2}\right)^{\alpha+1}(ab^{-1})^{\frac{1}{2}}\sum_{i=0}^{\infty}\frac{(-\frac{1}{2}\ln(ab^{-1}))^{i-1}}{(\alpha+1)_i} \\
 & +2\left(|f'(a)|-|f'(b)|\right)(ab^{-1})\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(ab^{-1}))^{i-1}}{(\alpha+2)_i} \\
 & +|f'(b)|(ab^{-1})\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(ab^{-1}))^{i-1}}{(\alpha+1)_i} \\
 & -\left(\left(|f'(a)|-|f'(b)|\right)(ab^{-1})\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(ab^{-1}))^{i-1}}{(2)_i}+|f'(b)|\frac{(ab^{-1})-1}{\ln(ab^{-1})}\right)\Big].
 \end{aligned}$$

Case 2:  $\alpha \in [1, \infty)$ . By using Definition 2.1, Lemma 1.1, Lemma 2.1, Lemma 2.2, Lemma 2.3, Lemma 2.5, Lemma 2.6 and Hölder inequality again, we have

$$\begin{aligned}
 & \left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^\alpha}[{}_H J_{a^+}^\alpha f(b)+{}_H J_{b^-}^\alpha f(a)]\right| \\
 & =\left|\frac{\ln b-\ln a}{2}\int_0^1[(1-t)^\alpha-t^\alpha]a^t b^{1-t}f'(a^t b^{1-t})dt\right| \\
 & \leq\frac{\ln b-\ln a}{2}\int_0^1|(1-t)^\alpha-t^\alpha|a^t b^{1-t}|f'(a^t b^{1-t})|dt \\
 & \leq\frac{\ln b-\ln a}{2}\int_0^1|(1-t)^\alpha-t^\alpha|a^t b^{1-t}|f'(at+(1-t)b)|dt \\
 & \leq\frac{\ln b-\ln a}{2}\int_0^1|(1-t)^\alpha-t^\alpha|a^t b^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt \\
 & =\frac{\ln b-\ln a}{2}\left[\int_0^{\frac{1}{2}}((1-t)^\alpha-t^\alpha)a^t b^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt\right. \\
 & \quad \left.+\int_{\frac{1}{2}}^1(-(1-t)^\alpha+t^\alpha)a^t b^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt\right] \\
 & =\frac{\ln b-\ln a}{2}\left[\int_0^{\frac{1}{2}}(1-t)^\alpha a^t b^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt\right. \\
 & \quad -\int_0^{\frac{1}{2}}t^\alpha a^t b^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt \\
 & \quad -\int_{\frac{1}{2}}^1(1-t)^\alpha a^t b^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt \\
 & \quad \left.+\int_{\frac{1}{2}}^1t^\alpha a^t b^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt\right]
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\ln b - \ln a}{2} \left[ \int_0^{\frac{1}{2}} (1 - t^\alpha) a^t b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \right. \\
&\quad - \int_0^{\frac{1}{2}} t^\alpha a^t b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \\
&\quad - \int_{\frac{1}{2}}^1 (2^{1-\alpha} - t^\alpha) a^t b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \\
&\quad \left. + \int_{\frac{1}{2}}^1 t^\alpha a^t b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \right] \\
&= \frac{\ln b - \ln a}{2} \left[ \int_0^{\frac{1}{2}} a^t b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \right. \\
&\quad - 2 \int_0^{\frac{1}{2}} t^\alpha a^t b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \\
&\quad - \int_{\frac{1}{2}}^1 2^{1-\alpha} a^t b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \\
&\quad \left. + 2 \int_{\frac{1}{2}}^1 t^\alpha a^t b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \right] \\
&= \frac{\ln b - \ln a}{2} \left[ (2^{1-\alpha} + 1) \int_0^{\frac{1}{2}} a^t b^{1-t} (t(|f'(a)| - |f'(b)|) + |f'(b)|) dt \right. \\
&\quad - 4 \int_0^{\frac{1}{2}} t^\alpha a^t b^{1-t} (t(|f'(a)| - |f'(b)|) + |f'(b)|) dt \\
&\quad + 2 \int_0^1 t^\alpha a^t b^{1-t} (t(|f'(a)| - |f'(b)|) + |f'(b)|) dt \\
&\quad \left. - \int_0^1 2^{1-\alpha} a^t b^{1-t} (t(|f'(a)| - |f'(b)|) + |f'(b)|) dt \right] \\
&= \frac{\ln b - \ln a}{2} b \left[ (2^{1-\alpha} + 1) \left( \int_0^{\frac{1}{2}} \left(\frac{a}{b}\right)^t t(|f'(a)| - |f'(b)|) dt + \int_0^{\frac{1}{2}} \left(\frac{a}{b}\right)^t |f'(b)| dt \right) \right. \\
&\quad - 4 \left( \int_0^{\frac{1}{2}} t^{\alpha+1} \left(\frac{a}{b}\right)^t (|f'(a)| - |f'(b)|) dt + \int_0^{\frac{1}{2}} t^\alpha \left(\frac{a}{b}\right)^t |f'(b)| dt \right) \\
&\quad + 2 \left( \int_0^1 t^{\alpha+1} \left(\frac{a}{b}\right)^t (|f'(a)| - |f'(b)|) dt + \int_0^1 t^\alpha \left(\frac{a}{b}\right)^t |f'(b)| dt \right) \\
&\quad \left. - \left( \int_0^1 2^{1-\alpha} \left(\frac{a}{b}\right)^t t(|f'(a)| - |f'(b)|) dt + \int_0^1 2^{1-\alpha} \left(\frac{a}{b}\right)^t |f'(b)| dt \right) \right] \\
&= \frac{\ln b - \ln a}{2} b \left[ (2^{1-\alpha} + 1) \left( (|f'(a)| - |f'(b)|) \frac{1}{4} (ab^{-1})^{\frac{1}{2}} \right. \right.
\end{aligned}$$



$$\begin{aligned}
 & \sum_{i=0}^{\infty} \frac{(-\frac{1}{2} \ln(ab^{-1}))^{i-1}}{(2)_i} + |f'(b)| \frac{(ab^{-1})^{\frac{1}{2}} - 1}{\ln(ab^{-1})} \\
 & -4 \left( (|f'(a)| - |f'(b)|) \left(\frac{1}{2}\right)^{\alpha+2} (ab^{-1})^{\frac{1}{2}} \sum_{i=0}^{\infty} \frac{(-\frac{1}{2} \ln(ab^{-1}))^{i-1}}{(\alpha+2)_i} \right. \\
 & + |f'(b)| \left(\frac{1}{2}\right)^{\alpha+1} (ab^{-1})^{\frac{1}{2}} \sum_{i=0}^{\infty} \frac{(-\frac{1}{2} \ln(ab^{-1}))^{i-1}}{(\alpha+1)_i} \\
 & + 2 \left( (|f'(a)| - |f'(b)|) (ab^{-1}) \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(ab^{-1}))^{i-1}}{(\alpha+2)_i} \right. \\
 & \left. + |f'(b)| (ab^{-1}) \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(ab^{-1}))^{i-1}}{(\alpha+1)_i} \right) \\
 & \left. - 2^{1-\alpha} \left( (|f'(a)| - |f'(b)|) (ab^{-1}) \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(ab^{-1}))^{i-1}}{(2)_i} + |f'(b)| \frac{(ab^{-1}) - 1}{\ln(ab^{-1})} \right) \right].
 \end{aligned}$$

The proof is done.  $\square$

**Theorem 3.2.** *Let  $f : (0, b) \rightarrow \mathbb{R}$  be a differentiable mapping and  $1 < q < \infty$ . If  $|f'|^q$  is measurable and  $|f'|^q$  is convex and  $|f'|$  is the increasing function on  $(0, b)$  for some fixed  $\alpha \in (0, \infty)$ ,  $0 < a < b$ , then the following inequality for fractional integrals holds:*

$$\begin{aligned}
 & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha} [{}_H J_{b^-}^\alpha f(a) + {}_H J_{a^+}^\alpha f(b)] \right| \\
 & \leq \frac{\ln b - \ln a}{2} b \left( \frac{2 - 2(\frac{1}{2})^{\alpha p + 1}}{\alpha p + 1} \right)^{\frac{1}{p}} \left[ |f'(a)| \left( \left(\frac{a}{b}\right)^q \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(q \ln \frac{a}{b})^{i-1}}{(2)_i} \right)^{\frac{1}{q}} \right. \\
 & \left. + |f'(b)| \left( \frac{(\frac{a}{b})^q - 1}{q \ln \frac{a}{b}} - \left(\frac{a}{b}\right)^q \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(q \ln \frac{a}{b})^{i-1}}{(2)_i} \right)^{\frac{1}{q}} \right]
 \end{aligned}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Proof.** By using Definition 2.1, Lemma 1.1, Lemma 2.3, Lemma 2.5, Lemma 2.6 and Hölder inequality, we have

$$\begin{aligned}
 & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha} [{}_H J_{a^+}^\alpha f(b) + {}_H J_{b^-}^\alpha f(a)] \right| \\
 & = \left| \frac{\ln b - \ln a}{2} \int_0^1 [(1-t)^\alpha - t^\alpha] a^t b^{1-t} f'(a^t b^{1-t}) dt \right| \\
 & \leq \frac{\ln b - \ln a}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| a^t b^{1-t} |f'(a^t b^{1-t})| dt
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\ln b - \ln a}{2} b \int_0^1 |(1-t)^\alpha - t^\alpha| \left(\frac{a}{b}\right)^t |f'(at + (1-t)b)| dt \\
&\leq \frac{\ln b - \ln a}{2} b \left( \int_0^1 |(1-t)^\alpha - t^\alpha|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 \left(\frac{a}{b}\right)^{qt} |f'(at + (1-t)b)|^q dt \right)^{\frac{1}{q}} \\
&= \frac{\ln b - \ln a}{2} b \left( \int_0^{\frac{1}{2}} ((1-t)^\alpha - t^\alpha)^p dt + \int_{\frac{1}{2}}^1 (t^\alpha - (1-t)^\alpha)^p dt \right)^{\frac{1}{p}} \\
&\quad \cdot \left( \int_0^1 \left(\frac{a}{b}\right)^{qt} |f'(at + (1-t)b)|^q dt \right)^{\frac{1}{q}} \\
&\leq \frac{\ln b - \ln a}{2} b \left( \int_0^{\frac{1}{2}} ((1-t)^{\alpha p} - t^{\alpha p}) dt + \int_{\frac{1}{2}}^1 (t^{\alpha p} - (1-t)^{\alpha p}) dt \right)^{\frac{1}{p}} \\
&\quad \cdot \left( \int_0^1 \left(\frac{a}{b}\right)^{qt} |f'(at + (1-t)b)|^q dt \right)^{\frac{1}{q}} \\
&\leq \frac{\ln b - \ln a}{2} b \left( \int_0^{\frac{1}{2}} ((1-t)^{\alpha p} - t^{\alpha p}) dt + \int_{\frac{1}{2}}^1 (t^{\alpha p} - (1-t)^{\alpha p}) dt \right)^{\frac{1}{p}} \\
&\quad \cdot \left( \int_0^1 \left(\frac{a}{b}\right)^{qt} (t|f'(a)|^q + (1-t)|f'(b)|^q) dt \right)^{\frac{1}{q}} \\
&= \frac{\ln b - \ln a}{2} b \left( \int_0^{\frac{1}{2}} (1-t)^{\alpha p} dt - \int_0^{\frac{1}{2}} t^{\alpha p} dt + \int_{\frac{1}{2}}^1 t^{\alpha p} dt - \int_{\frac{1}{2}}^1 (1-t)^{\alpha p} dt \right)^{\frac{1}{p}} \\
&\quad \cdot \left( \int_0^1 \left(\frac{a}{b}\right)^{qt} t|f'(a)|^q dt + \int_0^1 \left(\frac{a}{b}\right)^{qt} (1-t)|f'(b)|^q dt \right)^{\frac{1}{q}} \\
&= \frac{\ln b - \ln a}{2} b \left( -\frac{(\frac{1}{2})^{\alpha p+1} - 1}{\alpha p + 1} - \frac{(\frac{1}{2})^{\alpha p+1}}{\alpha p + 1} + \frac{1 - (\frac{1}{2})^{\alpha p+1}}{\alpha p + 1} + \frac{(\frac{1}{2})^{\alpha p+1}}{\alpha p + 1} \right)^{\frac{1}{p}} \\
&\quad \cdot \left( \int_0^1 \left(\frac{a}{b}\right)^{qt} t|f'(a)|^q dt + \int_0^1 \left(\frac{a}{b}\right)^{qt} (1-t)|f'(b)|^q dt \right)^{\frac{1}{q}} \\
&\leq \frac{\ln b - \ln a}{2} b \left( \frac{2 - 2(\frac{1}{2})^{\alpha p+1}}{\alpha p + 1} \right)^{\frac{1}{p}} \left[ \left( \int_0^1 \left(\frac{a}{b}\right)^{qt} t|f'(a)|^q dt \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left( \int_0^1 \left(\frac{a}{b}\right)^{qt} (1-t)|f'(b)|^q dt \right)^{\frac{1}{q}} \right] \\
&= \frac{\ln b - \ln a}{2} b \left( \frac{2 - 2(\frac{1}{2})^{\alpha p+1}}{\alpha p + 1} \right)^{\frac{1}{p}} \left[ |f'(a)| \left( \left(\frac{a}{b}\right)^q \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(q \ln \frac{a}{b})^{i-1}}{(2)_i} \right)^{\frac{1}{q}} \right. \\
&\quad \left. + |f'(b)| \left( \int_0^1 \left(\frac{a}{b}\right)^{qt} dt - \int_0^1 t \left(\frac{a}{b}\right)^{qt} dt \right)^{\frac{1}{q}} \right]
\end{aligned}$$

$$\begin{aligned}
 &= \frac{\ln b - \ln a}{2} b \left( \frac{2 - 2(\frac{1}{2})^{\alpha p + 1}}{\alpha p + 1} \right)^{\frac{1}{p}} \left[ |f'(a)| \left( \left( \frac{a}{b} \right)^q \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(q \ln \frac{a}{b})^{i-1}}{(2)_i} \right)^{\frac{1}{q}} \right. \\
 &\quad \left. + |f'(b)| \left( \frac{(\frac{a}{b})^q - 1}{q \ln \frac{a}{b}} - \left( \frac{a}{b} \right)^q \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(q \ln \frac{a}{b})^{i-1}}{(2)_i} \right)^{\frac{1}{q}} \right].
 \end{aligned}$$

The proof is done.  $\square$

### 3.2. Main results via Lemma 1.2

**Theorem 3.3.** *Let  $f : (0, b) \rightarrow \mathbb{R}$  be a differentiable mapping. If  $|f'|$  is measurable and  $|f'|$  is convex and increasing function on  $(0, b)$  for some fixed  $\alpha \in (0, \infty)$ ,  $0 < a < b$ , then the following inequality for fractional integrals holds:*

$$\begin{aligned}
 &\left| \frac{(\ln x - \ln a)^\alpha + (\ln b - \ln x)^\alpha}{\ln b - \ln a} f(x) - \frac{\Gamma(\alpha + 1)}{\ln b - \ln a} [{}_H J_{x^-}^\alpha f(a) + {}_H J_{x^+}^\alpha f(b)] \right| \\
 &\leq \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a} \left( x(|f'(x)| - |f'(a)|) \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(xa^{-1}))^{i-1}}{(\alpha + 2)_i} \right. \\
 &\quad \left. + x|f'(a)| \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(xa^{-1}))^{i-1}}{(\alpha + 1)_i} \right) \\
 &\quad + \frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a} \left( x(|f'(x)| - |f'(b)|) \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(xb^{-1}))^{i-1}}{(\alpha + 2)_i} \right. \\
 &\quad \left. + x|f'(b)| \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(xb^{-1}))^{i-1}}{(\alpha + 1)_i} \right)
 \end{aligned}$$

for any  $x \in (a, b)$ .

**Proof.** By using Definition 2.1, Lemma 1.2, Lemma 2.1 and Lemma 2.6, we have

$$\begin{aligned}
 &\left| \frac{(\ln x - \ln a)^\alpha + (\ln b - \ln x)^\alpha}{\ln b - \ln a} f(x) - \frac{\Gamma(\alpha + 1)}{\ln b - \ln a} [{}_H J_{x^-}^\alpha f(a) + {}_H J_{x^+}^\alpha f(b)] \right| \\
 &= \left| \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a} \int_0^1 t^\alpha x^t a^{1-t} f'(x^t a^{1-t}) dt \right. \\
 &\quad \left. - \frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a} \int_0^1 t^\alpha x^t b^{1-t} f'(x^t b^{1-t}) dt \right| \\
 &\leq \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a} \int_0^1 t^\alpha x^t a^{1-t} |f'(x^t a^{1-t})| dt \\
 &\quad + \frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a} \int_0^1 t^\alpha x^t b^{1-t} |f'(x^t b^{1-t})| dt \\
 &\leq \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a} \int_0^1 t^\alpha x^t a^{1-t} |f'(tx + (1-t)a)| dt
 \end{aligned}$$

$$\begin{aligned}
& + \frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a} \int_0^1 t^\alpha x^t b^{1-t} |f'(tx + (1-t)b)| dt \\
\leq & \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a} \int_0^1 t^\alpha x^t a^{1-t} (t|f'(x)| + (1-t)|f'(a)|) dt \\
& + \frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a} \int_0^1 t^\alpha x^t b^{1-t} (t|f'(x)| + (1-t)|f'(b)|) dt \\
= & \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a} \int_0^1 \left( t^{\alpha+1} x^t a^{1-t} |f'(x)| + t^\alpha x^t a^{1-t} |f'(a)| - t^{\alpha+1} x^t a^{1-t} |f'(a)| \right) dt \\
& + \frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a} \int_0^1 \left( t^{\alpha+1} x^t b^{1-t} |f'(x)| + t^\alpha x^t b^{1-t} |f'(b)| - t^{\alpha+1} x^t b^{1-t} |f'(b)| \right) dt \\
= & \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a} \left( a|f'(x)| \int_0^1 t^{\alpha+1} (xa^{-1})^t dt + a|f'(a)| \int_0^1 t^\alpha (xa^{-1})^t dt \right. \\
& \left. - a|f'(a)| \int_0^1 t^{\alpha+1} (xa^{-1})^t dt \right) + \frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a} \left( b|f'(x)| \int_0^1 t^{\alpha+1} (xb^{-1})^t dt \right. \\
& \left. + b|f'(b)| \int_0^1 t^\alpha (xb^{-1})^t dt - b|f'(b)| \int_0^1 t^{\alpha+1} (xb^{-1})^t dt \right) \\
= & \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a} \left( a|f'(x)| xa^{-1} \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(xa^{-1}))^{i-1}}{(\alpha+2)_i} \right. \\
& + a|f'(a)| xa^{-1} \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(xa^{-1}))^{i-1}}{(\alpha+1)_i} - a|f'(a)| xa^{-1} \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(xa^{-1}))^{i-1}}{(\alpha+2)_i} \Big) \\
& + \frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a} \left( b|f'(x)| xb^{-1} \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(xb^{-1}))^{i-1}}{(\alpha+2)_i} \right. \\
& + b|f'(b)| xb^{-1} \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(xb^{-1}))^{i-1}}{(\alpha+1)_i} - b|f'(b)| xb^{-1} \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(xb^{-1}))^{i-1}}{(\alpha+2)_i} \Big) \\
= & \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a} \left( x(|f'(x)| - |f'(a)|) \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(xa^{-1}))^{i-1}}{(\alpha+2)_i} \right. \\
& + x|f'(a)| \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(xa^{-1}))^{i-1}}{(\alpha+1)_i} \Big) \\
& + \frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a} \left( x(|f'(x)| - |f'(b)|) \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(xb^{-1}))^{i-1}}{(\alpha+2)_i} \right. \\
& \left. + x|f'(b)| \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(xb^{-1}))^{i-1}}{(\alpha+1)_i} \right).
\end{aligned}$$

The proof is completed.  $\square$

**Theorem 3.4.** *Let  $f : (0, b) \rightarrow \mathbb{R}$  be a differentiable mapping and  $1 < q < \infty$ . If*

$|f'|^q$  is measurable and  $|f'|^q$  is convex and  $|f'|$  is the increasing function on  $(0, b)$  for some fixed  $\alpha \in (0, \infty)$ ,  $0 < a < b$ , then the following inequality for fractional integrals holds:

$$\begin{aligned} & \left| \frac{(\ln x - \ln a)^\alpha + (\ln b - \ln x)^\alpha}{\ln b - \ln a} f(x) - \frac{\Gamma(\alpha + 1)}{\ln b - \ln a} [{}_H J_{x^-}^\alpha f(a) + {}_H J_{x^+}^\alpha f(b)] \right| \\ & \leq \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a} \left( \frac{1}{\alpha p + 1} \right)^{\frac{1}{p}} \left[ a |f'(x)| \left( \left( \frac{x}{a} \right)^q \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(\frac{x}{a})^q)^{i-1}}{(2)_i} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + x |f'(a)| \left( \left( \frac{a}{x} \right)^q \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(\frac{a}{x})^q)^{i-1}}{(2)_i} \right)^{\frac{1}{q}} \right] \\ & \quad + \frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a} \left( \frac{1}{\alpha p + 1} \right)^{\frac{1}{p}} \left[ b |f'(x)| \left( \left( \frac{x}{b} \right)^q \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(\frac{x}{b})^q)^{i-1}}{(2)_i} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + x |f'(b)| \left( \left( \frac{b}{x} \right)^q \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(\frac{b}{x})^q)^{i-1}}{(2)_i} \right)^{\frac{1}{q}} \right] \end{aligned}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Proof.** By using Definition 2.1, Lemma 1.2, Lemma 2.1, Lemma 2.4, Lemma 2.6 and Hölder inequality, we have

$$\begin{aligned} & \left| \frac{(\ln x - \ln a)^\alpha + (\ln b - \ln x)^\alpha}{\ln b - \ln a} f(x) - \frac{\Gamma(\alpha + 1)}{\ln b - \ln a} [{}_H J_{x^-}^\alpha f(a) + {}_H J_{x^+}^\alpha f(b)] \right| \\ & = \left| \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a} \int_0^1 t^\alpha x^t a^{1-t} f'(x^t a^{1-t}) dt \right. \\ & \quad \left. - \frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a} \int_0^1 t^\alpha x^t b^{1-t} f'(x^t b^{1-t}) dt \right| \\ & \leq \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a} \int_0^1 t^\alpha x^t a^{1-t} |f'(x^t a^{1-t})| dt \\ & \quad + \frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a} \int_0^1 t^\alpha x^t b^{1-t} |f'(x^t b^{1-t})| dt \\ & \leq \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a} \left( \int_0^1 t^{\alpha p} dt \right)^{\frac{1}{p}} \left( \int_0^1 (x^t a^{1-t} |f'(x^t a^{1-t})|)^q dt \right)^{\frac{1}{q}} \\ & \quad + \frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a} \left( \int_0^1 t^{\alpha p} dt \right)^{\frac{1}{p}} \left( \int_0^1 (x^t b^{1-t} |f'(x^t b^{1-t})|)^q dt \right)^{\frac{1}{q}} \\ & \leq \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a} \left( \int_0^1 t^{\alpha p} dt \right)^{\frac{1}{p}} \left( \int_0^1 (x^t a^{1-t})^q |f'(tx + (1-t)a)|^q dt \right)^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
& + \frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a} \left( \int_0^1 t^{\alpha p} dt \right)^{\frac{1}{p}} \left( \int_0^1 (x^t b^{1-t})^q |f'(tx + (1-t)b)|^q dt \right)^{\frac{1}{q}} \\
\leq & \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a} \left( \int_0^1 t^{\alpha p} dt \right)^{\frac{1}{p}} \left( \int_0^1 x^{qt} a^{q(1-t)} (t|f'(x)|^q + (1-t)|f'(a)|^q) dt \right)^{\frac{1}{q}} \\
& + \frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a} \left( \int_0^1 t^{\alpha p} dt \right)^{\frac{1}{p}} \left( \int_0^1 x^{qt} b^{q(1-t)} (t|f'(x)|^q + (1-t)|f'(b)|^q) dt \right)^{\frac{1}{q}} \\
\leq & \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a} \left( \int_0^1 t^{\alpha p} dt \right)^{\frac{1}{p}} \left[ \left( \int_0^1 x^{tq} a^{(1-t)q} t |f'(x)|^q dt \right)^{\frac{1}{q}} \right. \\
& \left. + \left( \int_0^1 x^{tq} a^{(1-t)q} (1-t) |f'(a)|^q dt \right)^{\frac{1}{q}} \right] \\
& + \frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a} \left( \int_0^1 t^{\alpha p} dt \right)^{\frac{1}{p}} \left[ \left( \int_0^1 x^{tq} b^{(1-t)q} t |f'(x)|^q dt \right)^{\frac{1}{q}} \right. \\
& \left. + \left( \int_0^1 x^{tq} b^{(1-t)q} (1-t) |f'(b)|^q dt \right)^{\frac{1}{q}} \right] \\
= & \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a} \left( \frac{1}{\alpha p + 1} \right)^{\frac{1}{p}} \left[ a |f'(x)| \left( \int_0^1 \left( \frac{x}{a} \right)^{tq} t dt \right)^{\frac{1}{q}} \right. \\
& \left. + x |f'(a)| \left( \int_0^1 \left( \frac{a}{x} \right)^{t'q} t' dt' \right)^{\frac{1}{q}} \right] + \frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a} \left( \frac{1}{\alpha p + 1} \right)^{\frac{1}{p}} \\
& \cdot \left[ b |f'(x)| \left( \int_0^1 \left( \frac{x}{b} \right)^{tq} t dt \right)^{\frac{1}{q}} + x |f'(b)| \left( \int_0^1 \left( \frac{b}{x} \right)^{t'q} t' dt' \right)^{\frac{1}{q}} \right] \\
= & \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a} \left( \frac{1}{\alpha p + 1} \right)^{\frac{1}{p}} \left[ a |f'(x)| \left( \left( \frac{x}{a} \right)^q \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(\frac{x}{a})^q)^{i-1}}{(2)_i} \right)^{\frac{1}{q}} \right. \\
& \left. + x |f'(a)| \left( \left( \frac{a}{x} \right)^q \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(\frac{a}{x})^q)^{i-1}}{(2)_i} \right)^{\frac{1}{q}} \right] \\
& + \frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a} \left( \frac{1}{\alpha p + 1} \right)^{\frac{1}{p}} \left[ b |f'(x)| \left( \left( \frac{x}{b} \right)^q \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(\frac{x}{b})^q)^{i-1}}{(2)_i} \right)^{\frac{1}{q}} \right. \\
& \left. + x |f'(b)| \left( \left( \frac{b}{x} \right)^q \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(\frac{b}{x})^q)^{i-1}}{(2)_i} \right)^{\frac{1}{q}} \right].
\end{aligned}$$

The proof is done.  $\square$

#### 4. Applications to special means

Consider the following special means (see [26]) for arbitrary real numbers  $x, y, x \neq y$  as follows:

$$(i) A(x, y) = \frac{x+y}{2}, x, y \in \mathbb{R}.$$

$$(ii) L(x, y) = \frac{y-x}{\ln|y|-\ln|x|}, |x| \neq |y|, xy \neq 0.$$

$$(iii) L_n(x, y) = \left[ \frac{y^{n+1}-x^{n+1}}{(n+1)(y-x)} \right]^{\frac{1}{n}}, n \in \mathbb{Z} \setminus \{-1, 0\}, x, y \in \mathbb{R}, x \neq y.$$

Using the results in Section 3., we give some applications to special means of real numbers.

**Proposition 4.1.** *Let  $a, b \in \mathbb{R}^+ \setminus \{0\}, 0 < a < b, x \in [0, b]$ . Then*

$$\begin{aligned} & \left| A(a, b) - L(a, b) \right| \\ \leq & \frac{a(\ln a - \ln b) - (\sqrt{a} - \sqrt{b})^2}{\ln b - \ln a} + \frac{a+b}{2}; \\ & \left| A(a, b) - L(a, b) \right| \\ \leq & \frac{\ln b - \ln a}{2} \left( \frac{2 - 2(\frac{1}{2})^{p+1}}{p+1} \right)^{\frac{1}{p}} \left[ \left( \frac{a^q(\ln a - \ln b)q - a^q + b^q}{q^2(\ln a - \ln b)^2} \right)^{\frac{1}{q}} \right. \\ & \left. + \left( \frac{a^q - b^q}{q(\ln a - \ln b)} - \frac{a^q(\ln a - \ln b)q - a^q + b^q}{q^2(\ln a - \ln b)^2} \right)^{\frac{1}{q}} \right]; \\ & \left| A(x, x) - L(a, b) \right| \\ \leq & \frac{(2 \ln x - \ln a - \ln b - 2)x + a + b}{\ln b - \ln a}; \\ & \left| A(x, x) - L(a, b) \right| \\ \leq & \frac{(\ln x - \ln a)^2}{\ln b - \ln a} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left[ \left( \frac{x^q(\ln x - \ln a)q - x^q + a^q}{q^2(\ln x - \ln a)^2} \right)^{\frac{1}{q}} \right. \\ & \left. + \left( \frac{a^q(\ln a - \ln x)q - a^q + x^q}{q^2(\ln x - \ln a)^2} \right)^{\frac{1}{q}} \right] + \frac{(\ln b - \ln x)^2}{\ln b - \ln a} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \\ & \cdot \left[ \left( \frac{x^q(\ln x - \ln b)q - x^q + b^q}{q^2(\ln x - \ln b)^2} \right)^{\frac{1}{q}} + \left( \frac{b^q(\ln b - \ln x)q - b^q + x^q}{q^2(\ln x - \ln b)^2} \right)^{\frac{1}{q}} \right]. \end{aligned}$$

**Proof.** Applying Theorems 3.1, 3.2, 3.3, 3.4, for  $f(x) = x$  and  $\alpha = 1$ , one can obtain the results immediately.  $\square$

**Proposition 4.2.** *Let  $a, b \in \mathbb{R}^+ \setminus \{0\}$ ,  $0 < a < b$ ,  $x \in [0, b]$ ,  $n \geq 2$ . Then*

$$\begin{aligned}
& \left| A(a^n, b^n) - \frac{1}{2}L(a, b)L_{n-1}^{n-1}(a, b) \right| \\
& \leq \frac{1}{2}(na^{n-1} - 3nb^{n-1}) \left( a - \sqrt{ab} + \frac{(\sqrt{a} - \sqrt{b})^2}{\ln a - \ln b} \right) + \frac{1}{2}nb^{n-1}(\sqrt{a} - \sqrt{b})^2 \\
& \quad + (na^{n-1} - nb^{n-1}) \left( -a + \frac{\sqrt{ab}}{2} - \frac{2\sqrt{ab} - 2a}{\ln a - \ln b} - \frac{2(\sqrt{a} - \sqrt{b})^2}{(\ln a - \ln b)^2} \right) \\
& \left| A(a^n, b^n) - \frac{1}{2}L(a, b)L_{n-1}^{n-1}(a, b) \right|; \\
& \leq \frac{\ln b - \ln a}{2} \left( \frac{2 - 2(\frac{1}{2})^{p+1}}{p+1} \right)^{\frac{1}{p}} \left[ na^{n-1} \left( \frac{a^q(\ln a - \ln b)q - a^q + b^q}{q^2(\ln a - \ln b)^2} \right)^{\frac{1}{q}} \right. \\
& \quad \left. + nb^{n-1} \left( \frac{a^q - b^q}{q(\ln a - \ln b)} - \frac{a^q(\ln a - \ln b)q - a^q + b^q}{q^2(\ln a - \ln b)^2} \right)^{\frac{1}{q}} \right]; \\
& \left| A(x^n, x^n) - L(a, b)L_{n-1}^{n-1}(a, b) \right| \\
& \leq \frac{nx^n(\ln x - \ln a)^2 - n(\ln x - \ln a)(a^{n-1}x + a^n - 2x^n) + 2n(x - a)(-a^{n-1} + x^{n-1})}{(\ln b - \ln a)(\ln x - \ln a)} \\
& \quad + \frac{nx^n(\ln x - \ln b)^2 - n(\ln x - \ln b)(b^{n-1}x + b^n - 2x^n) + 2n(x - b)(-b^{n-1} + x^{n-1})}{(\ln b - \ln a)(\ln x - \ln b)}; \\
& \left| A(x^n, x^n) - L(a, b)L_{n-1}^{n-1}(a, b) \right| \\
& \leq \frac{(\ln x - \ln a)^2}{\ln b - \ln a} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left[ nx^{n-1} \left( \frac{x^q(\ln x - \ln a)q - x^q + a^q}{q^2(\ln x - \ln a)^2} \right)^{\frac{1}{q}} \right. \\
& \quad \left. + na^{n-1} \left( \frac{a^q(\ln a - \ln x)q - a^q + x^q}{q^2(\ln x - \ln a)^2} \right)^{\frac{1}{q}} \right] + \frac{(\ln b - \ln x)^2}{\ln b - \ln a} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \\
& \quad \cdot \left[ nx^{n-1} \left( \frac{x^q(\ln x - \ln b)q - x^q + b^q}{q^2(\ln x - \ln b)^2} \right)^{\frac{1}{q}} + nb^{n-1} \left( \frac{b^q(\ln b - \ln x)q - b^q + x^q}{q^2(\ln x - \ln b)^2} \right)^{\frac{1}{q}} \right].
\end{aligned}$$

**Proof.** Applying Theorems 3.1, 3.2, 3.3, 3.4, for  $f(x) = x^n$  and  $\alpha = 1$ , one can obtain the results immediately.  $\square$

**Proposition 4.3.** *Let  $a, b \in \mathbb{R}^+ \setminus \{0\}$ ,  $0 < a < b$ ,  $x \in [0, b]$ ,  $n \geq 2$ . Then*

$$\begin{aligned}
& \left| A\left(\frac{1}{a}, \frac{1}{b}\right) - \frac{1}{2}L\left(\frac{1}{a}, \frac{1}{b}\right) \right| \\
& \leq \frac{1}{2} \left( \frac{1}{a^2} - \frac{3}{b^2} \right) \left( a - \sqrt{ab} - \frac{(\sqrt{a} - \sqrt{b})^2}{\ln a - \ln b} \right) + \frac{1}{2b^2}(\sqrt{a} - \sqrt{b})^2
\end{aligned}$$



$$\begin{aligned}
 & + \left( \frac{1}{a^2} - \frac{1}{b^2} \right) \left( -a + \frac{1}{2}\sqrt{ab} - \frac{2\sqrt{ab} - 2a}{\ln a - \ln b} - \frac{2(\sqrt{a} - \sqrt{b})^2}{(\ln a - \ln b)^2} \right); \\
 & \left| A \left( \frac{1}{a}, \frac{1}{b} \right) - \frac{1}{2} L \left( \frac{1}{a}, \frac{1}{b} \right) \right| \\
 \leq & \frac{\ln b - \ln a}{2} \left( \frac{2 - 2(\frac{1}{2})^{p+1}}{p+1} \right)^{\frac{1}{p}} \left[ \frac{1}{a^2} \left( \frac{a^q(\ln a - \ln b)q - a^q + b^q}{q^2(\ln a - \ln b)^2} \right)^{\frac{1}{q}} \right. \\
 & \left. + \frac{1}{b^2} \left( \frac{a^q - b^q}{q(\ln a - \ln b)} - \frac{a^q(\ln a - \ln b)q - a^q + b^q}{q^2(\ln a - \ln b)^2} \right)^{\frac{1}{q}} \right]; \\
 & \left| A \left( \frac{1}{x}, \frac{1}{x} \right) - L \left( \frac{1}{a}, \frac{1}{b} \right) \right| \\
 \leq & \frac{1}{\ln b - \ln a} \left[ \left( \frac{1}{x^2} - \frac{1}{a^2} \right) \frac{x(\ln x - \ln a)^2 - 2x(\ln x - \ln a) + 2x - 2a}{\ln x - \ln a} \right. \\
 & \left. + \frac{x(\ln x - \ln a) - x + a}{a^2} \right] + \frac{1}{\ln b - \ln a} \left[ \left( \frac{1}{x^2} - \frac{1}{b^2} \right) \right. \\
 & \left. \frac{x(\ln x - \ln b)^2 - 2x(\ln x - \ln b) + 2x - 2b}{\ln x - \ln b} + \frac{x(\ln x - \ln b) - x + b}{b^2} \right]; \\
 & \left| A \left( \frac{1}{x}, \frac{1}{x} \right) - L \left( \frac{1}{a}, \frac{1}{b} \right) \right| \\
 \leq & \frac{(\ln x - \ln a)^2}{\ln b - \ln a} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left[ \frac{1}{x^2} \left( \frac{x^q(\ln x - \ln a)q - x^q + a^q}{q^2(\ln x - \ln a)^2} \right)^{\frac{1}{q}} \right. \\
 & \left. + \frac{1}{a^2} \left( \frac{a^q(\ln a - \ln x)q - a^q + x^q}{q^2(\ln x - \ln a)^2} \right)^{\frac{1}{q}} \right] + \frac{(\ln b - \ln x)^2}{\ln b - \ln a} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \\
 & \left[ \frac{1}{x^2} \left( \frac{x^q(\ln x - \ln b)q - x^q + b^q}{q^2(\ln x - \ln b)^2} \right)^{\frac{1}{q}} + \frac{1}{b^2} \left( \frac{b^q(\ln b - \ln x)q - b^q + x^q}{q^2(\ln x - \ln b)^2} \right)^{\frac{1}{q}} \right].
 \end{aligned}$$

**Proof.** Applying Theorems 3.1, 3.2, 3.3, 3.4, for  $f(x) = \frac{1}{x}$  and  $\alpha = 1$ , one can obtain the results immediately.  $\square$

**Proposition 4.4.** Let  $a, b \in \mathbb{R}^+ \setminus \{0\}, 0 < a < b, x \in [0, b], n \geq 2$ . Then

$$\begin{aligned}
 & \left| A(x, x) - L(b^{-1}, a^{-1}) \right| \\
 \leq & \frac{(2 \ln x + \ln a + \ln b - 2)x + a^{-1} + b^{-1}}{\ln b - \ln a}; \\
 & \left| A(x, x) - L(b^{-1}, a^{-1}) \right| \\
 \leq & \frac{(\ln x + \ln b)^2}{\ln b - \ln a} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left[ \left( \frac{x^q(\ln x + \ln b)q - x^q + b^{-q}}{q^2(\ln x + \ln b)^2} \right)^{\frac{1}{q}} \right.
 \end{aligned}$$

$$\begin{aligned}
& + \left( \frac{b^{-q}(-\ln b - \ln x)q - b^{-q} + x^q}{q^2(\ln x + \ln b)^2} \right)^{\frac{1}{q}} \Big] + \frac{(\ln a + \ln x)^2}{\ln b - \ln a} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \\
& \cdot \left[ \left( \frac{x^q(\ln x + \ln a)q - x^q + a^{-q}}{q^2(\ln x + \ln a)^2} \right)^{\frac{1}{q}} + \left( \frac{a^{-q}(-\ln a - \ln x)q - a^{-q} + x^q}{q^2(\ln x + \ln a)^2} \right)^{\frac{1}{q}} \right]; \\
& \left| A(b^{-1}, a^{-1}) - L(b^{-1}, a^{-1}) \right| \\
\leq & \frac{b^{-1}(\ln a - \ln b) - (\sqrt{b^{-1}} - \sqrt{a^{-1}})^2}{\ln b - \ln a} + \frac{b^{-1} + a^{-1}}{2}; \\
& \left| A(b^{-1}, a^{-1}) - L(b^{-1}, a^{-1}) \right| \\
\leq & \frac{\ln b - \ln a}{2} \left( \frac{2 - 2(\frac{1}{2})^{p+1}}{p+1} \right)^{\frac{1}{p}} \left[ \left( \frac{b^{-q}(\ln a - \ln b)q - b^{-q} + a^{-q}}{q^2(\ln a - \ln b)^2} \right)^{\frac{1}{q}} \right. \\
& \left. + \left( \frac{b^{-q} - a^{-q}}{q(\ln a - \ln b)} - \frac{b^{-q}(\ln a - \ln b)q - b^{-q} + a^{-q}}{q^2(\ln a - \ln b)^2} \right)^{\frac{1}{q}} \right]; \\
& \left| A(x^n, x^n) - L(b^{-1}, a^{-1})L_{n-1}^{n-1}(b^{-1}, a^{-1}) \right| \\
\leq & \frac{nx^n(\ln x + \ln b)^2 - n(\ln x + \ln b)(b^{1-n}x + b^{-n} - 2x^n) + 2n(x - b^{-1})(-b^{1-n} + x^{n-1})}{(\ln b - \ln a)(\ln x + \ln b)} \\
& + \frac{nx^n(\ln x + \ln a)^2 - n(\ln x + \ln a)(a^{1-n}x + a^{-n} - 2x^n) + 2n(x - a^{-1})(-a^{1-n} + x^{n-1})}{(\ln b - \ln a)(\ln x + \ln a)}; \\
& \left| A(x^n, x^n) - L(b^{-1}, a^{-1})L_{n-1}^{n-1}(b^{-1}, a^{-1}) \right| \\
\leq & \frac{(\ln x + \ln b)^2}{\ln b - \ln a} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left[ nx^{n-1} \left( \frac{x^q(\ln x + \ln b)q - x^q + b^{-q}}{q^2(\ln x + \ln b)^2} \right)^{\frac{1}{q}} \right. \\
& \left. + nb^{1-n} \left( \frac{b^{-q}(-\ln b - \ln x)q - b^{-q} + x^q}{q^2(\ln x + \ln b)^2} \right)^{\frac{1}{q}} \right] + \frac{(\ln x + \ln a)^2}{\ln b - \ln a} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \\
& \cdot \left[ nx^{n-1} \left( \frac{x^q(\ln x + \ln a)q - x^q + a^{-q}}{q^2(\ln x + \ln a)^2} \right)^{\frac{1}{q}} + na^{1-n} \left( \frac{a^{-q}(-\ln a - \ln x)q - a^{-q} + x^q}{q^2(\ln x + \ln a)^2} \right)^{\frac{1}{q}} \right]; \\
& \left| A(b^{-n}, a^{-n}) - \frac{1}{2}L(b^{-1}, a^{-1})L_{n-1}^{n-1}(b^{-1}, a^{-1}) \right| \\
\leq & \frac{1}{2}(nb^{1-n} - 3na^{1-n}) \left( b^{-1} - \sqrt{b^{-1}a^{-1}} + \frac{(\sqrt{b^{-1}} - \sqrt{a^{-1}})^2}{\ln a - \ln b} \right) + \frac{1}{2}na^{1-n}(\sqrt{b^{-1}} - \sqrt{a^{-1}})^2 \\
& + (nb^{1-n} - na^{1-n}) \left( -b^{-1} + \frac{\sqrt{b^{-1}a^{-1}}}{2} - \frac{2\sqrt{b^{-1}a^{-1}} - 2a^{-1}}{\ln a - \ln b} - \frac{2(\sqrt{b^{-1}} - \sqrt{a^{-1}})^2}{(\ln a - \ln b)^2} \right); \\
& \left| A(b^{-n}, a^{-n}) - \frac{1}{2}L(b^{-1}, a^{-1})L_{n-1}^{n-1}(b^{-1}, a^{-1}) \right|
\end{aligned}$$

$$\begin{aligned}
 &\leq \frac{\ln b - \ln a}{2} \left( \frac{2 - 2(\frac{1}{2})^{p+1}}{p+1} \right)^{\frac{1}{p}} \left[ nb^{1-n} \left( \frac{b^{-q}(\ln a - \ln b)q - b^{-q} + a^{-q}}{q^2(\ln a - \ln b)^2} \right)^{\frac{1}{q}} \right. \\
 &\quad \left. + na^{1-n} \left( \frac{b^{-q} - a^{-q}}{q(\ln a - \ln b)} - \frac{b^{-q}(\ln a - \ln b)q - b^{-q} + a^{-q}}{q^2(\ln a - \ln b)^2} \right)^{\frac{1}{q}} \right]; \\
 &\quad \left| A \left( \frac{1}{x}, \frac{1}{x} \right) - L(b, a) \right| \\
 &\leq \frac{1}{\ln b - \ln a} \left[ \left( \frac{1}{x^2} - b^2 \right) \frac{x(\ln x + \ln b)^2 - 2x(\ln x + \ln b) + 2x - 2b^{-1}}{\ln x + \ln b} \right. \\
 &\quad \left. + b^2 (x(\ln x + \ln b) - x + b^{-1}) \right] + \frac{1}{\ln b - \ln a} \left[ \left( \frac{1}{x^2} - a^2 \right) \right. \\
 &\quad \left. \frac{x(\ln x + \ln a)^2 - 2x(\ln x + \ln a) + 2x - 2a^{-1}}{\ln x + \ln a} + a^2 (x(\ln x + \ln a) - x + a^{-1}) \right]; \\
 &\quad \left| A \left( \frac{1}{x}, \frac{1}{x} \right) - L(b, a) \right| \\
 &\leq \frac{(\ln x + \ln b)^2}{\ln b - \ln a} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left[ \frac{1}{x^2} \left( \frac{x^q(\ln x + \ln b)q - x^q + b^{-q}}{q^2(\ln x + \ln b)^2} \right)^{\frac{1}{q}} \right. \\
 &\quad \left. + b^2 \left( \frac{b^{-q}(-\ln b - \ln x)q - b^{-q} + x^q}{q^2(\ln x + \ln b)^2} \right)^{\frac{1}{q}} \right] + \frac{(\ln a + \ln x)^2}{\ln b - \ln a} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \\
 &\quad \left[ \frac{1}{x^2} \left( \frac{x^q(\ln x + \ln a)q - x^q + a^{-q}}{q^2(\ln x + \ln a)^2} \right)^{\frac{1}{q}} + a^2 \left( \frac{a^{-q}(-\ln a - \ln x)q - a^{-q} + x^q}{q^2(\ln x + \ln a)^2} \right)^{\frac{1}{q}} \right]; \\
 &\quad \left| A(b, a) - \frac{1}{2}L(b, a) \right| \\
 &\leq \frac{1}{2}(b^2 - 3a^2) \left( b^{-1} - \sqrt{b^{-1}a^{-1}} - \frac{(\sqrt{b^{-1}} - \sqrt{a^{-1}})^2}{\ln a - \ln b} \right) + \frac{1}{2}a^2(\sqrt{b^{-1}} - \sqrt{a^{-1}})^2 \\
 &\quad + (b^2 - a^2) \left( -b^{-1} + \frac{1}{2}\sqrt{b^{-1}a^{-1}} - \frac{2\sqrt{b^{-1}a^{-1}} - 2b^{-1}}{\ln a - \ln b} - \frac{2(\sqrt{b^{-1}} - \sqrt{a^{-1}})^2}{(\ln a - \ln b)^2} \right); \\
 &\quad \left| A(b, a) - \frac{1}{2}L(b, a) \right| \\
 &\leq \frac{\ln b - \ln a}{2} \left( \frac{2 - 2(\frac{1}{2})^{p+1}}{p+1} \right)^{\frac{1}{p}} \left[ b^2 \left( \frac{b^{-q}(\ln a - \ln b)q - b^{-q} + a^{-q}}{q^2(\ln a - \ln b)^2} \right)^{\frac{1}{q}} \right. \\
 &\quad \left. + a^2 \left( \frac{b^{-q} - a^{-q}}{q(\ln a - \ln b)} - \frac{b^{-q}(\ln a - \ln b)q - b^{-q} + a^{-q}}{q^2(\ln a - \ln b)^2} \right)^{\frac{1}{q}} \right].
 \end{aligned}$$

**Proof.** Making the substitutions  $a \rightarrow b^{-1}$ ,  $b \rightarrow a^{-1}$  in Proposition 4.1, Proposition 4.2 and Proposition 4.3, one can obtain the desired inequalities respectively.  $\square$

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