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# INVEXITY AND A CLASS OF CONSTRAINED OPTIMIZATION PROBLEMS IN HILBERT SPACES

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**Abstract.** In this paper the notion of *invexity* has been introduced in Hilbert spaces. A class of constrained optimization problems has been proposed under the assumption of invexity. Some of the algebraic properties leading to the optimality criterion of such a class of problems has been studied.

**Keywords**: Convexity , Invexity , Frechet Derivative , Archimedean Order , Zorn's Lemma

## 1. Introduction

The mathematics of Convex Optimization was discussed by several authors for about a century [5, 10, 11, 12, 16, 17, 20, 22]. In the second half of the last century various generalizations of convex functions have been introduced [2, 14, 15, 18, 19, 21]. The *invex(invariant convex), pseudoinvex and quasiinvex functions* were introduced by M.A.Hanson in 1981 [13]. These functions are extremely significant in optimization theory mainly due to the properties regarding their global optima. For example, *a differentiable function is invex iff every stationary point is a global minima*[1]. Later in 1986, Craven defined the non-smooth invex functions [3]. For the last few decades generalized monotonicity, duality and optimality conditions in invex optimization theory have been discussed by several authors but mainly in $\mathbb{R}^n$ [1, 3, 4, 6, 8, 13, 18]. In this paper the concept of invex functions and a class of optimization problems involving invex functions have been introduced in Hilbert spaces. Some important theorems regarding the characterization of extreme points and optimal solution have also been discussed.

## 2. Prerequisites

**Definition 2.1:** A subset $C$ of $\mathbb{R}^n$ is *convex* if for every pair of points $x_1, x_2$ in $C$, the line segment

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$$[x_1, x_2] = \{x : x = \alpha x_1 + \beta x_2,\ \alpha \geq 0,\ \beta \geq 0, \alpha + \beta = 1\}$$

belongs to $C$.
The set C is said to be $\eta - invex$ if there exist a vector function $\eta : C \times C \to \mathbb{R}^n$ such that

$$x_1 + \lambda\eta(x_1, x_2) \in C \quad \forall\ x_1, x_2 \in C \text{ and } \forall\lambda \in [0, 1]$$

**Definition 2.2:** Let C be an open convex set in $\mathbb{R}^n$ and let $f$ be real valued and differentiable on C. Then $f$ is *convex* if

$$f(x) - f(y) \geq \langle \nabla f(y), x - y \rangle\ , \quad \forall\ x, y \in C$$

The function $f$ is said to be $\eta - invex$ if there is a vector function $\eta : C \times C \to \mathbb{R}^n$

$$f(x) - f(y) \geq \langle \nabla f(y), \eta(x, y) \rangle\ , \quad \forall\ x, y \in C$$

**Remark:** Clearly, every differentiable convex function is $\eta - invex$, since in that case one can take $\eta(x, y) = (x - y)$. It is to be further noted that a differentiable function which is $\eta$-invex for a specific $\eta(x, y)$ may not be the same for a different $\eta(x, y)$. Therefore, in a particular context of a discussion $\eta(x, y)$(if exists) is fixed and that depends upon the differentiable function to be studied.
**Definition 2.3:** A set $C \subseteq \mathbb{R}^n$ is said to be a *cone*, if $x \in C \Rightarrow \lambda x \in C\ ,\ \ \forall\ \lambda \geq 0$. In addition, if C is convex, then C is said to be a *convex cone*. A convex cone is said to be a *proper cone* if it is closed, having non-empty interior and pointed(i,e. $x \in C, -x \in C \Rightarrow x = 0$).
A *barrier cone* of a convex set C is defined to be the set of all vectors $x^*$ such that, for some $\beta \in \mathbb{R}$, $\langle x, x^* \rangle \leq \beta$ for every $x \in C$.
**Definition 2.4:** Let $X$ and $Y$ be two normed vector spaces. A continuous linear transformation $A$:$X \to Y$ is said to be the *Fréchet*(Strong) derivative of $f : X \to Y$ at $x$ if for every $\epsilon > 0,\ \exists\ \delta > 0$ such that

$$\parallel f(x + h) - f(x) - Ah \parallel_Y \leq \epsilon \parallel h \parallel_X \qquad \forall h \text{ with } \parallel h \parallel_X \leq \delta$$

When the derivative exists it is denoted by $Df(x)$.
**Remark :** It is to be noted that in $\mathbb{R}^n$, $Df(x) = \nabla f(x)$.
**Definition 2.5:** An ordering $\geq$ on a real vector space V is said to be *Archimedean* if $v \geq \theta_V$ whenever $u + \lambda v \geq \theta_V$ for some $u \in V$ and all $\lambda > 0$.
If $u \leq w$ and $u, w \in V$ then [u,w] will denote the set $\{v \in V : u \leq v \leq w\}$. Such a set is termed as *order interval*. A subset of V is *order bounded* if it is contained in some order interval.
**Remark :** The order relation $\geq$ in $\mathbb{R}^n$ is archimedean if $x + ny \geq \theta, n = 1, 2, 3, .....$ implies $y \geq \theta$. Most of the orderings that occur in practical problems are archimedean. Lexicographic orderings in sequence spaces are non-archimedean orderings.

## 3. Invex Programming Problem(IP)

**Definition 3.1:** Let $H_1$ and $H_2$ be two Hilbert spaces with some archimedean ordering " $\geq$ " and $X \subseteq H_1$ is an open set. The differentiable(Frechet) function $f : X \to H_2$ is $\eta - invex$ if there exist a vector function $\eta : X \times X \to H_2$ and some $\mathbf{e} \in H_2$ with $\|\mathbf{e}\|_{H_2} = 1$ such that,

$$f(\mathbf{x}) - f(\mathbf{y}) \geq \langle Df(\mathbf{y}) \ , \ \eta(\mathbf{x}, \mathbf{y})\rangle \mathbf{e} \qquad \forall \mathbf{x}, \mathbf{y} \in X \tag{3.1}$$

**Remark**: It is to be noted that if $H_1$ and $H_2$ are taken as $\mathbb{R}^n$, then if we choose $\mathbf{e} = (1, 1, 1, \ldots, 1)$ and $\eta(x, y) = (x - y)$, $f$ will become a convex function in $\mathbb{R}^n$. The norm in this case can be taken as $(n)^{-\frac{1}{2}}$-multiple of the usual euclidean norm.
**Example 3.1:** Let us consider the function $f : L^2[0, 1] \to L^2[0, 1]$ defined as,

$$(f(x))(t) = (x - sinx)(t) \ , \ x > 0, \ n \in \mathbb{N}$$

Clearly, f($x$)is non-convex in nature. But it can be verified that f($x$) is $\eta$-invex for $\eta(x, y) = \frac{4sin\frac{x-y}{2}}{cos(x)-1}$ whenever $x \neq 2n\pi$ and $\eta(x, y) = 0$ elsewhere.

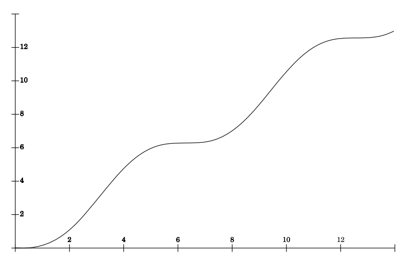


Fig. 3.1: Graph of f($x$)=$x - sin(x)$ , $x > 0$ , which is not convex but invex

**Definition 3.2:** Let $H_1$ and $H_2$ be two real Archimedean ordered separable Hilbert spaces. Let $\phi : H_1 \to H_2$ and $f : H_1 \to H_2$ be $\eta - invex$ functions (i.e. both the functions are invex with respect to the same $\eta$) such that $f(x) = f(y) \Rightarrow (x - y) \in Ker f$ . Let us consider the following program:

$$\begin{array}{lll} & Min \ \ \phi(x) & \\ s.t. & f(x) = y & x \in H_1 \ , \ y \in H_2 \\ & x \geq \theta_{H_1} & \end{array}$$

Let us denote the program by IP($H_1$, $H_2$,$\phi$, $f$) or simply by IP(if there is no confusion).
**Example 3.2:** The very well known Bottleneck Problem due to Bellman(1957)[9] is one of the examples of IP.

## 4. Extreme Points and Basic Feasible Solutions

**Definition 4.1:** For any IP we define that $x$ is feasible iff it satisfies the conditions of IP . The set of feasible solutions of IP is thus

$$F = \{x \in H_1 \ : \ x \geq \theta_{H_1} \ and \ f(x) = y\}$$

**Definition 4.2:** Let H be a real Hilbert space and X be a subset of H. Let C denote a positive cone in X. For any $x \in C$ define

$$B(x) = \{y \in X : x + \lambda y \in C \ and \ x - \lambda y \in C, \lambda \in \mathbb{R}\}$$

In particular if F is $\mathbb{R}^n$, $B(x)$ reduces to

$$B(x) = \{\xi \in \mathbb{R}^n \ : \ \xi_i = 0 \ for \ i \notin S(x)\}$$

where S($x$) is the support of $x$ (often called the basis)[9].
**Definition 4.3:** A *feasible solution* of IP is basic if $B(x) \cap N(f) = \{\theta\}$
**Definition 4.4:** $x$ is an *extreme point* of X if for any $\lambda \in \mathbb{R}$ either $x+\lambda y \in X \ or \ x-\lambda y \in X$ but not both.
**Theorem 4.1:** $x$ is a Basic Feasible Solution(BFS) for IP iff $x$ is an extreme point of F .
**Proof:** A BFS $x$ is not an extreme point of F iff there exist a $\lambda > 0$ and $z(\neq \theta) \in F$ such that $x + \lambda z \ and \ x - \lambda z$ both are feasible. .This is true iff $x + \lambda z \in C \ and \ x - \lambda z \in C$ and $f(x + \lambda z) = y = f(x - \lambda z)$ which implies $f(2\lambda z) = \theta \ i.e. \ 2\lambda z \in N(f)$. Now since $z \in B(x)$, $2\lambda z \in B(x)$, which implies that $2\lambda z \in B(x) \cap N(f)$. This is a contradiction as $x$ is a BFS. □
**Theorem 4.2:** A feasible solution $x$ is a BFS iff for any $\xi \in F$ , $B(\xi) \subseteq B(x)$implies $\xi = x$
**Proof:** Let $x$ be a basic feasible solution and $\xi$ is another feasible solution such that $B(\xi) \subseteq B(x)$. Since $\xi \in B(\xi)$, $\xi \in B(x)$. Since both x and $\xi$ are feasible, $f(x) = y = f(\xi)$. Which implies that $f(x - \xi) = \theta \Rightarrow (x - \xi) \in B(x) \cap N(f)$. Since x is a BFS, $(x - \xi)$ must be $\theta$, i,e. $x = \xi$.
Conversely, let $x$ is a feasible solution but not basic.Choose $\eta \neq \theta$ with $\eta \in B(x) \cap N(f) \ then \ x + \lambda\eta \geq \theta \ and \ x - \lambda\eta \geq \theta$.Let $\xi = x + \lambda\eta$. Let $\xi' \in B(\xi)$, then for some scalar $\mu$, $\xi + \mu\xi' \geq \theta \ and \ \xi - \mu\xi' \geq \theta \ i,e. \ x + \lambda\eta + \mu\xi' \geq \theta \ and \ x + \lambda\eta - \mu\xi' \geq \theta$. Now $x + (\frac{\mu}{2})\xi' = \frac{1}{2¢}[(x - \lambda\eta) + (x + \lambda\eta + \mu\xi')] \geq \theta$ and similarly $x - \frac{\mu}{2}\xi' \geq \theta$. Which implies that $\xi' \in B(x) \ \ i,e. \ \ B(\xi) \subseteq B(x)$ which is a contradiction since as per the assumption $\xi \neq x$. □
**Definition 4.5:** Let H be a Hilbert space with some ordering " $\geq$ ". A vector $x \in H$ is said to be non-negative (positive) if $x \geq \theta_H (x > \theta_H)$.

## 5. Main Results

**Theorem 5.1:** Let C be a proper cone in $H_1$ , then a feasible solution $x$ of IP is basic iff there is no other feasible solution $\xi$ such that $B(\xi) \subset B(x)$.
**Proof:** Let $x$ be a feasible solution of IP which is not basic. Let $y \in B(x) \cap N(f)$. Let L be the order interval $[x - \lambda y, x + \lambda y]$. Then clearly $L \cap F = L \cap C$ must be an order interval having at least a maximal or a minimal element ( using Zorn's

Lemma). Let $\xi$ be such an element of that order interval. Then $\xi \in B(x)$ and it has been shown in the proof of Theorem 4.2 that this implies $B(\xi) \subset B(x)$. The inclusion is strict because $y \notin B(\xi)$ (since $C \cap -C = \theta_{B_1}$ as C is a proper cone). $\square$

**Theorem 5.2:** Let C be a proper cone in $H_1$ and $x$ is an optimal solution of IP. If $\eta(H_1 \times L)$ , $D\phi(L)$ and $\alpha\mathbf{e}$ are all non-negative, where $\alpha \geq 0$, then IP has a basic optimal solution.

**Proof:** Let $x$ is not basic.Then as discussed in Theorem 5.1 we can find a $\xi \in B(x)$ which is feasible for IP. Now if the set of subspaces $\{B(\xi) : \xi \in F\}$ is inductively ordered by inclusion then using Zorn's Lemma we can find a $\xi$ for which $B(\xi)$ is minimal. Now since $\phi$ is $\eta$-invex , $\phi(x) - \phi(\xi) \geq D\phi(\xi)\ \eta(x,\xi) \geq \theta_{B_2}$ ,as per the assumption. Which implies that $\phi(x) \geq \phi(\xi)$. Now since x is optimal, $\phi(\xi) \geq \phi(x)$.This implies that $\phi(\xi) = \phi(x)$.Therefore $\xi$ is a basic optimal solution. $\square$

## 6. Conclusion

Theorem 4.1 provides the characterization of basic solutions as the extreme point of the set of feasible solutions. Theorem 4.2 shows that if there is a feasible solution for an IP then there must be a basic feasible solution. This property has been strengthened in Theorem 5.1. Theorem 5.2 characterizes the basic optimal solutions of an IP and guarantees that a basic optimal solution can be constructed from an optimal solution using the method discussed in Theorem 5.1. But here it is worth mentioning that the ordering in the context is very important. There are examples of spaces with non-Archimedean ordering where existence of optimal solution does not imply the existence of basic optimal solution.

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