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SOME NEW Q-ESTIMATES FOR CERTAIN INTEGRAL INEQUALITIES

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Abstract. In this paper, we consider a newly introduced class of convex functions that is η-convex functions. We give some new quantum analogues for Hermite-Hadamard, Iynger and Ostrowski type inequalities via η-convex functions. Some special cases are also discussed.

Keywords: Convex; quantum, differentiable; Hermite-Hadamard's inequalities; η-convex functions.

1. Introduction

A set $K \subset \mathbb{R}$ is said to be convex, if the line segment joining any pair of points of K entirely lies in K.
A set function $f : K \to \mathbb{R}$ is said to be convex, if

$$f((1-t)u + tv) \le (1-t)f(u) + tf(v), \quad \forall u, v \in K, t \in [0,1]. \tag{1.1}$$

In recent years many researchers have shown their interest in theory of convexity and as a result theory of convexity has experienced rapid development. One of the main reason for this development is the close relationship between theory of convexity and theory of inequalities. Many famous inequalities known in the literature are proved via convex functions, for example an intensively studied inequality in the literature which is mainly due to Hermite and Hadamard is Hermite-Hadmard's inequality. This classical result of Hermite and Hadamard reads as follows:
Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function, where $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2}.$$

This inequality provides a necessary and sufficient condition for a function to be convex. For some recent studies on Hermite-Hadamard's type of inequalities and

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other inequalities which can be obtained by using convexity property, see [1, 2, 3, 4, 5, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22].
Many researchers recently generalized and extended the classical concepts of convex sets and convex functions in different directions, see [2, 4, 12, 13, 14, 15, 18, 19, 20]. Recently Gordji et al. [9] introduced the a new class of convexity which is called as η-convexity. It has been shown that the class of η-convexity reduces to classical convexity under suitable conditions. Gordji et al. [9] derived a new inequality of Hermite-Hadamard type for η-convex functions, which reads as:
Let $f:[a,b]\to\mathbb{R}$ be η-convex function such that η is bounded above on $f[a,b]\times f[a,b]$, then

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)-\frac{1}{2}\int_a^b \eta(f(a+b-x),f(x))\mathrm{d}x \\ \leq\ & \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}+\frac{\eta(f(a),f(b))+\eta(f(b),f(a))}{4}. \end{aligned}$$

For some other details on η-convex functions, see [9, 10].
Quantum calculus is an emerging field of special interest for different researchers. In this subject we basically obtain quantum analogue of different mathematical object, which can be recaptured as classical when $q\to 1$. Interested readers are referred to [6, 7]. Tariboon et al. [22, 23] recently introduced the notions of quantum integrals and quantum derivatives on finite intervals. For some recent investigations on quantum analogues of some classical inequalities, see [6, 7, 8, 16, 17, 18, 19, 20, 21, 22, 23]. Motivated by this ongoing research, we in this paper consider the class of η-convex functions. We derive some quantum analogues for Hermite-Hadamard and Ostrowski type inequalities via η-convex functions. Some special cases are also discussed. This is the main motivation of this paper.

2. Preliminaries

In this section, we recall some previously known concepts and basic results for η-convex functions and quantum calculus.

Definition 2.1. [9, 10] A function $f:K\to\mathbb{R}$ is said to be η-convex function, if there exists a bifunction $\eta(.,.)$, such that

$$f((1-t)u+tv)\leq f(u)+t\eta(f(v),f(u)),\quad \forall u,v\in K, t\in[0,1]. \tag{2.1}$$

Note that, if $\eta(f(v),f(u))=f(v)-f(u)$ in the above inequality, then, we have (1.1).
Authors [9, 10] have shown that there exists certain functions which areη-convex functions but not convex in the classical sense.

Theorem 2.1. *[9, 10] A function $f : I \to \mathbb{R}$ is η-convex if and only if*

$$\begin{vmatrix} 1 & x_1 & \eta(f(x_1), f(x_3)) \\ 1 & x_2 & f(x_2) - f(x_3) \\ 1 & x_3 & 0 \end{vmatrix} \geq 0; \quad x_1 < x_2 < x_3,$$

and

$$f(x_2) \leq f(x_3) + \eta(f(x_1), f(x_3)).$$

For some useful details on η-convex functions, see [9, 10].
Now we give some preliminary details of quantum calculus on finite intervals.
Let $J = [a, b] \subseteq \mathbb{R}$ be an interval and $0 < q < 1$ be a constant. The q-derivative of a function $f : J \to \mathbb{R}$ at a point $x \in J$ on $[a, b]$ is defined as follows.

Definition 2.2. [22, 23] Let $f : J \to \mathbb{R}$ be a continuous function and let $x \in J$. Then q-derivative of f on J at x is defined as

$$(2.2) \qquad {}_a\mathrm{D}_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \neq a.$$

It is obvious that ${}_a\mathrm{D}_q f(a) = \lim_{x\to a} {}_a\mathrm{D}_q f(x)$.
A function f is q-differentiable on J if ${}_aD_q f(x)$ exists for all $x \in J$. Also if $a = 0$ in (2.2), then ${}_0\mathrm{D}_q f = \mathrm{D}_q f$, where D_q is the q-derivative of the function f [6] defined as

$$\mathrm{D}_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}$$

.

Remark 2.1. [22, 23] Let $f : J \to \mathbb{R}$ be a continuous function. Let us define the second-order q-derivative on interval J, which is denoted by ${}_a\mathrm{D}_q^2 f$, provided ${}_a\mathrm{D}_q f$ is q-differentiable on J with ${}_a\mathrm{D}_q^2 f = {}_a\mathrm{D}_q({}_a\mathrm{D}_q f) : J \to \mathbb{R}$. Similarly, one can defined higher order q-derivative on J, ${}_a\mathrm{D}_q^n : J \to \mathbb{R}$.

Let us elaborate above definitions with the help of an example.

Example 2.1. [22, 23] Let $x \in [a, b]$ and $0 < q < 1$. Then, for $x \neq a$, we have

$$\begin{aligned} {}_a\mathrm{D}_q x^2 &= \frac{x^2 - (qx + (1-q)a)^2}{(1-q)(x-a)} \\ &= \frac{(1+q)x^2 - 2qax - (1-q)a^2}{x-a} \\ &= (1+q)x + (1-q)a. \end{aligned}$$

Note that when $x = a$, we have $\lim_{x\to a}({}_a\mathrm{D}_q x^2) = 2a$.

Definition 2.3. [22, 23] Let $f : J \to \mathbb{R}$ be a continuous function. A second-order q-derivative on J, which is denoted as ${}_a\mathrm{D}_q^2 f$, provided ${}_a\mathrm{D}_q f$ is q-differentiable on J is defined as ${}_a\mathrm{D}_q^2 f = {}_a\mathrm{D}_q({}_a\mathrm{D}_q f) : J \to \mathbb{R}$. Similarly higher order q-derivative on J is defined by ${}_a\mathrm{D}_q^n f =: J \to \mathbb{R}$.

Lemma 2.1. [22, 23] *Let $\alpha \in \mathbb{R}$, then*

$$ {}_a\mathrm{D}_q(x-a)^\alpha = \left(\frac{1-q^\alpha}{1-q}\right)(x-a)^{\alpha-1}. $$

Tariboon et al. [22, 23] defined the q-integral as follows

Definition 2.4. [22, 23] Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. Then q-integral on I is defined as

$$ \int_a^x f(t)\,{}_a\mathrm{d}_q t = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a), \tag{2.3} $$

for $x \in I$.

If $a = 0$ in (2.3), then we have the classical q-integral, that is

$$ \int_0^x f(t)\,{}_0\mathrm{d}_q t = (1-q)x\sum_{n=0}^{\infty} q^n f(q^n x), \quad x \in [0,\infty). $$

For more details, see [6].
Moreover, if $c \in (a,x)$, then the definite q-integral on J is defined by

$$ \begin{aligned} \int_c^x f(t)\,{}_a\mathrm{d}_q t &= \int_a^x f(t)\,{}_a\mathrm{d}_q t - \int_a^c f(t)\,{}_a\mathrm{d}_q t \\ &= (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) \\ &- (1-q)(c-a)\sum_{n=0}^{\infty} q^n f(q^n c + (1-q^n)a). \end{aligned} $$

Example 2.2. [22, 23] Let a constant $c \in J$, then

$$ \begin{aligned} \int_c^b (t-c)\,{}_a\mathrm{d}_q t &= \int_a^b (t-c)\,{}_a\mathrm{d}_q t - \int_a^c (t-c)\,{}_a\mathrm{d}_q t \\ &= \left[\frac{(t-a)(t+qa)}{1+q} - ct\right]_a^b - \left[\frac{(t-a)(t+qa)}{1+q} - ct\right]_a^c \\ &= \frac{b^2-(1+q)bc+qc^2}{1+q} - \frac{a(1-q)(b-c)}{1+q}. \end{aligned} $$

Note that when $q \to 1$, then the above integral reduces to the classical integration

$$ \int_c^b (t-c)\mathrm{d}t = \frac{(b-c)^2}{2}. $$

Theorem 2.2. [22, 23] *Let $f : I \to \mathbb{R}$ be a continuous function, then*

1. ${}_a\mathrm{D}_q \int\limits_a^x f(t)\, {}_a\mathrm{d}_q t = f(x)$

2. $\int\limits_c^x {}_a\mathrm{D}_q f(t) {}_a\mathrm{d}_q t = f(x) - f(c)$ *for* $c \in (a, x)$.

Theorem 2.3. [22, 23] *Let $f, g : I \to \mathbb{R}$ be a continuous functions, $\alpha \in \mathbb{R}$, then $x \in J$*

1. $\int\limits_a^x [f(t) + g(t)]\, {}_a\mathrm{d}_q t = \int\limits_a^x f(t)\, {}_a\mathrm{d}_q t + \int\limits_a^x g(t)\, {}_a\mathrm{d}_q t$

2. $\int\limits_a^x (\alpha f)(t)\, {}_a\mathrm{d}_q t = \alpha \int\limits_a^x f(t)\, {}_a\mathrm{d}_q t$

3. $\int\limits_c^x f(t)\, {}_a\mathrm{D}_q g(t)\, {}_a\mathrm{d}_q t = (fg)|_c^x - \int\limits_c^x g(qt + (1-q)a) {}_a\mathrm{D}_q f(t) {}_a\mathrm{d}_q t$ *for* $c \in (a, x)$.

Lemma 2.2. [22, 23] *Let $\alpha \in \mathbb{R} \setminus \{-1\}$, then*

$$\int\limits_a^x (t-a)^\alpha\, {}_a\mathrm{d}_q t = \Big(\frac{1-q}{1-q^{\alpha+1}}\Big)(x-a)^{\alpha+1}.$$

Proof. Let $f(x) = (x-a)^{\alpha+1}$, $x \in J$ and $\alpha \in \mathbb{R} \setminus \{-1\}$, then by definition, we have

$$\begin{aligned} {}_a\mathrm{D}_q f(x) &= \frac{(x-a)^{\alpha+1} - (qx + (1-q)a - a)^{\alpha+1}}{(1-q)(x-a)} \\ &= \frac{(x-a)^{\alpha+1} - q^{\alpha+1}(x-a)^{\alpha+1}}{(1-q)(x-a)} \\ &= \left(\frac{1-q^{\alpha+1}}{1-q}\right)(x-a)^\alpha. \qquad (2.4) \end{aligned}$$

Applying q-integral on J for (2.4), we obtain the required result. □

Example 2.3. [22, 23] Let $f(x) = x$ for $x \in J$, then, we have

$$\begin{aligned} \int\limits_a^x f(t) {}_a\mathrm{d}_q t = \int\limits_a^x t {}_a\mathrm{d}_q t &= (1-q)(x-a) \sum_{n=0}^{\infty} q^n (q^n x + (1-q^n)a) \\ &= \frac{(x-a)(x+qa)}{1+q}. \end{aligned}$$

Our next result is useful in proving some of our main results. This auxiliary result is mainly due to [16, 21].

Lemma 2.3. *Let $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on I° (the interior of I) with ${}_a\mathrm{D}_q$ be continuous and integrable on I where $0 < q < 1$, then*

$$\begin{aligned} H_f(a, b; q) &= \frac{1}{b-a}\int_a^b f(x)_a\mathrm{d}_qx - \frac{qf(a)+f(b)}{1+q} \\ &= \frac{q(b-a)}{1+q}\int_0^1 (1-(1+q)t)_a\mathrm{D}_qf((1-t)a+tb)\,{}_0\mathrm{d}_qt. \end{aligned}$$

Our next result is also an auxiliary result which is due to Noor et al. [17].

Lemma 2.4. *Let $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on I° (the interior of I) with ${}_a\mathrm{D}_q$ be continuous and integrable on I where $0 < q < 1$, then*

$$\begin{aligned} K_f(a, b; q) &= f(x) - \frac{1}{b-a}\int_a^b f(u)_a\mathrm{d}_qu \\ &= \frac{q(x-a)^2}{b-a}\int_0^1 t_a\mathrm{D}_qf(tx+(1-t)a)_0\mathrm{d}_qt \\ &\quad +\frac{q(b-x)^2}{b-a}\int_0^1 t_a\mathrm{D}_qf(tx+(1-t)b)_0\mathrm{d}_qt \end{aligned}$$

3. Main Results

In this section, we establish our main results.

Theorem 3.1. *Let $f, g : I \to \mathbb{R}$ be two η-convex functions, then*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(x)g(x)_a\mathrm{d}_qx \\ \leq\ & f(a)g(a) + \frac{1}{1+q}T(a, b; \eta; f; g) + \frac{1}{1+q+q^2}R(a, b; \eta; f; g), \end{aligned}$$

where

$$T(a, b; \eta; f; g) = f(a)\eta(g(b), g(a)) + g(a)\eta(f(b), f(a)),$$

and

$$R(a, b; \eta; f; g) = \eta(g(b), g(a))\eta(f(b), f(a)).$$

Proof. Let f and g be two η-convex functions, then

$$f((1-t)a+tb) \leq f(a) + t\eta(f(b), f(a)),$$

and

$$g((1-t)a+tb) \leq g(a) + t\eta(g(b), g(a)),$$

Multiplying above inequalities, we have

$$\begin{aligned} & f((1-t)a+tb)g((1-t)a+tb) \\ \leq\ & f(a)g(a) + tf(a)\eta(g(b), g(a)) + tg(a)\eta(f(b), f(a)) + t^2\eta(g(b), g(a))\eta(f(b), f(a)). \end{aligned}$$

Now q-integrating above inequality with respect to t on $[0, 1]$, we have

$$\begin{aligned} & \frac{1}{b-a}\int_a^b f(x)g(x)\,{}_a\mathrm{d}_q x \\ \leq\ & f(a)g(a) + \frac{1}{1+q}\left[f(a)\eta(g(b), g(a)) + g(a)\eta(f(b), f(a))\right] \\ & + \frac{1}{1+q+q^2}\eta(g(b), g(a))\eta(f(b), f(a)). \end{aligned}$$

This completes the proof. □

Theorem 3.2. *Let $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on the interior of I with ${}_a\mathrm{D}_q$ be continuous and integrable on I where $0 < q < 1$. If $|{}_a\mathrm{D}_q f|$ is η-convex function, then*

$$\begin{aligned} & |H_f(a, b; q)| \\ \leq\ & \Omega(b-a)\Bigg[2(1+q)(1+q+q^2)|{}_a\mathrm{D}_q f(a)| + (1+4q+q^2)\eta(|{}_a\mathrm{D}_q f(b)|, |{}_a\mathrm{D}_q f(a)|)\Bigg], \end{aligned}$$

where

$$\Omega = \frac{q^2}{(1+q+q^2)(1+q)^4}.$$

Proof. Using Lemma 2.3 and the fact that $|{}_a\mathrm{D}_q f|$ is η-convex function, we have

$$\begin{aligned} & |H_f(a, b; q)| \\ =\ & \left|\frac{q(b-a)}{1+q}\int_0^1 (1-(1+q)t)\,{}_a\mathrm{D}_q f((1-t)a+tb)\,{}_0\mathrm{d}_q t\right| \\ \leq\ & \frac{q(b-a)}{1+q}\int_0^1 |1-(1+q)t||{}_a\mathrm{D}_q f((1-t)a+tb)|\,{}_0\mathrm{d}_q t \\ \leq\ & \frac{q(b-a)}{1+q}\int_0^1 |1-(1+q)t|[|{}_a\mathrm{D}_q f(a)| + t\eta(|{}_a\mathrm{D}_q f(b)|, |{}_a\mathrm{D}_q f(a)|)]\,{}_0\mathrm{d}_q t \end{aligned}$$

$$= \frac{q(b-a)}{1+q}\left[\int_0^1 |1-(1+q)t||_a\mathrm{D}_q f(a)|_0\mathrm{d}_q t\right.$$
$$\left.+\int_0^1 t|1-(1+q)t|\eta(|_a\mathrm{D}_q f(b)|,|_a\mathrm{D}_q f(a)|)_0\mathrm{d}_q t\right]$$
$$= \frac{q(b-a)}{1+q}\left[\frac{2q}{(1+q)^2}|_a\mathrm{D}_q f(a)| + \frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3}\eta(|_a\mathrm{D}_q f(b)|,|_a\mathrm{D}_q f(a)|)\right]$$
$$= \frac{q^2(b-a)}{(1+q+q^2)(1+q)^4}\left[2(1+q)(1+q+q^2)|_a\mathrm{D}_q f(a)|\right.$$
$$\left.+(1+4q+q^2)\eta(|_a\mathrm{D}_q f(b)|,|_a\mathrm{D}_q f(a)|)\right].$$

This completes the proof. □

Theorem 3.3. *Let $f: I=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a q-differentiable function on the interior of I with ${}_a\mathrm{D}_q$ be continuous and integrable on I where $0<q<1$. If $|_a\mathrm{D}_q f|^r$ is η-convex function where $r>1$, then*

$$|H_f(a,b;q)|$$
$$\leq \frac{q(b-a)}{1+q}\left(\frac{2q}{(1+q)^2}\right)^{1-\frac{1}{r}}\left[\theta_1|_a\mathrm{D}_q f(a)|^r + \theta_2\eta(|_a\mathrm{D}_q f(b)|^r,|_a\mathrm{D}_q f(a)|^r)\right]^{\frac{1}{r}},$$

where

$$\theta_1 = \frac{2q}{(1+q)^2}, \quad \text{and} \quad \theta_2 = \frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3}.$$

Proof. Since $|_a\mathrm{D}_q f|^r$ is η-convex function, so from Lemma 2.3 and using Holder's inequality, we have

$$|H_f(a,b;q)|$$
$$= \left|\frac{q(b-a)}{1+q}\int_0^1 (1-(1+q)t)_a\mathrm{D}_q f((1-t)a+tb)_0\mathrm{d}_q t\right|$$
$$\leq \frac{q(b-a)}{1+q}\left(\int_0^1 |1-(1+q)t|_0\mathrm{d}_q t\right)^{1-\frac{1}{r}}$$
$$\left(\int_0^1 |1-(1+q)t||_a\mathrm{D}_q f((1-t)a+tb)|^r{}_0\mathrm{d}_q t\right)^{\frac{1}{r}}$$

$$= \frac{q(b-a)}{1+q}\left(\frac{2q}{(1+q)^2}\right)^{1-\frac{1}{r}}$$

$$\times\left[\frac{2q}{(1+q)^2}|{}_a\mathrm{D}_q f(a)|^r + \frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3}\eta(|{}_a\mathrm{D}_q f(b)|^r, |{}_a\mathrm{D}_q f(a)|^r)\right]^{\frac{1}{r}}.$$

This completes the proof. □

Now we derive some quantum analogues for Iynger type inequalities via η-quasiconvex functions. For this, we first define η-quasiconvex functions.

Definition 3.1. [9] A function $f : I \to \mathbb{R}$ is said to be η-quasiconvex, if

$$f((1-t)u+tv) \leq \max\{f(u), f(u)+\eta(f(u), f(v))\}, \quad \forall u, v \in I, t \in [0,1].$$

Theorem 3.4. *Let $f : I \to \mathbb{R}$ be a q-differentiable function on I° (the interior of I) with ${}_aD_q$ be continuous and integrable on I where $0 < q < 1$. If $|{}_aD_q f|^r$ is η-quasiconvex function, where $r > 1$, then*

$$|H_f(a,b;q)| \leq \frac{2q^2(b-a)}{(1+q)^3}\left(\max\{|f(a)|^r, |f(a)+\eta(f(a), f(b))|^r\}\right)^{\frac{1}{r}}.$$

Proof. Using Lemma 2.3, power mean inequality and the fact that $|{}_aD_q f|^r$ is η-quasiconvex function, we have

$$\begin{aligned}
&|H_f(a,b;q)| \\
= &\left|\frac{q(b-a)}{1+q}\int_0^1 (1-(1+q)t)\,{}_aD_q f((1-t)a+tb)\,{}_0d_q t\right| \\
\leq &\frac{q(b-a)}{1+q}\left(\int_0^1 |1-(1+q)t|\,{}_0d_q t\right)^{1-\frac{1}{r}} \\
&\left(\int_0^1 |1-(1+q)t||{}_aD_q f((1-t)a+tb)|^r\,{}_0d_q t\right)^{\frac{1}{r}} \\
= &\frac{2q^2(b-a)}{(1+q)^3}\left(\max\{|f(a)|^r, |f(a)+\eta(f(a), f(b))|^r\}\right)^{\frac{1}{r}}.
\end{aligned}$$

This completes the proof. □

Next we give some quantum estimates for Ostrowski type inequalities via η-convex functions.

Theorem 3.5. *Let $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on the interior of I with ${}_a\mathrm{D}_q$ be continuous and integrable on I where $0 < q < 1$. If $|{}_a\mathrm{D}_q f|$ is η-convex function, then*

$$\begin{aligned} &|K_f(a, b; q)| \\ \le\ & \frac{q(x-a)^2}{b-a}\left[\frac{1}{1+q}|{}_a\mathrm{D}_q f(a)| + \frac{1}{1+q+q^2}\eta(|{}_a\mathrm{D}_q f(x)|, |{}_a\mathrm{D}_q f(a)|)\right] \\ & + \frac{q(b-x)^2}{b-a}\left[\frac{1}{1+q}|{}_a\mathrm{D}_q f(b)| + \frac{1}{1+q+q^2}\eta(|{}_a\mathrm{D}_q f(x)|, |{}_a\mathrm{D}_q f(b)|)\right]. \end{aligned}$$

Proof. Using Lemma 2.4 and the fact that $|{}_a\mathrm{D}_q f|$ is η-convex function, we have

$$\begin{aligned} &|K_f(a, b; q)| \\ =\ & \left|\frac{q(x-a)^2}{b-a}\int_0^1 t\,{}_a\mathrm{D}_q f(tx+(1-t)a)\,{}_0\mathrm{d}_q t + \frac{q(b-x)^2}{b-a}\int_0^1 t\,{}_a\mathrm{D}_q f(tx+(1-t)b)\,{}_0\mathrm{d}_q t\right| \\ \le\ & \frac{q(x-a)^2}{b-a}\int_0^1 t|{}_a\mathrm{D}_q f(tx+(1-t)a)|\,{}_0\mathrm{d}_q t + \frac{q(b-x)^2}{b-a}\int_0^1 t|{}_a\mathrm{D}_q f(tx+(1-t)b)|\,{}_0\mathrm{d}_q t \\ \le\ & \frac{q(x-a)^2}{b-a}\int_0^1 t[|{}_a\mathrm{D}_q f(a)| + t\eta(|{}_a\mathrm{D}_q f(x)|, |{}_a\mathrm{D}_q f(a)|)]\,{}_0\mathrm{d}_q t \\ & + \frac{q(b-x)^2}{b-a}\int_0^1 t[|{}_a\mathrm{D}_q f(b)| + t\eta(|{}_a\mathrm{D}_q f(x)|, |{}_a\mathrm{D}_q f(b)|)]\,{}_0\mathrm{d}_q t \\ =\ & \frac{q(x-a)^2}{b-a}\left[\frac{1}{1+q}|{}_a\mathrm{D}_q f(a)| + \frac{1}{1+q+q^2}\eta(|{}_a\mathrm{D}_q f(x)|, |{}_a\mathrm{D}_q f(a)|)\right] \\ & + \frac{q(b-x)^2}{b-a}\left[\frac{1}{1+q}|{}_a\mathrm{D}_q f(b)| + \frac{1}{1+q+q^2}\eta(|{}_a\mathrm{D}_q f(x)|, |{}_a\mathrm{D}_q f(b)|)\right]. \end{aligned}$$

This completes the proof. □

Theorem 3.6. *Let $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on I° (the interior of I) with ${}_a\mathrm{D}_q$ be continuous and integrable on I where $0 < q < 1$. If $|{}_a\mathrm{D}_q f|^r$ is η-convex function, then, for $p, r > 1$, $\frac{1}{p} + \frac{1}{r} = 1$, we have*

$$\begin{aligned} &|K_f(a, b; q)| \\ \le\ & \frac{q(x-a)^2}{b-a}\Big(\frac{1-q}{1-q^{p+1}}\Big)^{\frac{1}{p}}\Big(|{}_a\mathrm{D}_q f(a)|^r + \frac{1}{1+q+q^2}\eta(|{}_a\mathrm{D}_q f(x)|^r, |{}_a\mathrm{D}_q f(a)|^r)\Big)^{\frac{1}{r}} \\ & + \frac{q(b-x)^2}{b-a}\Big(\frac{1-q}{1-q^{p+1}}\Big)^{\frac{1}{p}}\Big(|{}_a\mathrm{D}_q f(b)|^r + \frac{1}{1+q}\eta(|{}_a\mathrm{D}_q f(x)|^r, |{}_a\mathrm{D}_q f(b)|^r)\Big)^{\frac{1}{r}}. \end{aligned}$$

Proof. Using Lemma 2.4, Holder's inequality and the fact that $|{}_a\mathrm{D}_q f|^r$ is η-convex function, we have

$$
\begin{aligned}
&|K_f(a,b;q)| \\
= & \left|\frac{q(x-a)^2}{b-a}\int_0^1 t\,{}_a\mathrm{D}_q f(tx+(1-t)a)\,{}_0\mathrm{d}_q t + \frac{q(b-x)^2}{b-a}\int_0^1 t\,{}_a\mathrm{D}_q f(tx+(1-t)b)\,{}_0\mathrm{d}_q t\right| \\
\le & \frac{q(x-a)^2}{b-a}\Big(\int_0^1 t^p\,{}_0\mathrm{d}_q t\Big)^{\frac{1}{p}}\Big(\int_0^1 |{}_a\mathrm{D}_q f(tx+(1-t)a)|^r\,{}_0\mathrm{d}_q t\Big)^{\frac{1}{r}} \\
& \quad + \frac{q(b-x)^2}{b-a}\Big(\int_0^1 t^p\,{}_0\mathrm{d}_q t\Big)^{\frac{1}{p}}\Big(\int_0^1 |{}_a\mathrm{D}_q f(tx+(1-t)b)|^r\,{}_0\mathrm{d}_q t\Big)^{\frac{1}{r}} \\
\le & \frac{q(x-a)^2}{b-a}\Big(\frac{1-q}{1-q^{p+1}}\Big)^{\frac{1}{p}}\Big(\int_0^1 [|{}_a\mathrm{D}_q f(a)|^r + t\eta(|{}_a\mathrm{D}_q f(x)|^r, |{}_a\mathrm{D}_q f(a)|^r)]\,{}_0\mathrm{d}_q t\Big)^{\frac{1}{r}} \\
& \quad + \frac{q(b-x)^2}{b-a}\Big(\frac{1-q}{1-q^{p+1}}\Big)^{\frac{1}{p}}\Big(\int_0^1 [|{}_a\mathrm{D}_q f(b)|^r + t\eta(|{}_a\mathrm{D}_q f(x)|^r, |{}_a\mathrm{D}_q f(b)|^r)]\,{}_0\mathrm{d}_q t\Big)^{\frac{1}{r}} \\
= & \frac{q(x-a)^2}{b-a}\Big(\frac{1-q}{1-q^{p+1}}\Big)^{\frac{1}{p}}\Big(|{}_a\mathrm{D}_q f(a)|^r + \frac{1}{1+q}\eta(|{}_a\mathrm{D}_q f(x)|^r, |{}_a\mathrm{D}_q f(a)|^r)\Big)^{\frac{1}{r}} \\
& + \frac{q(b-x)^2}{b-a}\Big(\frac{1-q}{1-q^{p+1}}\Big)^{\frac{1}{p}}\Big(|{}_a\mathrm{D}_q f(b)|^r + \frac{1}{1+q}\eta(|{}_a\mathrm{D}_q f(x)|^r, |{}_a\mathrm{D}_q f(b)|^r)\Big)^{\frac{1}{r}}.
\end{aligned}
$$

This completes the proof. □

Conclusion

In this paper, we have obtained several new q-estimates for certain classical inequalities via η-convex functions. These quantum estimates and their variant forms are useful for quantum physics where lower and upper bounds for natural phenomena described by integrals (such as mechanical work) are frequently required. It is expected that the results obtained in this paper may motivate further research in this field.

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