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ENERGY DECAY RATES FOR THE BRESSE-CATTANEO SYSTEM WITH WEAK NONLINEAR BOUNDARY DISSIPATION

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Abstract. In this paper, we consider a one-dimensional Bresse system with Cattaneo's type heat conduction and a nonlinear weakly dissipative boundary feedback localized on a part of the boundary. We show the well-posedness, using the semigroup theory, and establish an explicit and general decay rate result without imposing a specific growth assumption on the behavior of damping terms near zero.

Keywords: Bresse system; Bresse system; decay rate

1. Introduction

In [3] a simple one dimensional Bresse model is usually considered in studying the elastic structures of the arcs type whose motion is governed by the following system of three wave equations:

$$
(1.1)\quad \begin{cases} \rho_1\varphi_{tt} - \kappa\left(\varphi_x + \psi + lw\right)_x - \kappa_0 l\left(w_x - l\varphi\right) = 0, & in \quad (0,L)\times(0,\infty) \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa\left(\varphi_x + \psi + lw\right) = 0, & in \quad (0,L)\times(0,\infty) \\ \rho_1 w_{tt} - \kappa_0\left(w_x - l\varphi\right)_x + \kappa l\left(\varphi_x + \psi + lw\right) = 0, & in \quad (0,L)\times(0,\infty) \end{cases}
$$

where the coefficients ρ_1, ρ_2 denote respectively the mass per unit length, the mass moment of inertia of a cross-section of the beam and the coefficients κ_0, κ, b and l are equal to EA, $\kappa^{'}GA$, EI and R^{-1}, respectively, where E is the Young's modulus, I is the moment of inertia of a cross-section of the beam, G is the modulus of elasticity in shear, A is the cross sectional area, $\kappa^{'}$ is the shear factor and R for the radius of the curvature. The functions φ, ψ and w represents the vertical, rotation angle, and longitudinal displacements, respectively, of the point x of the beam at the instant t.

Remark 1.1. We note that when $R \to \infty$, then $l \to 0$ and then this model reduces to the well-known Timoshenko beam equations (see [24]).

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There exist a few results about the stability of the Bresse system where the authors consider the different kinds of the dissipative mechanism. The case of one frictional damping has already been considered by Alabau Boussouira et al. [1], Noun and Wehbe [17] where the authors proved that the semi-group associated with the following Bresse system

$$(1.2) \quad \begin{cases} \rho_1\varphi_{tt} - \kappa\left(\varphi_x + \psi + lw\right)_x - \kappa_0 l\left(w_x - l\varphi\right) = 0, \quad in \quad (0,L)\times(0,\infty) \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa\left(\varphi_x + \psi + lw\right) + \gamma\psi_t = 0, \quad in \quad (0,L)\times(0,\infty) \\ \rho_1 w_{tt} - \kappa_0\left(w_x - l\varphi\right)_x + \kappa l\left(\varphi_x + \psi + lw\right) = 0, \quad in \quad (0,L)\times(0,\infty) \end{cases}$$

with boundary conditions of the Dirichlet-Dirichlet-Dirichlet type or mixed boundary conditions is polynomially stable provided

$$(1.3) \qquad \frac{\rho_1}{\rho_2} = \frac{\kappa}{b} \quad and \quad \kappa = \kappa_0.$$

(i.e., the equal-speed wave propagation condition) and moreover they proved the lack of exponential stability when they considered the Dirichlet-Neumann-Neumann type boundary condition. The equal-speed wave propagation condition has been used in many works in order to establish exponential decay rates. Fatori and Monteiro [6] showed the optimality of the polynomial decay rate for the Bresse system (1.2) with the Dirichlet-Neumann-Neumann type boundary condition. In [23], the authors considered the Bresse system with indefinite damping mechanism acting on the equation about the shear angle displacement. Under the equal speeds condition and only with Dirichlet.Neumann.Neumann boundary condition type, they proved the exponential stability of the system. Wehbe and Youssef [25]; Santos and Junior [20] showed the asymptotic stability without impose conditions about the equal-speed wave propagation for the Bresse system with linear dissipation by different methods. Soriano et al. [22] and Charles et al. [5] gave the asymptotic stability for the following Bresse system with nonlinear dissipation

$$\begin{cases} \rho_1\varphi_{tt} - \kappa\left(\varphi_x + \psi + lw\right)_x - \kappa_0 l\left(w_x - l\varphi\right) + \alpha_1(x)g_1(\varphi_t) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa\left(\varphi_x + \psi + lw\right) + \alpha_2(x)g_2(\psi_t) = 0, \\ \rho_1 w_{tt} - \kappa_0\left(w_x - l\varphi\right)_x + \kappa l\left(\varphi_x + \psi + lw\right) + \alpha_3(x)g_3(w_t) = 0, \end{cases}$$

in $(0,L)\times(0,\infty)$ by energy methods. However, to obtain the energy decay rate estimate, the authors required that α_i and the damping terms $g_i(.)$ satisfy the following growth rate:

$$(1.4) \quad \begin{cases} \alpha_i = \alpha_i(x) \in L^\infty(0,L), \quad \alpha_i(x) \geq C > 0, \\ g_i(s)s > 0, \quad for \quad s \neq 0, \quad cs \leq g_i(s) \leq ds \quad for \quad |s| > 1, \quad i = 1,2,3. \end{cases}$$

where C, c, d are constants. Li et al [11] extend the behavior of $\alpha_i, g_i(.)$ to more general cases which does not necessarily satisfy (1.4) and get the explicit energy decay rate estimate for the system.

Concerning stabilization via heat effect, Liu and Rao [12] considered the Bresse system with two different dissipative mechanism, given by two temperatures coupled to the system. The authors considered the problem

$$
\begin{cases}
\rho_1\varphi_{tt} - \kappa\left(\varphi_x + \psi + lw\right)_x - \kappa_0 l\left(w_x - l\varphi\right) + l\kappa_1\theta^1 = 0,\\
\rho_2\psi_{tt} - b\psi_{xx} + \kappa\left(\varphi_x + \psi + lw\right) + \kappa_1\theta^3_x = 0,\\
\rho_1 w_{tt} - \kappa_0\left(w_x - l\varphi\right)_x + \kappa l\left(\varphi_x + \psi + lw\right) + \kappa_1\theta^1_x = 0,\\
\rho_3\theta^1_t - \alpha\theta^1_{xx} + \kappa_1\left(w_x - l\varphi\right) = 0,\\
\rho_3\theta^3_t - \alpha\theta^3_{xx} + \kappa_1\psi_{tx} = 0,
\end{cases}
\tag{1.5}
$$

in $(0, L) \times \mathbb{R}^+$ and they proved that the exponential decay exists only when the velocities of the wave propagations are the same. If the wave speeds are different they showed that the energy of the system decays polynomially to zero with the rate $t^{-1/2}$ or $t^{-1/4}$, provided that the boundary conditions is of Dirichlet-Neumann-Neumann or Dirichlet-Dirichlet-Dirichlet type, respectively.

If $\theta^1 = 0$ in (1.5) Fatori and Munoz Rivera [7] analyzed the exponential stability of the obtained Bresse-Fourier system they showed that, in general, the system is not exponentially stable but that there exists polynomial stability with rates that depend on the wave propagations and the regularity of the initial data. Recently, Najdi and Wehbe in [16] extended and improved the results of [7] when the thermal dissipation is locally distributed.

In system (1.5), the heat equation is governed by Fourier's law of heat conduction, which states that the heat flux is proportional to the gradient of temperature. Moreover, it is well known that the model using the classic Fourier's law leads to the physical paradox of infinite speed of heat propagation. In other words, any thermal disturbance at one point will be instantaneously transferred to the other parts of the body. However, experiments showed that heat conduction in some dielectric crystals at low temperatures propagates with a finite speed [9]. To overcome this physical paradox but still keeping the essentials of a heat conduction process, many theories have subsequently emerged. One of which is the advent of the second sound effects observed experimentally in materials at a very low temperature. Second sound effects arise when heat is transported by a wave propagation process instead of the usual diffusion. This theory suggests replacing the classic Fourier's law

$$
q + \gamma\theta_x = 0 \tag{1.6}
$$

where q is the heat flux and γ is the coefficient of thermal conductivity by a modified law of heat conduction called Cattaneo's law

$$
\tau q_t + q + \gamma\theta_x = 0 \tag{1.7}
$$

Here, the parameter $\tau > 0$ represents the relaxation time describing the time lag in the response of the heat flux to a gradient in the temperature. The obtained heat system is of hyperbolic type and hence, automatically, eliminates the paradox of infinite speeds.

The coupling of the Bresse system (1.1) with the aforementioned theory is given by

$$\begin{cases} \rho_1\varphi_{tt} - \kappa\left(\varphi_x + \psi + lw\right)_x - \kappa_0 l\left(w_x - l\varphi\right) = 0, \quad 0 < x < 1, \quad t > 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa\left(\varphi_x + \psi + lw\right) + \delta\theta_x = 0, \quad 0 < x < 1, \quad t > 0 \\ \rho_1 w_{tt} - \kappa_0\left(w_x - l\varphi\right)_x + \kappa l\left(\varphi_x + \psi + lw\right) = 0, \quad 0 < x < 1, \quad t > 0 \\ \rho_3\theta_t + q_x + \delta\psi_{tx} = 0, \quad 0 < x < 1, \quad t > 0 \\ \tau_q q_t + \beta q + \theta_x = 0, \quad 0 < x < 1, \quad t > 0 \end{cases} \tag{1.8}$$

When thermal effects are considered, the asymptotic behavior of the Bresse system may become more complicated because of the coupling between the elasticity and heat conduction. Concerning to system (1.8), we found several papers that studies a reduced version which is known as Timoshenko system. Fernandez Sare and Racke [8] considered the following Timoshenko-Cattano system

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x = 0, \quad 0 < x < 1, \quad t > 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi\right) + \delta\theta_x = 0, \quad 0 < x < 1, \quad t > 0 \\ \rho_3\theta_t + q_x + \delta\psi_{tx} = 0, \quad 0 < x < 1, \quad t > 0 \\ \tau_q q_t + \beta q + \theta_x = 0, \quad 0 < x < 1, \quad t > 0 \end{cases} \tag{1.9}$$

with some initial and Dirichlet-Neumann-Dirichlet boundary conditions and proved that the system is not exponentially stable even if the propagation speeds are equal. Moreover, they showed that the presence of viscoelastic damping term of the form $\int_0^\infty g(s)\varphi_{xx}(t-s)ds$ in the second equation of (1.9) is also not sufficient to obtain exponential stability. Recently, Santos et al. [21] considered (1.9), and introduced a new stability number in the form

$$\mu = \left(\tau - \frac{\rho_1}{k\rho_3}\right)\left(\frac{\rho_2}{b} - \frac{\rho_1}{k}\right) - \frac{\tau\rho_1\delta^2}{kb\rho_3}$$

and used the semi-group method to obtain exponential decay result for $\mu = 0$ and a polynomial decay for $\mu \neq 0$. Also, a stability result, using this new number, was obtained by Said-Houari and Hamadouche in [19] by considering the Cauchy problem for the one-dimensional Bresse system coupled with heat conduction governed by the Cattaneo law.

In [2] Ayadi et al considered the nonlinear Timoshenko-Cattaneo system of the form

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x = 0, \quad 0 < x < 1, \quad t > 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi\right) + \delta\theta_x + \alpha(t)h(\psi_t) = 0, \quad 0 < x < 1, \quad t > 0 \\ \rho_3\theta_t + q_x + \delta\psi_{tx} = 0, \quad 0 < x < 1, \quad t > 0 \\ \tau_q q_t + \beta q + \theta_x = 0, \quad 0 < x < 1, \quad t > 0 \end{cases}$$

and proved an explicit and general decay results which depend on the stability number μ as identified previously by Santos et al. in [21].

The boundary feedback with a time-dependent coefficient has been used by Mustafa [15] for a wave equation.

In the present paper, we consider the Bresse-Cattaneo (1.8) under the following initial data:

$$(1.10) \qquad \begin{cases} \psi\left(x,0\right) = \psi_0\left(x\right), \varphi\left(x,0\right) = \varphi_0\left(x\right), w\left(x,0\right) = w_0\left(x\right) \\ \psi_t\left(x,0\right) = \psi_1\left(x\right), \varphi_t\left(x,0\right) = \varphi_1\left(x\right), w_t\left(x,0\right) = w_1\left(x\right) \\ \theta\left(x,0\right) = \theta_0\left(x\right), q\left(x,0\right) = q_0\left(x\right), \end{cases}$$

and the following boundary conditions

$$(1.11) \qquad \psi(0,t) = 0, \varphi(0,t) = 0, w(0,t) = 0, \theta(0,t) = \theta(1,t) = 0 \quad t > 0$$

$$(1.12) \qquad \begin{cases} \left(\varphi_x + \psi + lw\right)(1,t) = -\alpha(t)h(\varphi_t(1,t)), \quad t > 0 \\ \psi_x(1,t) = -\alpha(t)h(\psi_t(1,t)), \quad t > 0 \\ \left(w_x - l\varphi\right)(1,t) = -\alpha(t)h(w_t(1,t)), \quad t > 0 \end{cases}$$

The boundary conditions (1.11)-(1.12) states that the system is fixed at $x = 0$ and the other end is subjected to the effect of a nonlinear time-dependent frictional damping.

Our aim in this paper is to investigate (1.8),(1.10)-(1.12), in which the damping considered is modulated by a time-dependent coefficient $\alpha(t)$, and establish an explicit and general decay result, depending on h and α. The proof is based on the multiplier method and makes use of a lemma by Martinez [13]. This paper is organized as follows. In Section 2, we present some notations and materials needed for our work and establish the well-posedness of system (1.8),(1.10)-(1.12) by using the semi-group theory. The statement and the proof of our main result are given in Section 3. In the last section, we investigate some special cases.

2. Preliminaries and Well-posedness

In this section, we present some material needed for the proof of our main result and we prove the existence and the uniqueness of the solution of system (1.8),(1.10)-(1.12).

We consider the following hypotheses:

(A1) $\alpha : [0,+\infty) \to \mathbb{R}^+$ are non-increasing C^1-function satisfying

$$\int_0^\infty \alpha(t)dt = +\infty$$

(A2) $h : \mathbb{R} \to \mathbb{R}$ is a nondecreasing C^0 function such that there exists a strictly increasing differentiable function $h_0 \in C^1([0,+\infty))$, with $h_0(0) = 0$, and positive constants c_1, c_2 and M such that

$$\begin{cases} h_0(|t|) \le |h(t)| \le h_0^{-1}(|t|), & |t| \le M, \\ c_1|t| \le |h(t)| \le c_2|t|, & |t| > M \end{cases}$$

Remark 2.1. Hypothesis (A1) implies that α is bounded and (A2) implies that $sh(s) > 0$, for all $s \neq 0$.

We now discuss the well-posedness of (1.8),(1.10)-(1.12). For this purpose, we introduce the following spaces:

$$\begin{cases} V = \left\{ v \in H_0^1(0,1) : v(0) = 0 \right\} \\ L_*^2(0,1) = \left\{ w \in L^2(0,1) : \int_0^1 w(s)ds = 0 \right\}, \\ H_*^1(0,1) = H^1(0,1) \cap L_*^2(0,1) \end{cases}$$

Introducing the vector function

$$U = (\varphi, u, \psi, v, w, z, \theta, q)^T \tag{2.1}$$

where $u = \varphi_t$, $v = \psi_t$, $z = w_t$. The phase space of our problem is

$$\mathcal{H} = V \times L^2(0,1) \times V \times L^2(0,1) \times V \times L^2(0,1) \times L^2(0,1) \times L_*^2(0,1) \tag{2.2}$$

equipped with the inner product

$$\begin{aligned} (U, \tilde{U})_{\mathcal{H}} &= \int_0^1 \kappa(\varphi_x + \psi + lw)(\tilde{\varphi}_x + \tilde{\psi} + l\tilde{w})dx + \int_0^1 b\psi_x\tilde{\psi}_x dx \\ &+ \int_0^1 \kappa_0(w_x - l\varphi)(\tilde{w}_x - l\tilde{\varphi})dx + \int_0^1 \rho_1 u\tilde{u}dx + \int_0^1 \rho_2 v\tilde{v}dx \\ &+ \int_0^1 \rho_1 z\tilde{z}dx + \int_0^1 \rho_3\theta\tilde{\theta}dx + \int_0^1 \tau_q q\tilde{q}dx. \end{aligned}$$

and the corresponding norm is

$$\begin{aligned}\|\varphi,u,\psi,v,w,z,\theta,q\|_{\mathcal{H}}^2 &= \kappa\|\varphi_x+\psi+lw\|^2+b\|\psi_x\|^2+\kappa_0\|w_x-l\varphi\|^2+\rho_1\|u\|^2+\rho_2\|v\|^2\\ &\quad+\rho_1\|z\|^2+\rho_3\|\theta\|^2+\tau_q\|q\|^2.\end{aligned}$$

System (1.8),(1.10)-(1.12) can be written as a linear ordinary differential equation in $\mathcal{H}$ of the form

$$\frac{d}{dt}U(t)+\mathcal{A}U(t)=0,\quad U_0=(\varphi_0,\varphi_1,\psi_0,\psi_1,w_0,w_1,\theta_0,q_0)^T \tag{2.3}$$

where the domain $D(\mathcal{A})$ of the linear operator $\mathcal{A}:D(\mathcal{A})\subset\mathcal{H}\to\mathcal{H}$ is given by

$$D(\mathcal{A})=\left\{\begin{array}{c} U\in\mathcal{H}:\varphi,\psi,w\in H^2(0,1)\cap V, u,v,z\in V,\theta\in H_0^1(0,1), q\in H_*^1(0,1)\\ (\varphi_x+\psi+lw)(1)=\alpha(t)h(u(1)), \psi_x(1)=\alpha(t)h(v(1)), (w_x-l\varphi)(1)=\alpha(t)h(z(1))\end{array}\right\}$$

and

$$\mathcal{A}U=\begin{pmatrix} -u\\ -\frac{\kappa}{\rho_1}(\varphi_x+\psi+lw)_x-\frac{\kappa_0 l}{\rho_1}(w_x-l\varphi)\\ -v\\ -\frac{b}{\rho_2}\psi_{xx}+\frac{\kappa}{\rho_2}(\varphi_x+\psi+lw)+\frac{\delta}{\rho_2}\theta_x\\ -z\\ -\frac{\kappa_0}{\rho_1}(w_x-l\varphi)_x+\frac{\kappa l}{\rho_1}(\varphi_x+\psi+lw)\\ \frac{1}{\rho_3}q_x+\frac{\delta}{\rho_3}v_x\\ \frac{\beta}{\tau_q}q+\frac{1}{\tau_q}\theta_x\end{pmatrix}$$

It is not difficult to see that $\mathcal{H}$ is a Hilbert space and that $D(\mathcal{A})$ is dense in $\mathcal{H}$.

We have the following existence and uniqueness result.

Theorem 2.1. *The operator $\mathcal{A}$ is the infinitesimal generator of C_0-semigroup $S(t)$ of contraction in $\mathcal{H}$. Thus for any initial data $U_0\in\mathcal{H}$, there exists a unique solution $U\in C(\mathbb{R}^+,\mathcal{H})$ of problem* (1.8),(1.10)-(1.12). *Moreover if $U_0\in D(\mathcal{A})$, then $U\in C(\mathbb{R}^+,D(\mathcal{A}))\cap C^1(\mathbb{R}^+,\mathcal{H})$.*

Proof. To prove Theorem 2.1, we use the semigroup approach. For this purpose, we show firstly that the operator $\mathcal{A}$ is monotone in the phase space $\mathcal{H}$. Indeed, for any $U\in D(\mathcal{A})$, by definition of the operator $\mathcal{A}$ and the scalar product of $\mathcal{H}$, we have

$$\begin{aligned}(\mathcal{A}U,U)_{\mathcal{H}} &= \beta\int_0^1 q^2dx+\kappa\alpha(t)h(u(1))u(1)+b\alpha(t)h(v(1))v(1)\\ &\quad+\kappa_0\alpha(t)h(z(1))z(1)\geq 0.\end{aligned} \tag{2.4}$$

which implies that $\mathcal{A}$ is monotone in $\mathcal{H}$. Next, we prove that the operator $I-\mathcal{A}$ is surjective. Given

$$G=(g_1,g_2,g_3,g_4,g_5,g_6,g_7,g_8)^T\in\mathcal{H} \tag{2.5}$$

we prove that there exists $U \in D(\mathcal{A})$ satisfying,

$$(I+\mathcal{A})U = G. \tag{2.6}$$

Equation (2.6) is equivalent to

$$\begin{cases} -u+\varphi = g_1 \\ -\kappa(\varphi_x+\psi+lw)_x - \kappa_0 l(w_x - l\varphi) + \rho_1 u = \rho_1 g_2 \\ -v+\psi = g_3 \\ -b\psi_{xx} + \kappa(\varphi_x+\psi+lw) + \delta\theta_x + \rho_2 v = \rho_2 g_4 \\ -z+w = g_5 \\ -\kappa_0(w_x - l\varphi)_x + \kappa l(\varphi_x+\psi+lw) + \rho_1 z = \rho_1 g_6 \\ q_x + \delta v_x + \rho_3\theta = \rho_3 g_7 \\ (\beta+\tau_q)q + \theta_x = \tau_q g_8. \end{cases} \tag{2.7}$$

From the first, the third and the fifth equations of (2.7), we obtain

$$u = \varphi - g_1, \quad v = \psi - g_3, \quad z = w - g_5. \tag{2.8}$$

From the eighth equation, we have

$$\theta = -(\beta+\tau_q)\int_0^x q(y)dy + \tau_q \int_0^x g_8(y)dy \tag{2.9}$$

with

$$\theta(0,t) = \theta(1,t) = 0 \tag{2.10}$$

Substitutig u, v, z, θ into the second , the fourth, the sixth, the seventh equations in (2.7), we get

$$\begin{cases} -\kappa(\varphi_x+\psi+lw)_x - \kappa_0 l(w_x - l\varphi) + \rho_1\varphi = \rho_1(g_1+g_2) \\ -b\psi_{xx} + \kappa(\varphi_x+\psi+lw) - \delta(\beta+\tau_q)q + \rho_2\psi = \rho_2(g_3+g_4) - \delta\tau_q g_8 \\ -\kappa_0(w_x - l\varphi)_x + \kappa l(\varphi_x+\psi+lw) + \rho_1 w = \rho_1(g_5+g_6) \\ -q_x + \rho_3(\beta+\tau_q)\int_0^x q(y)dy - \delta\psi_x = \rho_3\tau_q\int_0^x g_8(y)dy - \rho_3 g_7 - \delta g_{3x} \end{cases} \tag{2.11}$$

The variational formulation corresponding to (2.11) takes the form

$$a((\varphi_1,\psi_1,w_1,q_1),(\varphi_2,\psi_2,w_2,q_2)) = \ell(\varphi_2,\psi_2,w_2,q_2) \tag{2.12}$$

where the bilinear form a is defined as follows:

$$
\begin{aligned}
& a((\varphi_1,\psi_1,w_1,q_1),(\varphi_2,\psi_2,w_2,q_2)) \\
& = \kappa \int_0^1 (\varphi_{1,x}+\psi_1+lw_1)(\varphi_{2,x}+\psi_2+lw_2)dx + b\int_0^1 \psi_{1,x}\psi_{2,x}dx \\
& + \kappa_0 \int_0^1 (w_{1,x}-l\varphi_1)(w_{2,x}-l\varphi_2)dx + (\beta+\tau_q)\int_0^1 q_1 q_2 dx \\
& + \rho_1 \int_0^1 \varphi_1\varphi_2 dx + \rho_2 \int_0^1 \psi_1\psi_2 dx + \rho_1 \int_0^1 w_1 w_2 dx \\
& - \delta(\beta+\tau_q)\int_0^1 q_1\psi_2 dx - \delta(\beta+\tau_q)\int_0^1 \psi_{1x}\int_0^x q_2(y)dydx \\
& + \rho_3(\beta+\tau_q)^2 \int_0^1 \left(\int_0^x q_1(y)dy \int_0^x q_2(y)dy\right) dx
\end{aligned}
$$

and the linear form

$$
\begin{aligned}
\ell((\varphi_2,\psi_2,w_2,q_2)) = {} & \rho_1 \int_0^1 (g_1+g_2)\varphi_2 dx + \rho_2 \int_0^1 (g_3+g_4)\psi_2 dx - \delta\tau_q \int_0^1 g_8\psi_2 dx \\
& + \rho_1 \int_0^1 (g_5+g_6)w_2 dx - \delta(\beta+\tau_q)\int_0^1 g_{3x}\int_0^x q_2(y)dydx \\
& + \rho_3\tau_q(\beta+\tau_q)\int_0^1 \left(\int_0^x g_8(y)dy - \rho_3 g_7\right)\int_0^x q_2(y)dydx \\
& + \kappa\alpha(t)h(u(1))\varphi_2(1) + b\alpha(t)h(v(1))\psi_2(1) + \kappa_0\alpha(t)h(z(1))w_2(1).
\end{aligned}
$$

Now, we set $X = V^3 \times L^2_*(0,1)$ equipped with norm

$$
\|(\varphi,\psi,w,q)\|_X = \|(\varphi_x+\psi+lw)\|_2^2 + \|\psi_x\|_2^2 + \|(w_x-l\varphi)\|_2^2 + \|q\|_2^2 \tag{2.13}
$$

It is clear that a is bounded and coercive and that ℓ is bounded. Then, using Lax-Milgram theorem ,we deduce that (2.12) has a unique solution $(\varphi,\psi,w,q) \in X$. Thus, using (2.8)-(2.9) and classical regularity arguments, we conclude that (2.6) admits a unique solution $U \in D(\mathcal{A})$. Consequently, the operator $\mathcal{A}$ is maximal. Hence, the result of Theorem (2.1) follows (see [4]). □

3. The main result

In this section, we state and prove our main result. For this purpose, we establish several lemmas.

The energy associated to the solution of system (1.8), (1.10)-(1.12) is given by the following formula:

$$E(t) = \frac{1}{2}\int_0^1 \left\{\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_1 w_t^2+\kappa_0(w_x-l\varphi)^2+\kappa(\varphi_x+\psi+lw)^2+b\psi_x^2+\rho_3\theta^2+\tau_q q^2\right\}dx$$

We can prove that the system (1.8), (1.11), (1.12) is dissipative as stated below

Lemma 3.1. *Let* $(\varphi,\psi,w,\theta,q)$ *be a global solution to problem* (1.8), (1.10)-(1.12) *on* $[0,+\infty[$. *Then we have*

$$\begin{aligned} E'(t) &= -\beta\int_0^1 q^2dx-\kappa\alpha(t)h(\varphi_t(1,t))\varphi_t(1,t)-b\alpha(t)h(\psi_t(1,t))\psi_t(1,t) \\ &\quad -\kappa_0\alpha(t)h(w_t(1,t))w_t(1,t)\le 0 \end{aligned} \tag{3.1}$$

Proof. By multiplying the first, the second, the third, the fourth and the fifth equations of (1.8) by φ_t, ψ_t, w_t, θ and q respectively, integrating by parts over $(0,1)$,adding these equalities and using hypotheses (A1)-(A2) and some manipulations we obtain (3.1). That is, the energy function $E(t)$ is nonincreasing. □

The following lemma will be of essential use in establishing our main results.

Lemma 3.2. *([13]) Let* E: $\mathbb{R}^+\to\mathbb{R}^+$ *be a nonincreasing function and* σ: $\mathbb{R}^+\to\mathbb{R}^+$ *be a strictly increasing* C^1*-function, with* $\sigma(t)\to+\infty$ *as* $t\to+\infty$. *Assume that there exist* $p,q>0$ *and* $c>0$ *such that*

$$\int_T^{+\infty}\sigma'(t)E(t)^{1+p}dt\le cE^{1+p}(T)+c\frac{E(T)}{\sigma^q(T)},\quad 0\le T<+\infty. \tag{3.2}$$

Then there exist positive constants k *and* ω *such that*

$$\begin{cases} E(t)\le ke^{-\omega\sigma(t)}, & if \quad p=0,\\ E(t)\le k\sigma(t)^{\frac{1+q}{p}}, & if \quad p>0. \end{cases} \tag{3.3}$$

Lemma 3.3. *Let* $(\varphi,\psi,w,\theta,q)$ *be a global solution of* (1.8), (1.10)-(1.12), *then for any* $N>0$ *the energy function satisfies*

$$\begin{aligned}
E(t) \quad \leq \quad & -(N+1)\frac{dL_1}{dt}-(N+1)\frac{dL_2}{dt}+\frac{N}{2}\frac{dL_3}{dt}+\frac{(N+1)}{2}\kappa_0\int_0^1(w_x-l\varphi)^2dx \\
&+\frac{(N{+}1)}{2}\kappa\int_0^1(\varphi_x+\psi+lw)^2dx+\frac{(N{+}1)}{2}c_1\rho_1\int_0^1\varphi_t^2dx+\frac{(N{+}1)}{2}c_2\int_0^1\psi_x^2dx \\
&+\frac{(N+1)}{2}c_3\rho_1\int_0^1 w_t^2dx+\frac{(N+1)}{2}\rho_1\int_0^1\psi_t^2dx+\frac{(N+1)}{2}\alpha^2(t)\kappa^2h^2(\varphi_t(1,t)) \\
&+\frac{(N+1)}{2}c_4\int_0^1 q^2dx+\frac{(N+1)}{2}c_5\int_0^1\theta^2dx+\frac{(N+1)}{2}\alpha^2(t)bh^2(\psi_t(1,t)) \\
&+\frac{(N+1)}{2}\alpha^2(t)\kappa_0^2h^2(w_t(1,t))-\frac{N}{2}\kappa h(\varphi_t(1,t))\varphi(1,t)-\frac{N}{2}bh\psi_t(1,t))\psi(1,t) \\
&-\frac{N}{2}\kappa_0h(w_t(1,t))w(1,t)+\frac{(N{+}1)}{2}\left(\rho_1\kappa\varphi_t^2(1,t)+\rho_2\psi_t^2(1,t){+}\rho_1\kappa_0w_t^2(1,t)\right). \qquad (3.4)
\end{aligned}$$

where c_i, $(i=1,...,5)$ *are positive constants and the functionals* L_i, $(i=1,...,3)$*are*

$$L_1(t)=\int_0^1\rho_1\kappa x(\varphi_x+\psi+lw)\varphi_tdx+\int_0^1\rho_1\kappa_0x(w_x-l\varphi)w_tdx,$$

$$L_2(t)=\int_0^1\rho_2x\psi_x\psi_tdx+\int_0^1\rho_3\tau_qx\theta qdx,$$

$$L_3(t)=\int_0^1\rho_1\varphi_t\varphi dx+\int_0^1\rho_2\psi_t\psi dx+\int_0^1\rho_1w_twdx.$$

Proof. Multiplying the first equation in (1.8) by $\kappa(N+1)x(\varphi_x+\psi+lw)-\frac{N}{2}\varphi$, and integrating it over (0,1) with respect to x, we have

$$\begin{aligned} 0 &= (N+1)\rho_1\kappa\int_0^1 x(\varphi_x+\psi+lw)\varphi_{tt}dx - \frac{N}{2}\rho_1\int_0^1 \varphi\varphi_{tt}dx \\ &-\kappa^2\int_0^1 (N+1)x(\varphi_x+\psi+lw)(\varphi_x+\psi+lw)_x dx \\ &+\kappa\frac{N}{2}\int_0^1(\varphi_x+\psi+lw)_x\varphi dx + \kappa_0 l\frac{N}{2}\int_0^1(w_x-l\varphi)\varphi dx \\ &-\kappa\kappa_0 l(N+1)\int_0^1 x(\varphi_x+\psi+lw)(w_x-l\varphi)dx \end{aligned} \tag{3.5}$$

By using integration by parts and using the boundary conditions, we get the following estimation

$$\begin{aligned} -\frac{\kappa^2}{2}\int_0^1(\varphi_x+\psi+lw)^2dx &= \rho_1\kappa(N+1)\int_0^1 \frac{d}{dt}\left[x(\varphi_x+\psi+lw)\varphi_t\right]dx \\ &+\frac{\rho_1[\kappa(N+1)+N]}{2}\int_0^1\varphi_t^2dx - \frac{N}{2}\rho_1\int_0^1\frac{d}{dt}\left[\varphi_t\varphi\right]dx \\ &-\rho_1\kappa(N+1)\int_0^1 x(\psi_t+lw_t)\varphi_t dx - \frac{\kappa N}{2}\int_0^1(\varphi_x+\psi+lw)\varphi_x dx \\ &-\kappa\kappa_0 l(N+1)\int_0^1 x(\varphi_x+\psi+lw)(w_x-l\varphi)dx \\ &+\kappa_0 l\frac{N}{2}\int_0^1(w_x-l\varphi)\varphi dx - \rho_1\frac{\kappa(N+1)}{2}\varphi_t^2(1,t) \\ &-\frac{\kappa N}{2}\alpha(t)h(\varphi_t(1,t))\varphi(1,t) - \frac{\kappa^2}{2}(N+1)\alpha^2(t)h^2(\varphi_t(1,t)) \end{aligned} \tag{3.6}$$

Next, multiplying the second equation in (1.8) by $(N+1)x\psi_x - \frac{N}{2}\psi$, and integrating it over $(0,1)$ with respect to x, we obtain

$$
\begin{aligned}
0 &= \int_0^1 \left[(N+1)x\psi_x - \frac{N}{2}\psi\right](\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi + lw) + \delta\theta_x)\,dx \\
&= \rho_2(N+1)\int_0^1 x\psi_x\psi_{tt}dx - \frac{N}{2}\rho_2\int_0^1 \psi_{tt}\psi dx \\
&\quad -b(N+1)\int_0^1 x\psi_x\psi_{xx}dx + b\frac{N}{2}\int_0^1 \psi_{xx}\psi dx \\
&\quad +\kappa(N+1)\int_0^1 x\psi_x(\varphi_x + \psi + lw)dx - \kappa\frac{N}{2}\int_0^1 \psi(\varphi_x + \psi + lw)dx \\
&\quad +\delta(N+1)\int_0^1 x\psi_x\theta_x dx - \frac{N}{2}\delta\int_0^1 \psi\theta_x dx
\end{aligned} \tag{3.7}
$$

Multiplying the fifth equation in (1.8) by $\delta(N+1)x\psi_x$ and integrating it over $(0,1)$ with respect to x, we obtain

$$
\delta(N+1)\int_0^1 x\psi_x\theta_x dx = -\delta(N+1)\tau_q\int_0^1 xq_t\psi_x dx - \beta(N+1)\delta\int_0^1 xq\psi_x dx \tag{3.8}
$$

We have

$$
\int_0^1 xq_t\psi_x dx = \int_0^1 \frac{d}{dt}[xq\psi_x]\,dx - \int_0^1 xq\psi_{xt}dx \tag{3.9}
$$

Multiplying the fourth equation in (1.8) by $\tau_q(N+1)xq$ and integrating it over $(0,1)$ with respect to x, we obtain

$$
\rho_3\tau_q(N+1)\int_0^1 x\theta_t q dx = -\tau_q(N+1)\int_0^1 xq_x q dx - \delta\tau_q(N+1)\int_0^1 x\psi_{tx}q dx \tag{3.10}
$$

We can rewrite the term in the left hand side of the last equation as follows

$$
\int_0^1 x\theta_t q dx = \int_0^1 \frac{d}{dt}[x\theta q]\,dx - \int_0^1 x\theta q_t dx \tag{3.11}
$$

Next, multiplying the fifth equation in (1.8) by $\rho_3(N+1)x\theta$ and integrating it over $(0,1)$ with respect to x, we obtain

$$
\rho_3(N+1)\tau_q\int_0^1 xq_t\theta dx + \beta\rho_3(N+1)\int_0^1 xq\theta dx + \rho_3(N+1)\int_0^1 x\theta\theta_x dx = 0 \tag{3.12}
$$

Substituting (3.9)-(3.12) in (3.8), we obtain

$$\begin{aligned}\delta(N+1)\int_0^1 x\psi_x\theta_x dx &= -\delta(N+1)\tau_q\int_0^1 \frac{d}{dt}\left[xq\psi_x\right]dx-\beta(N+1)\delta\int_0^1 xq\psi_x dx\\&+\frac{\tau_q(N+1)}{2}\int_0^1 q^2dx-\rho_3\tau_q(N+1)\int_0^1\frac{d}{dt}\left[x\theta q\right]dx\\&-\rho_3\beta(N+1)\int_0^1 xq\theta dx+\frac{\rho_3(N+1)}{2}\int_0^1\theta^2dx. \qquad (3.13)\end{aligned}$$

Replacing (3.13) in (3.7), using integration by parts and boundary conditions, we get

$$\begin{aligned}-\frac{b}{2}\int_0^1\psi_x^2dx &= \rho_2(N+1)\int_0^1\frac{d}{dt}\left[x\psi_x\psi_t\right]dx-\frac{\rho_2(N+1)}{2}\psi_t^2(1,t)\\&+\rho_2\frac{N+1}{2}\int_0^1\psi_t^2dx-\frac{N}{2}\rho_2\int_0^1\frac{d}{dt}\left[\psi_t\psi\right]dx\\&-b\frac{(N+1)}{2}\alpha^2(t)h^2(\psi_t(1,t))-b\frac{N}{2}\alpha(t)h(\psi_t(1,t))\psi(1,t)\\&+\kappa(N+1)\int_0^1 x\psi_x(\varphi_x+\psi+lw)dx-\kappa\frac{N}{2}\int_0^1\psi(\varphi_x+\psi+lw)dx\\&-\delta(N+1)\tau_q\int_0^1\frac{d}{dt}\left[xq\psi_x\right]dx-\beta(N+1)\delta\int_0^1 xq\psi_x dx\\&+\frac{\tau_q(N+1)}{2}\int_0^1 q^2dx-\rho_3\tau_q(N+1)\int_0^1\frac{d}{dt}\left[x\theta q\right]dx\\&-\rho_3\beta(N+1)\int_0^1 xq\theta dx+\frac{\rho_3(N+1)}{2}\int_0^1\theta^2dx+\frac{N}{2}\delta\int_0^1\psi_x\theta dx \qquad (3.14)\end{aligned}$$

Finally, multiplying the third equation in (1.8) by $\kappa_0(N+1)x(w_x-l\varphi)-\frac{N}{2}w$ and integrating it over $(0,1)$ with respect to x, we obtain

$$\begin{aligned} 0 \;=\; & \int_0^1 \rho_1\kappa_0(N+1)x(w_x-l\varphi)w_{tt}dx - \frac{N}{2}\rho_1\int_0^1 ww_{tt}dx \\ & -\kappa_0^2\int_0^1 (N+1)x(w_x-l\varphi)(w_x-l\varphi)_x dx \\ & +\kappa_0\frac{N}{2}\int_0^1 (w_x-l\varphi)_x w dx - \frac{N}{2}\kappa l\int_0^1 (\varphi_x+\psi+lw)wdx \\ & +\kappa_0\kappa l\int_0^1 (N+1)x(w_x-l\varphi)(\varphi_x+\psi+lw)dx \end{aligned} \tag{3.15}$$

Integrating by parts and using boundary conditions, we obtain

$$\begin{aligned} -\frac{\kappa_0^2}{2}\int_0^1 (w_x-l\varphi)^2 dx \;=\; & \rho_1\kappa_0(N+1)\int_0^1 \frac{d}{dt}\left[x(w_x-l\varphi)w_t\right]dx \\ & -\rho_1\kappa_0(N+1)\int_0^1 x(w_{xt}-l\varphi_t)w_t dx \\ & -\frac{N}{2}\rho_1\int_0^1 \frac{d}{dt}\left[w_t w\right]dx - \frac{\kappa_0^2(N+1)}{2}\alpha^2(t)h^2(w_t(1,t)) \\ & -\frac{\kappa_0 N}{2}\alpha(t)h(w_t(1,t))w(1,t) - \frac{\kappa_0 N}{2}\int_0^1 (w_x-l\varphi)w_x dx \\ & -\frac{N}{2}\kappa l\int_0^1 (\varphi_x+\psi+lw)wdx + \frac{N}{2}\rho_1\int_0^1 w_t^2 dx \\ & +\kappa_0\kappa l\int_0^1 (N+1)x(w_x-l\varphi)(\varphi_x+\psi+lw)dx \end{aligned} \tag{3.16}$$

Combining (3.6), (3.14) and (3.16), we obtain the inequality (3.4). This completes the proof of Lemma3.3. □

Lemma 3.4. *Let $(\varphi,\psi,w,\theta,q)$ be a global solution of (1.8), and $\Gamma:\mathbb{R}^+\to\mathbb{R}^+$ be a concave nondecreasing C^2-class function. Then, for any constant $p\geq 0$ the* *energy function $E(t)$ satisfies the following estimation,* $0 < T < S$

$$\begin{aligned} \int_T^S \Gamma'(t) E^{p+1} dt \leq\ & C E^{p+1}(T) + C \int_T^S \Gamma' E^p \int_0^1 q^2 dx dt \\ & + C \int_T^S \Gamma' E^p(t) \left(\varphi_t^2(1,t) + \alpha^2(t) h^2(\varphi_t(1,t)) \right) dt \\ & + C \int_T^S \Gamma' E^p(t) \left(\psi_t^2(1,t) + \alpha^2(t) h^2(\psi_t(1,t)) \right) dt \\ & + C \int_T^S \Gamma' E^p(t) \left(w_t^2(1,t) + \alpha^2(t) h^2(w_t(1,t)) \right) dt. \end{aligned} \tag{3.17}$$

Proof. Using the inequality (3.4), we have

$$\begin{aligned} \int_T^S \Gamma'(t) E^{p+1} dt \leq\ & \frac{N+1}{2} \int_T^S \Gamma' E^p(t) \left(\rho_1 \kappa \varphi_t^2(1,t) + \kappa^2 \alpha^2(t) h^2(\varphi_t(1,t)) \right) dt \\ & + \frac{N+1}{2} \int_T^S \Gamma' E^p(t) \left(\rho_2 \psi_t^2(1,t) + b \alpha^2(t) h^2(\psi_t(1,t)) \right) dt \\ & + \frac{N+1}{2} \int_T^S \Gamma' E^p(t) \left(\rho_1 \kappa_0 w_t^2(1,t) + \kappa_0^2 \alpha^2(t) h^2(w_t(1,t)) dt \right) dt \\ & + I_1 + I_2 + I_3 + I_4 + I_5 + I_6 \end{aligned} \tag{3.18}$$

where

$$I_1 = \int_T^S \Gamma'(t) E^p(t) \frac{N}{2} \int_0^1 \frac{d}{dt} \left[\rho_1 \varphi_t \varphi + \kappa \psi_t \psi + \rho_1 w_t w \right] dx dt$$

$$I_2 = \int_T^S \Gamma' E^p(t) (N+1)$$

$$\int_0^1 \frac{d}{dt} \left[\rho_1 \kappa x (\varphi_x + \psi + l w) \varphi_t + \rho_2 x \psi_x \psi_t + \rho_1 \kappa_0 x (w_x - l \varphi) w_t + \rho_3 \tau_q x \theta q \right] dx$$

$$I_3 = \int_T^S \Gamma'(t) E^p(t) \frac{N+1}{2} \rho_1 \int_0^1 \left(c_1 \varphi_t^2 + \psi_t^2 + c_2 w_t^2 + c_5 \theta^2 \right) dx.$$

$$I_4 = \int_T^S \Gamma'(t)E^p(t)\left(\frac{N}{2}\kappa h(\varphi_t(1,t))\varphi(1,t)\right)dt$$

$$I_5 = \int_T^S \Gamma'(t)E^p(t)\left(\frac{N}{2}\kappa h(\psi_t(1,t))\psi(1,t)\right)dt$$

$$I_6 = \int_T^S \Gamma'(t)E^p(t)\left(\frac{N}{2}\kappa h(w_t(1,t))w(1,t)\right)dt$$

to estimate the terms $I_1 - I_6$ as follows

$$\begin{aligned} I_1 &= \int_T^S \left[\Gamma' E^p(t)\left(\frac{N}{2}(\rho_1\varphi_t\varphi + \kappa\psi_t\psi + \rho_1 w w_t)\right)dx\right]_T^S \\ &\quad + \int_T^S (\Gamma'' E^p + p\Gamma' E^{p-1}E')\int_0^1 \frac{N}{2}(\rho_1\varphi_t\varphi + \kappa\psi_t\psi + \rho_1 w w_t)dxdt \\ &\leq C\left[\Gamma' E^{p+1}\right]_S^T + CE^{p+1}(T)\left|\int_T^S \Gamma''(t)dt\right| + C\Gamma'(t)\left|\int_T^S E^p E' dt\right| \\ &\leq C\Gamma'(T)E^{p+1}(T). \end{aligned} \tag{3.19}$$

We have by similar procedure

$$I_2 \leq C\Gamma'(T)E^{p+1}(T) \tag{3.20}$$

Also we estimate I_3 by

$$I_3 \leq C\int_T^S \Gamma'(t)E^{p+1}dt. \tag{3.21}$$

By boundary conditions we have

$$\varphi^2(1,t) \leq \int_0^1 \varphi_x^2 dx, \quad \psi^2(1,t) \leq \int_0^1 \psi_x^2 dx, \quad w^2(1,t) \leq \int_0^1 w_x^2 dx \tag{3.22}$$

Then

$$\begin{aligned} I_4 + I_5 + I_6 &\leq \varepsilon \int_T^S \Gamma'(t)E^{p+1}(t)dt \\ &\quad + C\int_T^S \Gamma' E^p(t)\left(h^2(\varphi_t(1,t)) + h^2(\psi_t(1,t)) + h^2(w_t(1,t))\right) \end{aligned} \tag{3.23}$$

By inserting (3.19),(3.20), (3.21) and (3.23) in estimation (3.18), we obtain the estimation (3.4). □

We are now ready to state and prove our main result.

Theorem 3.1. *Assume that* $(A1)-(A2)$ *hold. Then, there exist a constant* $c>0$ *such that, for* t *large, the solutions of* (1.8),(1.10)-(1.12) *satisfy*

$$E(t) \leq c\left[H_0^{-1}\left(\frac{1}{\int_0^t \alpha(s)ds}\right)\right]^2, t \geq t_0 \tag{3.24}$$

where $H_0(t)=th_0(t)$. *Moreover, if* h_0 *is strictly convex on* $[0,M]$ *and* $h_0^{'}(0)=0$, *then we have the improved estimate*

$$E(t) \leq c\left[h_0^{-1}\left(\frac{1}{\int_0^t \alpha(s)ds}\right)\right]^2, t \geq t_0 \tag{3.25}$$

Proof. Let us define the following function

$$\gamma(t)=1+\int_1^t \frac{1}{h_0(\frac{1}{s})}ds \quad t \geq t^{'} \tag{3.26}$$

for some $t^{'}>\max\{1,\frac{1}{M}\}$. Then

$$\gamma^{'}(t)=\frac{1}{h_0(\frac{1}{t})} \quad \forall t \geq t^{'} \tag{3.27}$$

It follows from assumption (A2) that $\gamma^{'}(t)$ is strictly increasing and $\gamma^{'}(t)\to\infty$ as $t\to\infty$. Thus γ is a convex and strictly increasing C^2-function, with $\gamma(t)\to\infty$ as $t\to\infty$.

Now we set

$$\sigma(t)=\gamma^{-1}(\int_0^t \alpha(s)ds). \tag{3.28}$$

then it is easy to cheak that σ is strictly increasing concave C^2-function, with $\sigma(t)\to\infty$ as $t\to\infty$ and

$$\sigma^{'}(t)=h_0(\frac{1}{\sigma(t)})\alpha(t)$$

is strictly decreasing.

Now, we take $\Gamma(t) = \sigma(t)$ in Lemma 3.4, we obtain

$$\begin{aligned}\int_T^S \sigma'(t)E^{p+1}dt \quad \leq \quad & CE^{p+1}(T) + C\int_T^S \sigma' E^p(t)\int_0^1 q^2 dxdt + C\int_T^S \sigma' E^p(t)\varphi_t^2(1,t)dt \\ & + C\int_T^S \sigma' E^p(t)\left(h^2(\varphi_t(1,t)) + h^2(\psi_t(1,t)) + h^2(w_t(1,t))\right)dt \\ & + C\int_T^S \sigma' E^p(t)\left(\psi_t^2(1,t) + w_t^2(1,t)\right)dt \qquad (3.29)\end{aligned}$$

Estimation for $\int_T^S \sigma'(t)E(t)\varphi_t^2(1,t)$ By considering the following cases:

$$\begin{aligned}& C1 : |\varphi_t(1,t)| > M, \\ & C2 : |\varphi_t(1,t)| \leq M, \quad and \quad |\varphi_t(1,t)| \leq \frac{1}{\sigma(t)}, \\ & C3 : |\varphi_t(1,t)| \leq M, \quad and \quad |\varphi_t(1,t)| > \frac{1}{\sigma(t)}\end{aligned}$$

According to hypothesis (A2), we obtain for $t \geq t'$ in case C1:

$$\sigma'(t)\varphi_t^2(1,t) \leq \sigma'(t')\frac{1}{c_1}h(\varphi_t(1,t))\varphi_t(1,t) \leq -CE'(t). \qquad (3.30)$$

in case C2

$$\sigma'(t)\varphi_t^2(1,t) \leq \sigma'(t)\frac{1}{\sigma^2(t)}. \qquad (3.31)$$

in case C3: we use the definition of $\sigma(t)$, we obtain

$$\begin{aligned}\sigma'(t)\varphi_t^2(1,t) \quad \leq \quad & M\sigma'(t)\varphi_t(1,t) = Mh_0(\frac{1}{\sigma(t)})\alpha(t)\varphi_t(1,t) \\ \leq \quad & Mh_0(|\varphi_t(1,t)|)\alpha(t)|\varphi_t(1,t)| \leq M\alpha(t)h(\varphi_t(1,t))\varphi_t(1,t) \\ \leq \quad & -CE'(t). \qquad (3.32)\end{aligned}$$

By using the estimations (3.30), (3.31) and (3.32), we arrive at

$$\begin{aligned}\int_T^S \sigma'(t)E(t)\varphi_t^2(1,t)dt &\leq -C\int_T^S E(t)E'(t)dt + \int_T^S \sigma'(t)E(t)\frac{1}{\sigma(t)^2}dt \\ &\leq CE^2(T) + E(T)\int_{\sigma(T)}^{\sigma(S)} \frac{1}{s^2}ds \\ &= CE^2(T) + E(T)\left(\frac{1}{\sigma(T)} - \frac{1}{\sigma(S)}\right). \end{aligned} \tag{3.33}$$

Similarly, we estimate

$$\int_T^S \sigma'(t)E(t)\psi_t^2(1,t) \leq CE^2(T) + E(T)\left(\frac{1}{\sigma(T)} - \frac{1}{\sigma(S)}\right) \tag{3.34}$$

$$\int_T^S \sigma'(t)E(t)w_t^2(1,t) \leq CE^2(T) + E(T)\left(\frac{1}{\sigma(T)} - \frac{1}{\sigma(S)}\right) \tag{3.35}$$

Estimation for $\int_T^S \sigma'(t)E(t)h^2(\varphi_t(1,t))$ By considering the following cases:

$$C'1 : |\varphi_t(1,t)| > M, \tag{3.36}$$

$$C'2 : |\varphi_t(1,t)| \leq M, \quad \textit{and} \quad |\varphi_t(1,t)| \leq \sigma'(t), \tag{3.37}$$

$$C'3 : |\varphi_t(1,t)| \leq M, \quad \textit{and} \quad |\varphi_t(1,t)| > \sigma'(t) \tag{3.38}$$

According to hypothesis (A2), we obtain for $t \geq t_1$ in case $C'1$:

$$\sigma'(t)h^2(\varphi_t(1,t)) \leq \sigma'(t')c_2 h\varphi_t(1,t))\varphi_t(1,t) \leq -CE'(t). \tag{3.39}$$

in case $C'2$: Since $\sigma'(t) = h_0(\frac{1}{\sigma(t)})\alpha(t)$, we get

$$h_0^{-1}(|\varphi_t(1,t)|) \leq h_0^{-1}(\sigma'(t)) = \frac{1}{\sigma(t)}\alpha(t),$$

so

$$\sigma'(t)h^2(\varphi_t(1,t)) \leq \sigma'(t)\alpha^2(t)\frac{1}{\sigma^2(t)} \leq C\sigma'(t)\frac{1}{\sigma^2(t)}. \tag{3.40}$$

in case $C'3$: Since

$$\sigma'(t) < |\varphi_t(1,t)| \quad \textit{and} \quad h(\varphi_t(1,t)) \leq h_0^{-1}(|\varphi_t(1,t)|) \leq h_0^{-1}(M)$$

we obtain

$$\begin{aligned} \sigma^{'}(t)h^2(\varphi_t(1,t)) &\leq |\varphi_t(1,t)|h_0^{-1}(M)h(\varphi_t(1,t)) \\ &\leq h_0^{-1}(M)\varphi_t(1,t)h(\varphi_t(1,t)) \leq -CE^{'}(t). \end{aligned} \tag{3.41}$$

By using the estimations (3.39), (3.40) and (3.41), we arrive at

$$\begin{aligned} \int_T^S \sigma^{'}(t)E(t)h^2(\varphi_t(1,t))dt &\leq -C\int_T^S E(t)E^{'}(t)dt + C\int_T^S \sigma^{'}(t)E(t)\frac{1}{\sigma(t)^2}dt \\ &\leq CE^2(T) + CE(T)\int_{\sigma(T)}^{\sigma(S)} \frac{1}{s^2}ds \\ &= CE^2(T) + CE(T)\left(\frac{1}{\sigma(T)} - \frac{1}{\sigma(S)}\right). \end{aligned} \tag{3.42}$$

Similarly, we estimate

$$\int_T^S \sigma^{'}(t)E(t)h^2(\psi_t(1,t)) \leq CE^2(T) + CE(T)\left(\frac{1}{\sigma(T)} - \frac{1}{\sigma(S)}\right) \tag{3.43}$$

$$\int_T^S \sigma^{'}(t)E(t)h^2(w_t(1,t)) \leq CE^2(T) + CE(T)\left(\frac{1}{\sigma(T)} - \frac{1}{\sigma(S)}\right) \tag{3.44}$$

Finally, inserting (3.33), (3.34), (3.35), (3.42), (3.43) and (3.44) into (3.29) as $S \to +\infty$, we find that

$$\int_T^{+\infty} \sigma^{'}(t)E^2(t) \leq CE^2(T) + \frac{C}{\sigma(T)}E(T) \tag{3.45}$$

Hence, we deduce from Lemma 3.2 that

$$E(t) \leq k\sigma(t)^{-2}, \quad t \geq t_1. \tag{3.46}$$

In order to obtain (3.24), we take $s_0 > t^{'}$ such that $h_0(\frac{1}{s_0}) \leq 1$. Since h_0 is increasing and $H_0 = sh_0(s)$ then we have

$$\begin{aligned} \gamma(s) &= 1 + \int_1^s \frac{1}{h_0(\frac{1}{\tau})}d\tau \leq 1 + (s-1)\frac{1}{h_0(\frac{1}{s})} \\ &\leq \frac{s}{h_0(\frac{1}{s})} = \frac{1}{H_0(\frac{1}{s})}, \quad s \geq s_0. \end{aligned} \tag{3.47}$$

So, with $t = \frac{1}{H_0(\frac{1}{s})}$, we easily see that

$$\frac{1}{\sigma(t)} \leq H_0^{-1}(\frac{1}{t}), \quad \forall t \geq t^{'}. \tag{3.48}$$

Therefore, using (3.46) and (3.48) ,we obtain estimate (3.24). To prove (3.25), we define a new function K as follows:

$$K(t) = \frac{h_0(t)}{t}, \quad t \geq t^{'}. \tag{3.49}$$

According to the strict convexity of h_0 on $[0, M]$ and the mean value theorem, we easily deduce that $K(t)$ is strictly increasing on $(0, M)$. Now, we take $\sigma = \gamma^{-1}$, where

$$\gamma(t) = 1 + \int_1^t \frac{1}{K(\frac{1}{s})} ds \quad t \geq t^{'}. \tag{3.50}$$

Then by the same procedure, one derives (3.25). This compeltes the proof of Theorem 3.1. □

As in [14] we give some examples to illustrate the energy decay rates obtained by our results.

Example1. **Exponential growth**

If $h_0(t) = e^{\frac{-1}{t}}$ near zero. Then by according to Theorem 3.1, we obtain the energy decay estimate

$$E(t) \leq k(ln(\int_0^t \alpha(s)ds))^{-2}, \quad t \geq t_0$$

Example 2. **Between polynomial and exponential growth**

If $h_0(t) = e^{(-lnt)^2}$ near zero. Then by according to Theorem 3.1, we obtain the energy decay estimate

$$E(t) \leq ke^{-2(ln(\int_0^t \alpha(s)ds)))^{\frac{1}{2}}} \tag{3.51}$$

Example 3. **Faster than exponential growth:**

If $h_0(t) = e^{-e^{\frac{1}{t}}}$ near zero. Then by according to Theorem 3.1, we obtain the energy decay estimate

$$E(t) \leq kln(ln(\int_0^t \alpha(s)ds))^{-2}, \quad t \geq t_0$$

4. Special cases

In this section, we consider the situation of polynomial growth rate near zero $h_0(t) = t^r$, $r \geq 1$. In other words, instead of (A2) we will give the following condition:

(A3) $h : \mathbb{R} \to \mathbb{R}$ are differentiable functions satisfying

$$c_1 \min\{|t|, |t|^r\} \leq |h(t)| \leq c_2 \min\{|t|, |t|^{\frac{1}{r}}\} \tag{4.1}$$

where the constants $c_1, c_2 > 0$ and $r \geq 1$.

According to Theorem 3.1, we obtain the energy decay estimate

$$E(t) \leq C \left(\int_0^t \alpha(s)ds \right)^{-\frac{2}{r}} \tag{4.2}$$

However, this estimate can be improved as follows:

Theorem 4.1. *Assume that (A1) and (A3) hold. Then there exist two constants $k_1, k_2 > 0$ such that for all $t \geq 0$,*

$$\begin{cases} E(t) \leq k_1 \exp\left\{ -k_2 \int_0^t \alpha(s)ds \right\}, & if r = 1 \\ E(t) \leq k_1 \left(\int_0^t \alpha(s)ds \right)^{-\frac{2}{r-1}}, & if r > 1. \end{cases} \tag{4.3}$$

Proof. We take $\Gamma(t) = \int_0^t \alpha(s)ds$ in Lemma 3.4 and furthermore the estimation (3.17) becomes

$$\begin{aligned} \int_T^S \alpha(t)E^{p+1}dt \quad \leq \quad & CE^{p+1}(T) + C\int_T^S \alpha(t)E^p \int_0^1 q^2 dxdt \\ & + C\int_T^S \alpha(t)E^p(t)\left(\varphi_t^2(1,t) + \psi_t^2(1,t) + w_t^2(1,t)\right)dt \\ & + C\int_T^S \alpha(t)E^p(t)\left(h^2(\varphi_t(1,t)) + h^2(\psi_t(1,t)) + h^2(w_t(1,t))\right)dt. \end{aligned} \tag{4.4}$$

We distinguish two cases related to the parameter r to establish the energy decay rate.

Case (I): $r = 1$. We choose $p = 0$. According to the hypothesis (A3), we know that for $t \geq 0$,

$$t^2 \leq \frac{1}{c_1}h(t)t, \quad h(t)^2 \leq c_2 h(t)t. \tag{4.5}$$

Hence, from inequality (4.4) and energy identity, we deduce that

$$\int_T^S \alpha(t)E(t)dt \leq CE(T) + C\int_T^S -E^{'}dt \leq CE(T). \tag{4.6}$$

Then by using Lemma (3.2), we obtain the first estimate in (4.3).

Case (II): $r > 1$. The hypothesis (A3) implies that

$$\begin{cases} t^2 \leq \left(\frac{1}{c_1}h(t)t\right)^{\frac{2}{r+1}}, & if|t| < 1 \\ t^2 \leq \frac{1}{c_1}h(t)t, & if|t| \geq 1. \end{cases} \tag{4.7}$$

It follows from energy identity (3.1),

$$\varphi_t^2(1,t) \leq \left[-\frac{1}{c_1}E^{'}(t)\right]^{\frac{2}{r+1}} - \frac{1}{c_1}E^{'}(t) \tag{4.8}$$

Then using Young's inequality and the fact that $E(t)$ is nonincreasing, we obtain, for any $\varepsilon > 0$

$$\int_T^S \alpha(t)E^p(t)\varphi_t^2(1,t)dt \leq \int_T^S \alpha(t)E^p(t)\left[-\frac{1}{c_1}E^{'}(t)\right]^{\frac{2}{r+1}}dt - \frac{1}{c_1}\int_T^S \alpha(t)E^p(t)E^{'}(t)dt$$

$$\leq \alpha(0)\frac{r+1}{r-1}\varepsilon^{\frac{r+1}{r-1}}\int_T^S E^{p\frac{r+1}{r-1}}(t)dt + \alpha(0)\frac{r+1}{2}\varepsilon^{-\frac{r+1}{2}}\int_T^S -\frac{1}{c_1}E^{'}(t)dt + \frac{E^{p+1}(T)}{c_1(p+1)}$$

$$\leq C\varepsilon^{\frac{r+1}{r-1}}\int_T^S E^{p\frac{r+1}{r-1}}(t)dt + C(\varepsilon^{-\frac{r+1}{2}} + 1)E(T). \tag{4.9}$$

Similarly, we obtain, for any $\varepsilon > 0$

$$\int_T^S \alpha(t)E^p(t)\psi_t^2(1,t)dt \leq C\varepsilon^{\frac{r+1}{r-1}}\int_T^S E^{p\frac{r+1}{r-1}}(t)dt + C(\varepsilon^{-\frac{r+1}{2}} + 1)E(T). \tag{4.10}$$

and

$$\int_T^S \alpha(t)E^p(t)w_t^2(1,t)dt \leq C\varepsilon^{\frac{r+1}{r-1}}\int_T^S E^{p\frac{r+1}{r-1}}(t)dt + C(\varepsilon^{-\frac{r+1}{2}} + 1)E(T). \tag{4.11}$$

Also we have from energy identity (3.1),

$$\int_0^1 q^2dx \leq -cE^{'}(t) \leq -cE^{'}(t) + \left[-cE^{'}(t)\right]^{\frac{2}{r+1}} \tag{4.12}$$

then we obtain

$$\int_T^S \alpha(t)E^p(t)\int_0^1 q^2dxdt \le C\varepsilon^{\frac{r+1}{r-1}}\int_T^S E^{p\frac{r+1}{r-1}}(t)dt + C(\varepsilon^{-\frac{r+1}{2}}+1)E(T). \tag{4.13}$$

On the other hand, the hypothesis (A3) implies that

$$\begin{cases} h^2(t) \le \left(c_2^{2r}h(t)t\right)^{\frac{2}{r+1}}, & if|t|<1 \\ h^2(t) \le c_2h(t)t, & if|t|\ge 1. \end{cases} \tag{4.14}$$

Then, similarly to (4.9), we obtain, for $\varepsilon > 0$

$$\begin{aligned} &\int_T^S \alpha(t)E^p(t)\left(h^2(\varphi_t(1,t)) + h^2(\psi_t(1,t)) + h^2(w_t(1,t))\right)dt \\ &\le C\varepsilon^{\frac{r+1}{r-1}}\int_T^S E^{p\frac{r+1}{r-1}}(t)dt + C(\varepsilon^{-\frac{r+1}{2}}+1)E(T). \end{aligned} \tag{4.15}$$

Finally, inserting (4.9)-(4.15) into (4.4), we get

$$\int_T^S \alpha(t)E^{p+1}(t)dt \le CE^{p+1}(T) + C\varepsilon^{\frac{r+1}{r-1}}\int_T^S E^{p\frac{r+1}{r-1}}(t)dt + C(\varepsilon^{-\frac{r+1}{2}}+1)E(T). \tag{4.16}$$

Now we choose $p = \frac{r-1}{2}$, then $p+1 = p\frac{r+1}{r-1} = \frac{r+1}{2}$. Choosing ε small enough, we find

$$\int_T^S \alpha(t)E^{p+1}(t)dt \le CE^{p+1}(T) + CE(T). \tag{4.17}$$

Finally, by using Lemma (3.2), we obtain the second estimate in (4.3). This completes the proof of Theorem 4.1. □

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