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η-RICCI SOLITONS IN (ε, δ)-TRANS-SASAKIAN MANIFOLDS

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Abstract. The objective of the present paper is to study (ε, δ)-trans-Sasakian manifolds admitting η-Ricci solitons. It is shown that a symmetric second order covariant tensor in an (ε, δ)-trans-Sasakian manifold is a constant multiple of the metric tensor. Also, an example of an η-Ricci soliton in a 3-diemsional (ε, δ)-trans-Sasakian manifold is provided in the region where (ε, δ)-Trans Sasakian manifold is expanding.
Keywords: Sasakian manifolds; Ricci soliton; Tensor.

1. Introduction

In 1985, J. A. Oubina [22] introduced a new class of almost contact metric manifolds known as trans-Sasakian manifolds. An almost contact metric structure on a manifold M is called a trans-Sasakian structure if the product manifold $M \times \mathbb{R}$ belongs to the class W_4, where the classification of almost Hermition manifolds appears as a class W_4 of Hermitian manifolds which are closely related to locally conformal Kähler manifolds studied by Gray and Hervella [14]. The class $C_5 \oplus C_6$ [22] coincides with the class of trans-Sasakian structure of type (α, β). This class consists of both Sasakian and Kenmotsu structures. If $\alpha = 1$, $\beta = 0$ then the class turn into Sasakian and when $\alpha = 0$, $\beta = 1$ then it turn into Kenmotsu. The above manifolds are studied by many authors like D. E. Blair and J. C. Marrero [1], K. Kenmotsu [17], C. S. Bagewadi and Venkatesha [8], U. C. De and M. M. Tripathi [12].

The differential geometry of manifolds with indefinite metric plays an interesting role in physics. Manifolds with indefinite metric have been studied by several authors. The concept of (ϵ)-Sasakian manifolds was initiated by A. Bejancu and K. L. Duggal [2] and further investigation was taken up by X. Xufeng and C. Xiaoli [30]. U. C. De and A. Sarkar [11] studied (ε)-Kenmotsu manifolds with indefinite metric. S. S. Shukla and D. D. Singh [25] extended with indefinite metric which is a natural generalization of both (ε)-Sasakian and (ε)-Kenmotsu manifolds. The

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authors H. G. Nagaraja et al. [20] studied (ε, δ)-trans-Sasakian manifolds which are extensions of (ε)-trans-Sasakian manifolds. M. D. Siddiqi et al. also studied some properties of (ε, δ)-trans-Sasakian manifolds in [26].

In 1982, R. S. Hamilton [15] stated that Ricci solitons move under the Ricci flow simply by diffeomorphisms of the initial metric, that is, they are stationary points of the Ricci flow which is given by

$$\frac{\partial g}{\partial t} = -2Ric(g). \tag{1.1}$$

Definition 1.1. A Ricci soliton (g, V, λ) on a Riemannian manifold is defined by

$$\mathcal{L}_V g + 2S + 2\lambda = 0, \tag{1.2}$$

where S is the Ricci tensor, $\mathcal{L}_V$ is the Lie derivative along the vector field V on M and λ is a real scalar. The Ricci soliton is said to be shrinking, steady or expanding depending on whether $\lambda < 0, \lambda = 0$ and $\lambda > 0$, respectively.

In 1925, Levy [18] obtained necessary and sufficient conditions for the existence of such tensors. later, R. Sharma [24] initiated a study of Ricci solitons in contact Riemannian geometry . After that, Tripathi [27], Nagaraja et al. [21] and others like C. S. Bagewadi et al. ([7], [16]) extensively studied Ricci solitons in almost (ϵ)-contact metric manifolds. In 2009, J. T. Cho and M. Kimura [10] introduced the notion of η-Ricci soliton and gave a classification of real hypersurfaces in non-flat complex space forms admitting η-Ricci solitons. Later η-Ricci solitons in (ε)-almost paracontact metric manifolds were studied by A. M. Blaga et. al. in [5]. Moreover, η-Ricci solitons have been studied by various authors for different structures (see [3], [4], [23], [9], [28]). Recently, K. Venu et al. [29] studied the η-Ricci solitons in trans-Sasakian manifolds. Motivated by these studies in the present paper we investigate η-Ricci solitons in 3-dimensional (ε, δ)-trans-Sasakian manifolds and derive the expression for the scalar curvature.

1.1. Preliminaries

Let M be an almost contact metric manifold equipped with the almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1, 1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying

$$\phi^2 = -I + \eta \otimes \xi, \ \ \eta(\xi) = 1, \ \ \eta \circ \phi = 0, \ \ \phi\xi = 0, \tag{1.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y), \qquad \eta(X) = \varepsilon g(X, \xi), \qquad g(\xi, \xi) = \varepsilon, \tag{1.4}$$

for all X, Y vector fields on M, where ε is 1 or -1 according as ξ is space-like or time-like. In particular, if the metric g is positive definite, then the (ε)-almost contact metric manifold is the usual almost contact metric manifold [25].

An (ε)-almost contact metric metric manifold is called an (ε)-trans Sasakian manifold [25] if

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \varepsilon\eta(Y)X) + \beta(g(\phi X, Y)\xi - \varepsilon\eta(Y)\phi X) \tag{1.5}$$

holds for some smooth functions α and β on M. According to the characteristic vector field ξ we have two classes of (ε)-trans-Sasakian manifolds. When $\varepsilon = -1$ and index of g is odd, then M is a time-like trans-Sasakian manifold and when $\varepsilon = 1$ and index of g is even, then M is a space-like trans-Sasakian manifold. Further, M is a usual trans-Sasakian manifold for $\varepsilon = 1$ and the index of g is 0 and M is a Lorentzian trans-Sasakian manifold for $\varepsilon = -1$ and the index of g is 1. An ε-almost contact metric manifold is said to be a (ε, δ)-trans-Sasakian manifold if it satisfies

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \varepsilon\eta(Y)X) + \beta(g(\phi X, Y)\xi - \delta\eta(Y)\phi X) \tag{1.6}$$

holds for some smooth functions α and β on M, where ε is 1 or -1 according as ξ is space-like or time-like and δ is alike ε.
From (1.6), we have

$$\nabla_X \xi = -\varepsilon\alpha\phi X - \delta\beta\phi^2 X, \tag{1.7}$$

and

$$(\nabla_X \eta)Y = \delta\beta[\varepsilon g(X,Y) - \eta(X)\eta(Y)] - \alpha g(\phi X, Y). \tag{1.8}$$

In (ε, δ)-trans-Sasakian manifold M, we have the following relations [7]:

$$\begin{aligned} R(X,Y)\xi = {} & (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] \\ & +2\varepsilon\delta\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] \\ & +\varepsilon[(Y\alpha)\phi X - (X\alpha)\phi Y] \\ & +\delta[(Y\beta)\phi^2 X - (X\beta)\phi^2 Y] \\ & +2\alpha\beta(\delta - \varepsilon)g(\phi X, Y)\xi, \end{aligned} \tag{1.9}$$

$$\begin{aligned} S(X,\xi) = {} & [((n-1)(\varepsilon\alpha^2 - \delta\beta^2) - (\xi\beta)]\eta(X) \\ & -\varepsilon((\phi X)\alpha) - (n-2)\varepsilon(X\beta)), \end{aligned} \tag{1.10}$$

$$Q\xi = ((n-1)(\varepsilon\alpha^2 - \delta\beta^2) - (\xi\beta))\xi + \varepsilon\phi(grad\alpha) - \varepsilon(n-2)(grad\beta), \tag{1.11}$$

where R is the curvature tensor, S is the Ricci tensor and Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$.

Further in a (ε, δ)-trans-Sasakian manifold, we have

$$\varepsilon\phi(grad\alpha) = \varepsilon(n-2)(grad\beta), \tag{1.12}$$

and

$$\varepsilon(\xi\alpha) + 2\varepsilon\delta\alpha\beta = 0. \tag{1.13}$$

Using (1.9) and (1.12), for constants α and β , we have

$$R(\xi, X)Y = (\alpha^2 - \beta^2)[\varepsilon g(X, Y)\xi - \eta(Y)X], \tag{1.14}$$

$$R(X, Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y], \tag{1.15}$$

$$\eta(R(X, Y)Z) = (\alpha^2 - \beta^2)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{1.16}$$

$$S(X, \xi) = [((n-1)(\varepsilon\alpha^2 - \delta\beta^2) - (\xi\beta)]\eta(X), \tag{1.17}$$

$$Q\xi = [(n-1)(\varepsilon\alpha^2 - \delta\beta^2) - (\xi\beta)]\xi. \tag{1.18}$$

An important consequence of (1.7) is that ξ is a geodesic vector field

$$\nabla_\xi \xi = 0. \tag{1.19}$$

For an arbitrary vector field X, we have that

$$d\eta(\xi, X) = 0. \tag{1.20}$$

The ξ-sectional curvature K_ξ of M is the sectional curvature of the plane spanned by ξ and a unit vector field X. From (1.15), we have

$$K_\xi = g(R(\xi, X), \xi, X) = (\alpha^2 - \beta^2) - \delta(\xi\beta). \tag{1.21}$$

It follows from (1.21) that ξ-sectional curvature does not depend on X.

1.2. η-Ricci solitons on (M, ϕ, ξ, η, g)

Fix h a symmetric tensor field of $(0, 2)$-type which we suppose to be parallel with respect to the Levi-Civita connection ∇, that is, $\nabla h = 0$. Applying the Ricci commutation identity [20]

$$\nabla^2 h(X, Y; Z, W) - \nabla^2 h(X, Y; W, Z) = 0, \tag{1.22}$$

we obtain the relation

$$h(R(X, Y)Z, W) + h(Z, R(X, Y)W) = 0. \tag{1.23}$$

Replacing $Z = W = \xi$ in (1.23) and using (1.9) and the symmetry of h, we have

$$\begin{aligned} &2(\alpha^2 - \beta^2)[\eta(Y)h(X, \xi) - \eta(X)h(Y, \xi)] \\ &+2\varepsilon[(Y\alpha)h(\phi X, \xi) - (X\alpha)h(\phi Y, \xi)] + 2\delta[(Y\beta)h(\phi^2 X, \xi) - (X\beta)h(\phi^2 Y, \xi)] \\ &+4\varepsilon\delta\alpha\beta[\eta(Y)h(\phi X, \xi) - \eta(X)h(\phi Y, \xi)] + 4\alpha\beta(\delta - \varepsilon)g(\phi X, Y)h(\xi, \xi) = 0. \end{aligned} \tag{1.24}$$

Putting $X = \xi$ in (1.24) and by virtue of (1.3), we obtain

$$-2[\varepsilon(\xi\alpha) + 2\varepsilon\delta\alpha\beta]h(\phi Y, \xi) \tag{1.25}$$

$$+2[(\alpha^2 - \beta^2) - \delta(\xi\beta)][\eta(Y)h(\xi, \xi) - h(Y, \xi)] = 0.$$

By using (1.13) in (1.25), we have

$$[(\alpha^2 - \beta^2) - \delta(\xi\beta)][\eta(Y)h(\xi, \xi) - h(Y, \xi)] = 0. \tag{1.26}$$

Suppose $(\alpha^2 - \beta^2) - \delta(\xi\beta) \neq 0$; it results in

$$h(Y, \xi) = \eta(Y)h(\xi, \xi). \tag{1.27}$$

Now, we can call a regular (ε, δ)-trans-Sasakian manifold if $(\alpha^2 - \beta^2) - \delta(\xi\beta) \neq 0$, where regularity, means the non-vanishing of the Ricci curvature with respect to the generator of the (ε, δ)-trans-Sasakian manifold.

Differentiating (1.27) covariantly with respect to X, we have

$$\begin{aligned}(\nabla_X h)(Y, \xi) + h(\nabla_X Y, \xi) + h(Y, \nabla_X \xi) \\ = [\varepsilon g(\nabla_X Y, \xi) + \varepsilon g(Y, \nabla_X \xi)]h(\xi, \xi) \\ +\eta(Y)[(\nabla_X h)(Y, \xi) + 2h(\nabla_X \xi, \xi)].\end{aligned} \tag{1.28}$$

By using the parallel condition $\nabla h = 0$, $\eta(\nabla_X \xi) = 0$ and by virtue of (1.27) in (1.28), we get

$$h(Y, \nabla_X \xi) = \varepsilon g(Y, \nabla_X \xi)h(\xi, \xi).$$

Now using (1.7) in the above equation, we get

$$-\varepsilon\alpha h(Y, \phi X) + \delta\beta h(Y, X) = -\alpha g(Y, \phi X)h(\xi, \xi) + \varepsilon\delta\beta g(Y, X)h(\xi, \xi). \tag{1.29}$$

Replacing $X = \phi X$ in (1.29) and after simplification, we get

$$h(X, Y) = \varepsilon g(X, Y)h(\xi, \xi), \tag{1.30}$$

which together with the standard fact that the parallelism of h implies that $h(\xi, \xi)$ is a constant, via (1.27). Now by considering the above equations, we can give the conclusion:

Theorem 1.1. *Let (M, ϕ, ξ, η, g) be a (ε, δ)-trans-Sasakian manifold with a non-vanishing ξ-sectional curvature and endowed with a tensor field $h \in \Gamma T_2^0(M)$ which is symmetric and ϕ-skew-symmetric. If h is parallel with respect to ∇, then it is a constant multiple of the metric tensor g.*

Let (M, ϕ, ξ, η, g) be an (ε)-almost contact metric manifold. Consider the equation [10]

$$\mathcal{L}_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.31}$$

where $\mathcal{L}_\xi$ is the Lie derivative operator along the vector field ξ, S is the Ricci curvature tensor field of the metric g, and λ and μ are real constants. Writing $\mathcal{L}_\xi g$ in terms of the Levi-Civita connection ∇, we obtain:

$$2S(X,Y) = -g(\nabla_X \xi, Y) - g(X, \nabla_X \xi) - 2\lambda g(X,Y) - 2\mu\eta(X)\eta(Y), \tag{1.32}$$

for any $X, Y \in \chi(M)$.

Definition 1.2. The data (g, ξ, λ, μ) which satisfy the equation (3.10) is said to be η- *Ricci soliton* on M [10]; in particular, if $\mu = 0$ then (g, ξ, λ) is the Ricci soliton [10] and it is called shrinking, steady or expanding following $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively [10].

Now, from (1.7) , the equation (1.31) becomes:

$$S(X,Y) = -(\lambda + \delta\beta)g(X,Y) + (\varepsilon\delta\beta - \mu)\eta(X)\eta(Y). \tag{1.33}$$

The above equations yields

$$S(X,\xi) = -[(\lambda + \mu) + (1 - \varepsilon)\delta\beta]\eta(X) \tag{1.34}$$

$$QX = -(\lambda + \beta\delta)X + (\varepsilon\delta\beta - \mu)\xi \tag{1.35}$$

$$Q\xi = -[(\lambda + \mu) + (1 - \varepsilon)\delta\beta]\xi \tag{1.36}$$

$$r = -\lambda n - (n-1)\varepsilon\delta\beta - \mu, \tag{1.37}$$

where r is the scalar curvature. Off the two natural situations regrading the vector field V: $V \in Span\xi$ and $V \perp \xi$, we investigate only the case $V = \xi$.

Our interest is in the expression for $\mathcal{L}_\xi g + 2S + 2\mu\eta \otimes \eta$. A direct computation gives

$$\mathcal{L}_\xi g(X,Y) = 2\delta\beta[g(X,Y) - \varepsilon\eta(X)\eta(Y)]. \tag{1.38}$$

In a 3-dimensional (ε, δ)-trans-Sasakian manifold the Riemannian curvature tensor is given by

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \tag{1.39}$$

$$-\frac{r}{2}[g(Y,Z)X - g(X,Z)Y].$$

Putting $Z = \xi$ in (1.39) and using (1.9) and (1.10) for 3-dimensional (ε, δ)-trans-Sasakian manifold, we get

$$(\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] + 2\varepsilon\delta\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] \tag{1.40}$$

$$+\varepsilon[(Y\alpha)\phi X - (X\alpha)\phi Y] + \delta[(Y\beta)\phi^2 X - (X\beta)\phi^2 Y]$$

$$+2(\delta-\varepsilon)\alpha\beta g(\phi X,Y)$$

$$=\varepsilon[(\varepsilon\alpha^2-\delta\beta^2)-(\xi\beta)]\eta(Y)X-\eta(X)Y]$$

$$+\varepsilon\eta(Y)QX-\varepsilon\eta(X)QY-\varepsilon[((\phi Y)\alpha)X+(Y\beta)X]+\varepsilon[((\phi X)\alpha)Y+(X\beta)Y].$$

Again, putting $Y=\xi$ in (1.40) and using (1.3) and (1.13), we obtain

$$QX=\Big[\frac{r}{2}-2(\varepsilon\alpha^2-\delta\beta^2)+\varepsilon(\alpha^2-\beta^2)\Big]X. \tag{1.41}$$

$$+\Big[4(\varepsilon\alpha^2-\delta\beta^2)-\frac{r}{2}-(\alpha^2-\beta^2)\Big]\eta(X)\xi$$

From (1.41), we have

$$S(X,Y)=\Big[\frac{r}{2}-2(\varepsilon\alpha^2-\delta\beta^2)+\varepsilon(\alpha^2-\beta^2)\Big]g(X,Y) \tag{1.42}$$

$$+\Big[4(\varepsilon\alpha^2-\delta\beta^2)-\frac{r}{2}-(\alpha^2-\beta^2)\Big]\varepsilon\eta(X)\eta(Y).$$

Equation (1.42) shows that a 3-dimensional (ε,δ)-trans-Sasakian manifold is η-*Einstein*.
Next, we consider the equation

$$h(X,Y)=(\mathcal{L}_\xi g)(X,Y)+2S(X,Y)+2\mu\eta(X)\eta(Y). \tag{1.43}$$

By Using (1.48) and (1.42) in (1.43), we have

$$h(X,Y)=\big[r-4(\varepsilon\alpha^2-\delta\beta^2)+2\varepsilon(\alpha^2-\beta^2)+2\delta\beta\big]g(X,Y) \tag{1.44}$$

$$+\big[8(\varepsilon\alpha^2-\delta\beta^2)-2\varepsilon(\alpha^2-\beta^2)-2\delta\beta-r\big]\varepsilon\eta(X)\eta(Y)+2\mu\eta(X)\eta(Y).$$

Putting $X=Y=\xi$ in (1.5), we get

$$h(\xi,\xi)=2[2\varepsilon(\varepsilon\alpha^2-\delta\beta^2)-2\mu]. \tag{1.45}$$

Now, (1.30) becomes

$$h(X,Y)=2[2\varepsilon(\varepsilon\alpha^2-\delta\beta^2)-2\mu]\varepsilon g(X,Y). \tag{1.46}$$

From (1.43) and (1.46), it follows that (g,ξ,μ) is an η-Ricci soliton.
Therefore, we can state as:

Theorem 1.2. *Let (M,ϕ,ξ,η,g) be a 3-dimensional (ε,δ)-trans-Sasakian manifold. Then (g,ξ,μ) yields an η-Ricci soliton on M.*

Let V be pointwise collinear with ξ, i.e., $V = b\xi$, where b is a function on the 3-dimensional (ε, δ)-trans-Sasakian manifold. Then

$$g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X) + 2S(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0$$

or

$$bg((\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y \xi, X) + (Yb)\eta(X)$$

$$+2S(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0.$$

Using (1.7), we obtain

$$bg(-\varepsilon\alpha\phi X - \delta\beta(-X + \eta(X)\xi, Y) + (Xb)\eta(Y) + bg(-\varepsilon\alpha\phi Y - \delta\beta(-Y + \eta(Y)\xi, X)$$

$$+(Yb)\eta(X) + 2S(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0.$$

which yields

$$2b\delta\beta g(X, Y) - 2b\delta\beta\eta(X)\eta(Y) + (Xb)\eta(Y) \tag{1.47}$$

$$+(Yb)\eta(X) + 2S(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0.$$

Replacing Y by ξ in (1.47), we obtain

$$(Xb) + (\xi b)\eta(X) + 2[2(\varepsilon\alpha^2 - \delta\beta^2) - (\xi\beta) + \lambda + \mu]\eta(X) = 0. \tag{1.48}$$

Again putting $X = \xi$ in (1.48), we obtain

$$\xi b = -2(\varepsilon\alpha^2 - \delta\beta^2) + (\xi\beta) - \lambda - \mu.$$

Plugging this in (1.48), we get

$$(Xb) + 2[2(\varepsilon\alpha^2 - \delta\beta^2) - (\xi\beta) + \lambda + \mu]\eta(X) = 0,$$

or

$$db = -\left\{\lambda + \mu + (\xi\beta) + 2(\varepsilon\alpha^2 - \delta\beta^2)\right\}\eta = 0. \tag{1.49}$$

Applying d on (1.49), we get $\left\{\lambda + \mu + (\xi\beta) + 2(\varepsilon\alpha^2 - \delta\beta^2)\right\} d\eta$. Since $d\eta \neq 0$ we have

$$\lambda + \mu + (\xi\beta) + 2(\varepsilon\alpha^2 - \delta\beta^2) = 0. \tag{1.50}$$

Equation (1.50) in (1.49) yields b as a constant. Therefore from (1.47), it follows that

$$S(X, Y) = -(\lambda + \delta\beta)g(X, Y) + (\varepsilon\delta b\beta - \mu)\eta(X)\eta(Y),$$

which implies that M is of constant scalar curvature for the constant $\delta\beta$. This leads to the following:

Theorem 1.3. *If in a 3-dimensional (ε, δ)-trans-Sasakian manifold the metric g is an η-Ricci soliton and V is pointwise collinear with ξ, then V is a constant multiple of ξ and g is of constant scalar curvature provided $\delta\beta$ is a constant.*

Tanking $X = Y = \xi$ in (1.30) and (1.42) and comparing, we get

$$\lambda = -2(\epsilon\alpha^2 - \delta\beta^2) + (\xi\beta) + \mu = -2K_\xi - \mu. \tag{1.51}$$

From (1.37) and (1.51), we obtain

$$r = 6(\epsilon\alpha^2 - \delta\beta^2) + 3(\xi\beta) - 2\varepsilon\delta\beta + 2\mu. \tag{1.52}$$

Since λ is a constant, it follows from (1.51) that K_ξ is a constant.

Theorem 1.4. *Let (g, ξ, μ) be an η-Ricci soliton in the 3-dimensional (ε, δ)-trans Sasaakian manifold (M, ϕ, ξ, η, g). Then the scalar $\lambda + \mu = -2K_\xi$, $r = 6K_\xi + 2\mu + 3(\xi\beta) - 2\varepsilon\delta\beta$.*

Remark 1.1. For $\mu = 0$, (1.51) reduces to $\lambda = -2K_\xi$, so the Ricci soliton in a 3-dimensional (ε, δ)-trans-Sasakian manifold is shrinking.

2. Example of η-Ricci solitons on (ε, δ)-Trans-Sasakian manifolds

Example 2.1. Consider the three dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 z \neq 0\}$, where (x, y, z) are the cartesian coordinates in $\mathbb{R}^3$ and let the vector fields

$$e_1 = \frac{e^x}{z^2}\frac{\partial}{\partial x}, \qquad e_2 = \frac{e^y}{z^2}\frac{\partial}{\partial y}, \qquad e_3 = \frac{-(\epsilon + \delta)}{2}\frac{\partial}{\partial z},$$

where e_1, e_2, e_3 are linearly independent at each point of M. Let g be the Riemannian metric defined by
$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = \varepsilon$, $g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0$,
where $\epsilon = \pm 1$.

Let η be the 1-form defined by $\eta(X) = \varepsilon g(X, \xi)$, for any vector field X on M, let ϕ be the (1,1)-tensor field defined by $\phi(e_1) = e_2$, $\phi(e_2) = -e_1$, $\phi(e_3) = 0$. Then by using the linearity of ϕ and g, we have $\phi^2 X = -X + \eta(X)\xi$, with $\xi = e_3$. Further $g(\phi X, \phi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y)$, for any vector fields X and Y on M. Hence for $e_3 = \xi$, the structure defines an (ε)-almost contact structure in $\mathbb{R}^3$.

Let ∇ be the Levi-Civita connection with respect to the metric g, then we have

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\ & - g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned}$$

which is known as Koszul's formula.

$\nabla_{e_1}e_3 = -\frac{(\varepsilon+\delta)}{z}e_1, \quad \nabla_{e_2}e_3 = -\frac{(\varepsilon+\delta)}{z}e_2, \quad \nabla_{e_1}e_2 = 0,$
using the above relation, for any vector X on M, we have $\nabla_X\xi = -\varepsilon\alpha\phi X - \beta\delta\phi^2 X$, where $\alpha = \frac{1}{z}$ and $\beta = -\frac{1}{z}$. Hence (ϕ, ξ, η, g) structure defines the (ε, δ)-tran-Sasakian structure in $\mathbb{R}^3$.

Here ∇ is the Levi-Civita connection with respect to the metric g , so we have $[e_1, e_2] = 0, \quad [e_1, e_3] = -\frac{(\varepsilon+\delta)}{z}e_1, \; [e_2, e_3] = -\frac{(\varepsilon+\delta)}{z}e_2$.

Thus we have

$$\nabla_{e_1}e_3 = -\frac{(\varepsilon+\delta)}{z}e_1 + e_2, \nabla_{e_1}e_2 = 0$$

$$\nabla_{e_2}e_1 = 0, \quad \nabla_{e_2}e_2 = -\frac{(\varepsilon+\delta)}{z}e_2, \quad \nabla_{e_2}e_3 = -\frac{(\varepsilon+\delta)}{z}e_2e_1$$

$$\nabla_{e_3}e_1 = 0, \qquad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_3 = -\frac{(\varepsilon+\delta)}{z}e_1 + e_2.$$

The manifold M satisfies (1.7) with $\alpha = \frac{1}{z}$ and $\beta = -\frac{1}{z}$. Hence M is a (ε, δ)-trans-Sasakian manifolds. Then the non-vanishing components of the curvature tensor fields are computed as follows:

$$R(e_1, e_3)e_3 = \frac{(\varepsilon+\delta)}{z^2}e_1, \quad R(e_3, e_1)e_3 = -\frac{(\varepsilon+\delta)}{z^2}e_1, \quad R(e_1, e_2)e_2 = \frac{(\varepsilon+\delta)}{z^2}e_1$$

$$R(e_2, e_3)e_3 = \frac{(\varepsilon+\delta)}{z^2}e_1, \quad R(e_3, e_2)e_3 = -\frac{(\varepsilon+\delta)}{z^2}e_1, \; R(e_2, e_1)e_1 = -\frac{(\varepsilon+\delta)}{z^2}e_1.$$

From the above expression of the curvature tensor we can also obtain

$$S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = \frac{(\varepsilon^2 + \delta\varepsilon)}{z^2}$$

since $g(e_1, e_3) = g(e_1, e_2) = 0$.
Therefore, we have

$$S(e_i, e_i) = -\frac{(\varepsilon+\delta)}{z^2}g(e_i, e_i),$$

for $i = 1, 2, 3$, and $\alpha = \frac{1}{z}$, $\beta = -\frac{1}{z}$. Hence M is also an *Einstein* manifold. In this case, from (1.32), we have

(2.1) $2\delta\beta[g(e_i, e_i - \varepsilon\eta(e_i)\eta(e_i)] + 2S(e_i, e_i) + 2\lambda g(e_i, e_i) + 2\mu\eta(e_i)\eta(e_i) = 0.$

Now, from (2.1), we get $\lambda = \frac{\varepsilon[\delta(1+z)-\varepsilon]}{z^2}$ (i.e, $\lambda > 0$) and $\mu = -\frac{\varepsilon[\varepsilon^2-\varepsilon-\delta(1+\varepsilon+\varepsilon z)]}{z^2}$, the data (g, ξ, λ, μ) is an η-Ricci soliton on (M, ϕ, ξ, η, g) i. e., expanding.

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