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FIXED POINTS OF ALMOST CONTRACTIVE TYPE MAPPINGS IN PARTIALLY ORDERED B-METRIC SPACES AND APPLICATIONS TO QUADRATIC INTEGRAL EQUATIONS

Mina Dinarvand

Abstract. The purpose of this paper is to present a new class of almost contractive mappings called almost generalized $(\psi, \varphi, \theta)_s$-contractive mappings and to establish some fixed point and common fixed point results for this class of mappings in partially ordered b-metric spaces. Our main results of the paper significantly generalize and improve many well known comparable results in the recent literature. Moreover, some examples and an application to the existence of a solution for a class of nonlinear quadratic integral equations is given here to illustrate the usability of the obtained results.

Keywords: common fixed point; fixed point; almost contractive mapping; partially ordered b-metric space

1. Introduction

Fixed point theory is one of the most powerful and productive tools from the nonlinear analysis and it can be considered the kernel of the nonlinear analysis. Not only is it used on a daily basis in pure and applied mathematics, but it also serves as a bridge between analysis and topology, and provides a very fruitful area of interaction between the two. Although the concept of a fixed point theorem may appear as an abstract notion in metric spaces, it has remarkable influence on applications such as the theory of differential and integral equations [38], the game theory relevant to the military, sports and medicine as well as economics [9]. Besides, it has applications in physics, engineering, boundary value problems and variational inequalities (see, e.g., [8, 16]).

The Banach contraction principle [7], which shows that every contractive mapping defined on a complete metric space has a unique fixed point, is one of the famous theorems in classical functional analysis. This theorem supplies a method for solving a variety of applied problems in mathematical sciences and engineering.

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A huge literature on this subject exist and this is a very active area of research at present.

The concept of a weakly contractive mapping was introduced by Alber and Guerre-Delabriere [2] in the setup of Hilbert spaces. Rhoades [33] generalized the Banach contraction principle by considering this class of mappings in the setup of metric spaces and proved that every weakly contractive mapping defined on a complete metric space has a unique fixed point.

Ran and Reurings initiated the studying of fixed point results on partially ordered sets in [32], where they gave many useful results on matrix equations. Recently, many researchers have focused on different contractive conditions in complete metric spaces endowed with a partial order and obtained many fixed point results in such spaces. For more details on fixed point results, their applications, comparison of different contractive conditions and related results in ordered metric spaces we refer the reader to [13, 17, 18, 27, 28, 29, 31] and the references mentioned therein.

The concept of metric spaces has been generalized in many directions. The notion of a b-metric space was introduced by Bakhtin in [6], and later extensively used by Czerwik in [14, 15]. Since then, several papers have been published on the fixed point theory of various classes of single-valued and multi-valued operators in (ordered) b-metric spaces. For further works in this direction, we refer to [1, 5, 10, 11, 12, 19, 20, 23, 24, 30, 34, 36]. Recently, Hussain and Shah [21] obtained some results on KKM mappings in cone b-metric spaces.

In the present paper, we introduce the notion of an almost generalized $(\psi, \varphi, \theta)_s$-contractive mapping and then derive fixed point and common fixed point theorems for this class of mappings in the setup of partially ordered b-complete b-metric spaces. Our main results generalize and improve many recent fixed point theorems in the literature. We show by examples that the obtained extensions are proper. Moreover, we apply our results to study the existence of a solution to a large class of nonlinear quadratic integral equations.

2. Preliminaries

Khan et al. [25] introduced the concept of an altering distance function as follows.

Definition 2.1. ([25]) A function $\psi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following properties are satisfied:

(1) ψ is continuous and nondecreasing;

(2) $\psi(t) = 0$ if and only if $t = 0$.

So far, many authors have studied fixed point theorems, which are based on altering distance functions. Recently, Harjani and Sadarangani proved some fixed point theorems for weak contraction and generalized contractions in partially ordered metric spaces by using the altering distance function in [17, 18] respectively.

Theorem 2.1. ([17]) *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \to X$ be a nondecreasing mapping such that*

$$d(fx, fy) \le d(x, y) - \psi\big(d(x, y)\big)$$

for all comparable $x, y \in X$, where $\psi : [0, +\infty) \to [0, +\infty)$ is continuous and nondecreasing function such that ψ is positive in $(0, +\infty)$, $\psi(0) = 0$ and $\lim_{t\to+\infty} \psi(t) = +\infty$. Assume that either f is continuous or the space $(X, \preceq, d)$ is regular. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Here, the ordered metric space $(X, \preceq, d)$ is called regular if for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to x \in X$, as $n \to +\infty$, one has $x_n \preceq x$ for all $n \in \mathbb{N}$.

Theorem 2.2. ([18]) *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \to X$ be a nondecreasing mapping such that*

$$\psi\big(d(fx, fy)\big) \le \psi\big(d(x, y)\big) - \phi\big(d(x, y)\big)$$

for all comparable $x, y \in X$, where ψ and ϕ are altering distance functions. Assume that either f is continuous or the space $(X, \preceq, d)$ is regular. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

In 1973, Geraghty [22] proved a fixed point result, generalizing Banach contraction principle. Several authors proved later various results using Geraghty-type conditions. Recently, Amini-Harandi and Emami [4] generalized the result of Geraghty to the framework of partially ordered metric spaces. Before, we introduce the set $\mathcal{F}$ of all functions $\beta : [0, +\infty) \to [0, 1)$ satisfying the following condition:

$$\beta(t_n) \to 1 \quad \text{as} \quad n \to +\infty \quad \text{implies} \quad t_n \to 0 \quad \text{as} \quad n \to +\infty.$$

Theorem 2.3. ([4]) *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \to X$ be an increasing mapping such that there exists $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that there exists $\beta \in \mathcal{F}$ such that*

$$d(Tx, Ty) \le \beta\big(d(x, y)\big)d(x, y)$$

for all comparable $x, y \in X$. Assume that either f is continuous or the space $(X, \preceq, d)$ is regular. Then f has a fixed point in X. Besides, if for each $x, y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point of f is unique.

Before stating and proving our results, we recall some notations, definitions, and examples required in b-metric spaces.

Consistent with [15, 26] and [36], the following definitions and results will be needed in the sequel.

Definition 2.2. ([15]) Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, +\infty)$ is a b-metric if, for all $x, y, z \in X$, the following conditions hold:

(b_1) $d(x, y) = 0$ if and only if $x = y$;

(b_2) $d(x, y) = d(y, x)$;

(b_3) $d(x, z) \leq s\big[d(x, y) + d(y, z)\big]$.

In this case, the pair (X, d) is called a b-metric space.

It should be noted that, the class of b-metric spaces is effectively larger than the class of metric spaces, since every metric is a b-metric with $s = 1$.

The following example shows that in general a b-metric space does not necessarily need to be a metric space. (see also [36, p. 264]).

Example 2.1. Let (X, ρ) be a metric space and $d(x, y) = (\rho(x, y))^p$, where $p > 1$ is a real number. We show that d is a b-metric with $s = 2^{p-1}$.

Obviously, conditions (b_1) and (b_2) of Definition 2.2 are satisfied. If $1 < p < +\infty$, then the convexity of the function $f(x) = x^p$ $(x > 0)$ implies

$$\left(\frac{a+b}{2}\right)^p \leq \frac{1}{2}(a^p + b^p),$$

and hence, $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ holds.

Thus, for each $x, y, z \in X$, we obtain

$$\begin{aligned} d(x, y) &= \Big(\rho(x, y)\Big)^p \leq \Big(\rho(x, z) + \rho(z, y)\Big)^p \\ &\leq 2^{p-1}\Big(\Big(\rho(x, z)\Big)^p + \Big(\rho(z, y)\Big)^p\Big) \\ &= 2^{p-1}\Big(d(x, z) + d(z, y)\Big). \end{aligned}$$

So, condition (b_3) of Definition 2.2 is also satisfied and d is a b-metric.

However, if (X, ρ) is a metric space, then (X, d) is not necessarily a metric space.

For example, if $X = \mathbb{R}$ is the set of real numbers and $\rho(x, y) = |x - y|$ is the usual Euclidean metric, then $d(x, y) = (x - y)^2$ is a b-metric on $\mathbb{R}$ with $s = 2$. But is not a metric on $\mathbb{R}$.

Also, the following example of a b-metric space is given in [24].

Example 2.2. ([24]) Let X be the set of Lebesgue measurable functions on $[0, 1]$ such that

$$\int_0^1 |f(x)|^2 dx < +\infty.$$

Define $D : X \times X \to [0, +\infty)$ by

$$D(f, g) = \int_0^1 |f(x) - g(x)|^2 dx.$$

Since

$$\left(\int_0^1 |f(x) - g(x)|^2 dx \right)^{\frac{1}{2}}$$

is a metric on X, it follows from the previous example that D is a b-metric on X with $s = 2$.

Definition 2.3. Let X be a nonempty set. Then $(X, \preceq, d)$ is called a partially ordered b-metric space if and only if d is a b-metric on a partially ordered set $(X, \preceq)$.

Definition 2.4. ([10]) Let (X, d) be a b-metric space, and let $\{x_n\}$ be a sequence in X. Then one has the following:

(1) The sequence $\{x_n\}$ in X is called b-convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to +\infty$. In this case, we write $\lim_{n\to+\infty} x_n = x$.

(2) The sequence $\{x_n\}$ is said to be b-Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to +\infty$.

(3) The b-metric space (X, d) is called b-complete if every b-Cauchy sequence $\{x_n\}$ in X be b-converges.

Proposition 2.1. ([10, Remark 2.1]) *In a b-metric space (X, d) the following assertions hold:*

(1) *A b-convergent sequence has a unique limit.*

(2) *Each b-convergent sequence is b-Cauchy.*

(3) *In general, a b-metric is not continuous.*

Note that a b-metric is not always a continuous function of its variables (see, e.g., [20, Example 2]), whereas an ordinary metric is.

Since in general a b-metric is not continuous, we need the following simple lemma about the b-convergent sequences in the proof of our main result.

Lemma 2.1. ([1]) *Let (X, d) be a b-metric space with the coefficient $s \geq 1$, and suppose that the sequences $\{x_n\}$ and $\{y_n\}$ are b-convergent to x, y, respectively. Then, we have*

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n\to+\infty} d(x_n, y_n) \leq \limsup_{n\to+\infty} d(x_n, y_n) \leq s^2 d(x, y).$$

In particular, if $x = y$, then $\lim_{n\to+\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{s} d(x, z) \leq \liminf_{n\to+\infty} d(x_n, z) \leq \limsup_{n\to+\infty} d(x_n, z) \leq s d(x, z).$$

3. Main Results

Use of auxiliary functions to generalize the contractive conditions on maps have been a subject of interest in fixed point theory. We start to this section by defining some sets of auxiliary functions which will be used densely in the sequel.

$$\Psi = \Big\{\psi : [0,+\infty) \to [0,+\infty) \;\big|\; \psi \text{ is an altering distance function}\Big\},$$

$$\begin{aligned}\Phi \;&= \Big\{\varphi : [0,+\infty) \to [0,+\infty) \;\big|\; \varphi \text{ is continuous with the condition} \\ &\qquad \varphi(t) < \psi(t) \text{ for all } t > 0, \text{ where } \psi \in \Psi\Big\}\end{aligned}$$

and

$$\Theta = \Big\{\theta : [0,+\infty) \to [0,+\infty) \;\big|\; \theta \text{ is continuous and } \theta(t) = 0 \text{ if and only if } t = 0\Big\}.$$

Remark 3.1. It is worth mentioning that if $\varphi \in \Phi$, then

$$0 \leq \varphi(0) = \lim_{t\to 0} \varphi(t) \leq \lim_{t\to 0} \psi(t) = \psi(0) = 0,$$

that is, $\varphi(0) = 0$.

Definition 3.1. Let $(X, \preceq, d)$ be a partially ordered b-metric space with the coefficient $s \geq 1$. We say that a mapping $f : X \to X$ is an almost generalized $(\psi, \varphi, \theta)_s$-contractive mapping with respect to a mapping $g : X \to X$ if there exist the functions $\psi \in \Psi$, $\varphi \in \Phi$, $\theta \in \Theta$ and a constant $L \geq 0$ such that

$$\psi\big(s^2 d(fx, gy)\big) \leq \varphi\big(M_s(x,y)\big) + L\theta\big(N(x,y)\big) \tag{3.1}$$

for all comparable elements $x, y \in X$, where

$$M_s(x,y) = \max\left\{d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy) + d(y,fx)}{2s}\right\}$$

and

$$N(x,y) = \min\big\{d(x,fx), d(y,fx), d(x,gy)\big\}.$$

Definition 3.2. ([3]) Let $(X, \preceq)$ be a partially ordered set. Then two mappings $f, g : X \to X$ are said to be weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ for all $x \in X$.

Now, we are ready to state and prove our main results.

Theorem 3.1. *Let $(X, \preceq, d)$ be a partially ordered b-complete b-metric space with the coefficient $s \geq 1$, and let $f, g : X \to X$ be two weakly increasing mappings with respect to $\preceq$. Suppose that*

(a) *f is an almost generalized $(\psi, \varphi, \theta)_s$-contractive mapping with respect to g;*

(b) *f or g is continuous.*

Then f and g have a common fixed point.

Proof. We prove that u is a fixed point of f if and only if u is a fixed point of g. Suppose that u is a fixed point of f, that is, $fu = u$. As $u \preceq u$, by applying 3.1, we have

$$\begin{aligned} \psi\big(d(u, gu)\big) &\leq \psi\big(s^2 d(u, gu)\big) \\ &= \psi\big(s^2 d(fu, gu)\big) \\ &\leq \varphi\left(\max\left\{d(u,u), d(u,fu), d(u,gu), \frac{1}{2s}\Big(d(u,gu)+d(u,fu)\Big)\right\}\right) \\ &\quad +L\theta\big(\min\left\{d(u,fu), d(u,gu)\right\}\big) \\ &= \varphi\big(d(u,gu)\big), \end{aligned}$$

since ψ is nondecreasing. By using the condition $\psi(t) > \varphi(t)$ for $t > 0$, we obtain $d(u, gu) = 0$. Therefore, $gu = u$. Similarly, we can show that if u is a fixed point of g, then u is a fixed point of f.

Let $x_0 \in X$. We construct a sequence $\{x_n\}$ in X such that $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all nonnegative integers. As f and g are weakly increasing with respect to $\preceq$, we have

$$\begin{aligned} x_1 &= fx_0 \preceq gfx_0 = x_2 = gx_1 \preceq fgx_1 = x_3 \preceq \cdots \\ &\preceq x_{2n+1} = fx_{2n} \preceq gfx_{2n} = x_{2n+2} \preceq \cdots . \end{aligned}$$

If $x_{2n} = x_{2n+1}$ for some $n \in \mathbb{N}$, then $x_{2n} = fx_{2n}$. Thus, x_{2n} is a fixed point of f. By the first part, we conclude that x_{2n} is also a fixed point of g.

If $x_{2n+1} = x_{2n+2}$ for some $n \in \mathbb{N}$, then $x_{2n+1} = gx_{2n+1}$. Thus, x_{2n+1} is a fixed point of g. By the first part, we conclude that x_{2n+1} is also a fixed point of f. Therefore, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Now, we complete the proof in the following steps.

Step 1. We will show that

$$\lim_{n \to +\infty} d(x_n, x_{n+1}) = 0. \tag{3.2}$$

Since x_{2n} and x_{2n+1} are comparable, by applying the inequality 3.1, we have

$$\begin{aligned} \psi\big(d(x_{2n+1}, x_{2n+2})\big) &\leq \psi\big(s^2 d(x_{2n+1}, x_{2n+2})\big) \\ &= \psi\big(s^2 d(fx_{2n}, gx_{2n+1})\big) \\ &\leq \varphi\big(M_s(x_{2n}, x_{2n+1})\big) + L\theta\big(N(x_{2n}, x_{2n+1})\big), \end{aligned} \tag{3.3}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned} M_s(x_{2n}, x_{2n+1}) &= \max\Big\{ d(x_{2n}, x_{2n+1}), d(x_{2n}, fx_{2n}), d(x_{2n+1}, gx_{2n+1}), \\ &\quad \frac{d(x_{2n}, gx_{2n+1}) + d(x_{2n+1}, fx_{2n})}{2s} \Big\} \\ &= \max\Big\{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{d(x_{2n}, x_{2n+2})}{2s} \Big\} \\ &\leq \max\Big\{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \\ &\quad \frac{s\big[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})\big]}{2s} \Big\} \\ &= \max\big\{ d(x_{2n}, x_{2n+1}), d(x_{2n+1},_{2n+2}) \big\} \qquad (3.4) \end{aligned}$$

and

$$\begin{aligned} N(x_{2n}, x_{2n+1}) &= \min\big\{ d(x_{2n}, fx_{2n}), d(x_{2n+1}, fx_{2n}), d(x_{2n}, gx_{2n+1}) \big\} \\ &= \min\big\{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+1}), d(x_{2n}, x_{2n+2}) \big\} \\ &= 0. \qquad (3.5) \end{aligned}$$

By applying the inequality 3.3 and using 3.4 and 3.5, we have

$$\psi\big(d(x_{2n+1}, x_{2n+2})\big) \leq \varphi\big(\max\big\{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \big\}\big). \tag{3.6}$$

Now, if for some $n \in \mathbb{N}$,

$$\max\big\{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \big\} = d(x_{2n+1}, x_{2n+2}),$$

then by using 3.6 and the properties of the function φ, we get

$$\psi\big(d(x_{2n+1}, x_{2n+2})\big) \leq \varphi\big(d(x_{2n+1}, x_{2n+2})\big) < \psi\big(d(x_{2n+1}, x_{2n+2})\big),$$

which gives a contradiction. Thus,

$$\max\big\{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \big\} = d(x_{2n}, x_{2n+1}).$$

Hence, the inequality 3.6 yields that

$$\psi\big(d(x_{2n+1}, x_{2n+2})\big) \leq \varphi\big(d(x_{2n}, x_{2n+1})\big) < \psi\big(d(x_{2n}, x_{2n+1})\big). \tag{3.7}$$

Similarly, we can show that

$$\psi\big(d(x_{2n+1}, x_{2n})\big) \leq \varphi\big(d(x_{2n}, x_{2n-1})\big) < \psi\big(d(x_{2n}, x_{2n-1})\big). \tag{3.8}$$

for all $n \in \mathbb{N}$. Since ψ is nondecreasing, it follows by 3.7 and 3.8 that $\{d(x_n, x_{n+1}) : n \in \mathbb{N} \cup \{0\}\}$ is a nonincreasing sequence of nonnegative real numbers which is bounded from below. Then there exists $r \geq 0$ such that

$$\lim_{n \to +\infty} d(x_n, x_{n+1}) = r.$$

Now, we claim that $r = 0$. On the contrary, assume that $r > 0$. Since ψ and φ are continuous, it follows by taking limit as $n \to +\infty$ in 3.7 that

$$\psi(r) \leq \varphi(r).$$

Now, by using the condition $\psi(t) > \varphi(t)$ for $t > 0$, we have $r = 0$, which is a contradiction. Hence, we conclude that $r = 0$.

Step 2. We will prove that $\{x_n\}$ is a b-Cauchy sequence in X. Because of 3.2, it is sufficient to show that the subsequence $\{x_{2n}\}$ is a b-Cauchy sequence. Assume on the contrary that $\{x_{2n}\}$ is not a b-Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{2m_k}\}$ and $\{x_{2n_k}\}$ of $\{x_n\}$ such that n_k is the smallest index for which

$$n_k > m_k > k, \qquad d(x_{2m_k}, x_{2n_k}) \geq \varepsilon. \tag{3.9}$$

This means that

$$d(x_{2m_k}, x_{2n_k-2}) < \varepsilon. \tag{3.10}$$

From 3.9 and 3.10 and by using the triangular inequality, we have

$$\begin{aligned}
\varepsilon &\leq d(x_{2m_k}, x_{2n_k}) \leq sd(x_{2m_k}, x_{2n_k-2}) + sd(x_{2n_k-2}, x_{2n_k}) \\
&\leq sd(x_{2m_k}, x_{2n_k-2}) + s^2 d(x_{2n_k-2}, x_{2n_k-1}) + s^2 d(x_{2n_k-1}, x_{2n_k}) \\
&< s\varepsilon + s^2 d(x_{2n_k-2}, x_{2n_k-1}) + s^2 d(x_{2n_k-1}, x_{2n_k}).
\end{aligned}$$

By taking the upper limit as $k \to +\infty$ and thanks to 3.2, we get

$$\varepsilon \leq \limsup_{k \to +\infty} d(x_{2m_k}, x_{2n_k}) \leq s\varepsilon. \tag{3.11}$$

Further, from

$$d(x_{2m_k}, x_{2n_k-1}) \leq sd(x_{2m_k}, x_{2n_k-2}) + sd(x_{2n_k-2}, x_{2n_k-1})$$

and by using 3.2 and 3.10, we get

$$\limsup_{k \to +\infty} d(x_{2m_k}, x_{2n_k-1}) \leq s\varepsilon.$$

Again, by using the triangular inequality, we have

$$d(x_{2m_k}, x_{2n_k}) \leq sd(x_{2m_k}, x_{2n_k-1}) + sd(x_{2n_k-1}, x_{2n_k}).$$

By taking the upper limit as $k \to +\infty$ in the above inequality and using 3.2 and 3.11, we get

$$\frac{\varepsilon}{s} \leq \limsup_{k \to +\infty} d(x_{2m_k}, x_{2n_k-1}).$$

Thus, we have

$$\frac{\varepsilon}{s} \leq \limsup_{k\to+\infty} d(x_{2m_k}, x_{2n_k-1}) \leq s\varepsilon. \tag{3.12}$$

From 3.10 and by using the triangular inequality again, we have

$$\begin{aligned} d(x_{2m_k+1}, x_{2n_k-1}) &\leq sd(x_{2m_k+1}, x_{2m_k}) + sd(x_{2m_k}, x_{2n_k-1}) \\ &\leq sd(x_{2m_k+1}, x_{2m_k}) + s^2d(x_{2m_k}, x_{2n_k-2}) + s^2d(x_{2n_k-2}, x_{2n_k-1}) \\ &< sd(x_{2m_k+1}, x_{2m_k}) + s^2\varepsilon + s^2d(x_{2n_k-2}, x_{2n_k-1}). \end{aligned}$$

By taking the upper limit as $k \to +\infty$ in the above inequality and using 3.2, we get

$$\limsup_{k\to+\infty} d(x_{2m_k+1}, x_{2n_k-1}) \leq s^2\varepsilon.$$

Again, by using the triangular inequality, we have

$$\begin{aligned} d(x_{2m_k}, x_{2n_k}) &\leq sd(x_{2m_k}, x_{2m_k+1}) + sd(x_{2m_k+1}, x_{2n_k}) \\ &\leq sd(x_{2m_k}, x_{2m_k+1}) + s^2d(x_{2m_k+1}, x_{2n_k-1}) + s^2d(x_{2n_k-1}, x_{2n_k}). \end{aligned}$$

By taking the upper limit as $k \to +\infty$ in the above inequality and using 3.2 and 3.11, we get

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to+\infty} d(x_{2m_k+1}, x_{2n_k-1}).$$

Hence, we have

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to+\infty} d(x_{2m_k+1}, x_{2n_k-1}) \leq s^2\varepsilon. \tag{3.13}$$

Moreover, from

$$d(x_{2m_k}, x_{2n_k}) \leq sd(x_{2m_k}, x_{2m_k+1}) + sd(x_{2m_k+1}, x_{2n_k})$$

and thanks to 3.2 and 3.11, we get

$$\frac{\varepsilon}{s} \leq \limsup_{k\to+\infty} d(x_{2m_k+1}, x_{2n_k}).$$

On the other hand, from

$$d(x_{2m_k+1}, x_{2n_k}) \leq sd(x_{2m_k+1}, x_{2m_k}) + sd(x_{2m_k}, x_{2n_k})$$

and by using 3.2 and 3.11, we obtain

$$\limsup_{k\to+\infty} d(x_{2m_k+1}, x_{2n_k}) \leq s^2\varepsilon.$$

Therefore, we have

$$\frac{\varepsilon}{s} \leq \limsup_{k\to+\infty} d(x_{2m_k+1}, x_{2n_k}) \leq s^2\varepsilon. \tag{3.14}$$

Since x_{2m_k} and x_{2n_k-1} are comparable, by applying the inequality 3.1, we have

$$\begin{aligned} \psi\big(s^2 d(x_{2m_k+1}, x_{2n_k})\big) &= \psi\big(s^2 d(fx_{2m_k}, gx_{2n_k-1})\big) \\ &\leq \varphi\big(M_s(x_{2m_k}, x_{2n_k-1})\big) \\ &\quad +L\theta\big(N(x_{2m_k}, x_{2n_k-1})\big), \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} M_s(x_{2m_k}, x_{2n_k-1}) &= \max\Big\{d(x_{2m_k}, x_{2n_k-1}), d(x_{2m_k}, fx_{2m_k}), d(x_{2n_k-1}, gx_{2n_k-1}), \\ &\qquad \frac{d(x_{2m_k}, g_{2n_k-1}) + d(fx_{2m_k}, x_{2n_k-1})}{2s}\Big\} \\ &= \max\Big\{d(x_{2m_k}, x_{2n_k-1}), d(x_{2m_k}, x_{2m_k+1}), d(x_{2n_k-1}, x_{2n_k}), \\ &\qquad \frac{d(x_{2m_k}, x_{2n_k}) + d(x_{2m_k+1}, x_{2n_k-1})}{2s}\Big\} \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} N(x_{2m_k}, x_{2n_k-1}) &= \min\big\{d(x_{2m_k}, fx_{2m_k}), d(x_{2n_k-1}, fx_{2n_k-1}), d(x_{2m_k}, gx_{2n_k-1})\big\} \\ &= \min\big\{d(x_{2m_k}, x_{2m_k+1}), d(x_{2n_k-1}, x_{2n_k}), d(x_{2m_k}, x_{2n_k})\big\}. \end{aligned} \tag{3.17}$$

By taking the upper limit as $k \to +\infty$ in 3.16 and using 3.2, 3.11, 3.12 and 3.13, we get

$$\begin{aligned} \frac{\varepsilon}{2s} + \frac{\varepsilon}{2s^3} &= \min\left\{\frac{\varepsilon}{s}, \frac{\varepsilon + \frac{\varepsilon}{s^2}}{2s}\right\} \\ &\leq \limsup_{k\to+\infty} M_s(x_{2m_k}, x_{2n_k-1}) \\ &= \max\Big\{\limsup_{k\to+\infty} d(x_{2m_k}, x_{2n_k-1}), 0, 0, \\ &\qquad \frac{\limsup_{k\to+\infty} d(x_{2m_k}, x_{2n_k}) + \limsup_{k\to+\infty} d(x_{2m_k+1}, x_{2n_k-1})}{2s}\Big\} \\ &\leq \max\left\{s\varepsilon, \frac{s\varepsilon + s^2\varepsilon}{2s}\right\} = s\varepsilon. \end{aligned}$$

So, we have

$$\frac{\varepsilon}{2s} + \frac{\varepsilon}{2s^3} \leq \limsup_{k\to+\infty} M_s(x_{2m_k}, x_{2n_k-1}) \leq s\varepsilon, \tag{3.18}$$

and from 3.17,

$$\limsup_{k\to+\infty} N(x_{2m_k}, x_{2n_k-1}) = 0. \tag{3.19}$$

Now, taking the upper limit as $k \to +\infty$ in 3.15 and using 3.14, 3.18 and 3.19, we obtain

$$\begin{aligned}
\psi(s\varepsilon) &= \psi\left(s^2\frac{\varepsilon}{s}\right) \le \psi\big(s^2 \limsup_{k\to+\infty} d(x_{2m_k+1}, x_{2n_k})\big) \\
&\le \varphi\big(\limsup_{k\to+\infty} M_s(x_{2m_k}, x_{2n_k-1})\big) \\
&\le \varphi(s\varepsilon) \\
&< \psi(s\varepsilon),
\end{aligned}$$

which is a contradiction. So, we deduce that $\{x_n\}$ is a b-Cauchy sequence.

Step 3. Existence of a common fixed point for f and g.

As $\{x_n\}$ is a b-Cauchy sequence in X which is a b-complete b-metric space, there exists $u \in X$ such that $\lim_{n\to+\infty} x_n = u$ and

$$\lim_{n\to+\infty} x_{2n+1} = \lim_{n\to+\infty} fx_{2n} = u.$$

Now, without any loss of generality, we may assume that f is continuous. By using the triangular inequality, we have

$$d(u, fu) \le sd(u, fx_{2n}) + sd(fx_{2n}, fu).$$

Now, by taking the upper limit as $n \to +\infty$ in the above inequality and using the continuity of f, we get

$$d(u, fu) \le s\limsup_{n\to+\infty} d(u, fx_{2n}) + s\limsup_{n\to+\infty} d(fx_{2n}, fu) = 0.$$

Thus, we have $fu = u$. Hence, u is a fixed point of f. By the first part of proof, we conclude that u is also a fixed point of g. □

The assumption of continuity of one of the mappings f or g in Theorem 3.1 can be replaced by another condition, which is often used in similar situations. Namely, we shall use the notion of a regular ordered b-metric space, which is defined analogously to the case of the standard metric (see the paragraph following Theorem 2.1).

Theorem 3.2. *Let $(X, \preceq, d)$ be a partially ordered b-complete b-metric space with the coefficient $s \ge 1$, and let $f, g : X \to X$ be two weakly increasing mappings with respect to $\preceq$. Suppose that*

(a) *f is an almost generalized $(\psi, \varphi, \theta)_s$-contractive mapping with respect to g;*

(b) *the space $(X, \preceq, d)$ is regular.*

Then f and g have a common fixed point in X.

Proof. Repeating the proof of Theorem 3.1, we construct an increasing sequence $\{x_n\}$ in X such that $\lim_{n\to+\infty} x_n = u$ for some $u \in X$. By using the given assumption on X, we have $x_n \preceq u$ for all $n \in \mathbb{N}$. Now, we show that $fu = gu = u$. Putting $x = x_{2n}$ and $y = u$ in 3.1, we obtain

$$\begin{aligned} \psi\big(s^2 d(x_{2n+1}, gu)\big) &= \psi\big(s^2 d(fx_{2n}, gu)\big) \\ &\leq \varphi\big(M_s(x_{2n}, u)\big) + L\theta\big(N(x_{2n}, u)\big), \end{aligned} \tag{3.20}$$

where

$$\begin{aligned} M_s(x_{2n}, u)\big) &= \max\Big\{d(x_{2n}, u), d(x_{2n}, fx_{2n}), d(u, gu), \\ &\qquad \frac{d(x_{2n}, gu) + d(fx_{2n}, u)}{2s}\Big\} \\ &= \max\Big\{d(x_{2n}, u), d(x_{2n}, x_{2n+1}), d(u, gu), \\ &\qquad \frac{d(x_{2n}, gu) + d(x_{2n+1}, u)}{2s}\Big\} \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} N(x_{2n}, u)\big) &= \min\big\{d(x_{2n}, fx_{2n}), d(u, fx_{2n}), d(x_{2n}, gu)\big\} \\ &= \min\big\{d(x_{2n}, x_{2n+1}), d(u, x_{2n+1}), d(x_{2n}, gu)\big\}. \end{aligned} \tag{3.22}$$

Letting $n \to +\infty$ in 3.21 and 3.22 and by using Lemma 2.1, we get

$$\begin{aligned} \frac{d(u, gu)}{2s^2} &= \min\left\{d(u, gu), \frac{\frac{d(u,gu)}{s}}{2s}\right\} \\ &\leq \limsup_{n\to+\infty} M_s(x_{2n}, u) \\ &\leq \max\left\{d(u, gu), \frac{sd(u, gu)}{2s}\right\} = d(u, gu) \end{aligned} \tag{3.23}$$

and

$$N(x_{2n}, u) \to 0. \tag{3.24}$$

Now, taking the upper limit as $n \to +\infty$ in 3.20 and by using Lemma 2.1, 3.23 and 3.24, we obtain

$$\begin{aligned} \psi\big(d(u, gu)\big) &\leq \psi\big(sd(u, gu)\big) = \psi\left(s^2 \frac{1}{s} d(u, gu)\right) \\ &\leq \psi\big(s^2 \limsup_{n\to+\infty} d(x_{2n+1}, gu)\big) \\ &\leq \varphi\big(\limsup_{n\to+\infty} M_s(x_{2n}, u)\big) \\ &\leq \varphi\big(d(u, gu)\big), \end{aligned}$$

since ψ is nondecreasing. By using the condition $\psi(t) > \varphi(t)$ for $t > 0$, we have $d(u, gu) = 0$, that is, $u = gu$. Thus, u is a fixed point of g. On the other hand, similar to the first part of the proof of Theorem 3.1, we can show that $fu = u$. Hence, u is a common fixed point of f and g. $\square$

By putting $f = g$ in Theorems 3.1 and 3.2, we obtain the following result.

Corollary 3.1. *Let $(X, \preceq, d)$ be a partially ordered b-complete b-metric space with the coefficient $s \geq 1$, and let $f : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that*

(a) *there exist the functions $\psi \in \Psi$, $\varphi \in \Phi$, $\theta \in \Theta$ and a constant $L \geq 0$ such that*

$$\psi\big(s^2 d(fx, fy)\big) \leq \varphi\big(M_s(x, y)\big) + L\theta\big(N(x, y)\big) \tag{3.25}$$

for all comparable elements $x, y \in X$, where

$$M_s(x, y) = \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2s}\right\}$$

and

$$N(x, y) = \min\big\{d(x, fx), d(y, fx), d(x, fy)\big\};$$

(b) *f is continuous, or*

(b′) *the space $(X, \preceq, d)$ is regular.*

If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

By taking $\varphi(t) = \psi(t) - \phi(t)$, where $\phi \in \Psi$ in Theorems 3.1 and 3.2, we obtain the following result.

Corollary 3.2. *Let $(X, \preceq, d)$ be a partially ordered b-complete b-metric space with the coefficient $s \geq 1$, and let $f, g : X \to X$ be two weakly increasing mappings with respect to $\preceq$. Suppose that*

(a) *there exist the functions $\psi, \phi \in \Psi$ and a constant $L \geq 0$ such that*

$$\psi\big(s^2 d(fx, gy)\big) \leq \psi\big(M_s(x, y)\big) - \phi\big(M_s(x, y)\big) + L\theta\big(N(x, y)\big) \tag{3.26}$$

for all comparable elements $x, y \in X$, where

$$M_s(x, y) = \max\left\{d(x, y), d(x, fx), d(y, gy), \frac{d(x, gy) + d(y, fx)}{2s}\right\}$$

and

$$N(x, y) = \min\big\{d(x, fx), d(y, fx), d(x, gy)\big\};$$

(b) *f or g is continuous, or*

(b′) *the space $(X, \preceq, d)$ is regular.*

Then f and g have a common fixed point in X.

By putting $\psi(t) = t$ and $\varphi(t) = \beta(t)t$, where $\beta \in \mathcal{F}$, we get the following result.

Corollary 3.3. *Let $(X, \preceq, d)$ be a partially ordered b-complete b-metric space with the coefficient $s \geq 1$, and let $f, g : X \to X$ be two weakly increasing mappings with respect to $\preceq$. Suppose that*

(a) *there exist the function $\beta \in \mathcal{F}$ and a constant $L \geq 0$ such that*

$$s^2 d(fx, gy) \leq \beta\big(M_s(x, y)\big) M_s(x, y) + L\theta\big(N(x, y)\big) \tag{3.27}$$

for all comparable elements $x, y \in X$, where

$$M_s(x, y) = \max\left\{d(x, y), d(x, fx), d(y, gy), \frac{d(x, gy) + d(y, fx)}{2s}\right\}$$

and

$$N(x, y) = \min\big\{d(x, fx), d(y, fx), d(x, gy)\big\};$$

(b) *f or g is continuous, or*

(b′) *the space $(X, \preceq, d)$ is regular.*

Then f and g have a common fixed point in X.

By taking $f = g$ and $L = 0$ in Corollaries 3.2 and 3.3, we get immediately the following results.

Corollary 3.4. *Let $(X, \preceq, d)$ be a partially ordered b-complete b-metric space with the coefficient $s \geq 1$, and let $f : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that*

(a) *there exist the functions $\psi, \phi \in \Psi$ such that*

$$\psi\big(s^2 d(fx, fy)\big) \leq \psi\big(M_s(x, y)\big) - \phi\big(M_s(x, y)\big) \tag{3.28}$$

for all comparable elements $x, y \in X$, where

$$M_s(x, y) = \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2s}\right\};$$

(b) *f is continuous, or*

(b′) *the space* $(X, \preceq, d)$ *is regular.*

If there exists $x_0 \in X$ *such that* $x_0 \preceq fx_0$*, then* f *has a fixed point.*

Corollary 3.5. *Let* $(X, \preceq, d)$ *be a partially ordered* b*-complete* b*-metric space with the coefficient* $s \geq 1$*, and let* $f : X \to X$ *be a nondecreasing mapping with respect to* $\preceq$*. Suppose that*

(a) *there exists the function* $\beta \in \mathcal{F}$ *such that*

$$s^2 d(fx, fy) \leq \beta\big(M_s(x, y)\big) M_s(x, y) \tag{3.29}$$

for all comparable elements $x, y \in X$*, where*

$$M_s(x, y) = \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2s}\right\};$$

(b) f *is continuous, or*

(b′) *the space* $(X, \preceq, d)$ *is regular.*

If there exists $x_0 \in X$ *such that* $x_0 \preceq fx_0$*, then* f *has a fixed point.*

If we take $\psi(t) = t$ in Corollary 3.4, then we get the following result.

Corollary 3.6. *Let* $(X, \preceq, d)$ *be a partially ordered* b*-complete* b*-metric space with the coefficient* $s \geq 1$*, and let* $f : X \to X$ *be a nondecreasing mapping with respect to* $\preceq$*. Suppose that*

(a) *there exists the function* $\phi \in \Psi$ *such that*

$$s^2 d(fx, fy) \leq M_s(x, y) - \phi\big(M_s(x, y)\big) \tag{3.30}$$

for all comparable elements $x, y \in X$*, where*

$$M_s(x, y) = \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2s}\right\}.$$

(b) f *is continuous, or*

(b′) *the space* $(X, \preceq, d)$ *is regular.*

If there exists $x_0 \in X$ *such that* $x_0 \preceq fx_0$*, then* f *has a fixed point.*

Remark 3.2. Recall that a subset W of a partially ordered set X is said to be well ordered if every two elements of W are comparable. Note that (common) fixed points of the given mappings in Theorems 3.1 and 3.2 and Corollary 3.1 need not be unique (see further Example 4.3). However, it is easy to show that they must be unique in the case that the respective sets of (common) fixed points are well ordered.

Remark 3.3. Corollary 3.2 improves the main results (Theorems 5 and 6) of Roshan et al. [34] (note that we have s^2 instead of s^4 in the contractive condition). Also, Corollary 3.4 (with $L = 0$ and s^2 instead of s) corresponds to Theorems 3 and 4 of Roshan et al. [34].

Remark 3.4. Since a b-metric is a metric when $s = 1$, so our results can be viewed as the generalization and extension of corresponding results in [4, 13, 17, 18, 35, 37] and several other comparable results.

4. Some Examples

In this section, we present some examples which illustrate our obtained results.

Example 4.1. Let $X = [0, +\infty)$ be equipped with the b-metric defined by

$$d(x,y) = \begin{cases} [\max\{x,y\}]^2, & x \neq y, \\ 0, & x = y, \end{cases}$$

for all $x, y \in X$. Obviously, (X, d) is a b-complete b-metric space with $s = 2^{2-1} = 2$. Define the partial order "$\preceq$" by

$$x \preceq y \qquad \Longleftrightarrow \qquad x = y \vee \Big(x, y \in [0,1] \wedge x \leq y\Big).$$

Consider the mapping $f : X \to X$ given by

$$fx = \begin{cases} \frac{x}{2\sqrt{1+x}}, & x \in [0,1], \\ \frac{x}{2\sqrt{2}}, & x > 1. \end{cases}$$

It is easy to see that f is continuous and increasing, and $0 \preceq f0$. Take altering distance functions

$$\psi(t) = t, \qquad\qquad \phi(t) = \begin{cases} \frac{t\sqrt{t}}{1+\sqrt{t}}, & t \in [0,1], \\ \frac{t}{2}, & t > 1. \end{cases}$$

In order to check the contractive condition 3.28, without loss of generality, we may take $x, y \in X$ such that $y \preceq x$. Thus, we have the following cases.

Case 1: If $x \in [0,1]$ (and hence also $y \in [0,1]$ and $y \leq x$), then

$$d(fx, fy) = \left[\max\left\{\frac{x}{2\sqrt{1+x}}, \frac{y}{2\sqrt{1+y}}\right\}\right]^2 = \frac{x^2}{4(1+x)}$$

and

$$M_s(x,y) = \max\left\{x^2, x^2, y^2, \frac{x^2 + \max^2\{y, \frac{x}{2\sqrt{1+x}}\}}{2s}\right\} = x^2.$$

Hence, 3.28 reduces to

$$\begin{aligned} \psi\Big(s^2 d(fx, fy)\Big) &= \psi\bigg(4 \cdot \frac{x^2}{4(1+x)}\bigg) = \frac{x^2}{1+x} \leq x^2 - \frac{x^3}{1+x} \\ &= \psi(x^2) - \phi(x^2) = \psi\Big(M_s(x,y)\Big) - \phi\Big(M_s(x,y)\Big). \end{aligned}$$

Case 2: If $x > 1$ (and hence $y = x$), then $d(fx, fy) = 0$ and $M_s(x, y) = x^2$. Hence, 3.28 reduces to

$$\begin{aligned}\psi\Big(s^2 d(fx, fy)\Big) &= \psi(4 \cdot 0) = 0 \leq x^2 - \frac{x^2}{2} \\ &= \psi(x^2) - \phi(x^2) = \psi\Big(M_s(x, y)\Big) - \phi\Big(M_s(x, y)\Big).\end{aligned}$$

Thus, all the hypotheses of Corollary 3.4 are satisfied and hence f has a unique fixed point. In fact, 0 is the unique fixed point of f.

Example 4.2. Let $X = [0, +\infty)$ be endowed with the b-metric defined by

$$d(x, y) = \begin{cases} (x + y)^2, & x \neq y, \\ 0, & x = y, \end{cases}$$

for all $x, y \in X$ and the standard order. Clearly, (X, d) is a b-complete b-metric space with $s = 2^{2-1} = 2$. Define the mapping $f : X \to X$ by

$$fx = \begin{cases} \frac{1}{8}x^2, & x \in [0, 1), \\ \frac{1}{8}x, & x \in [1, 2), \\ \frac{1}{4}, & x \in [2, +\infty). \end{cases}$$

It is easy to see that f is continuous and increasing, and $0 \preceq f0$. Take the function $\beta : [0, +\infty) \to [0, \frac{1}{2})$ given by $\beta(t) = \frac{1}{4}$. In order to check the contractive condition 3.29, without loss of generality, let $x, y \in X$ and, for example, $x \leq y$. Thus, the following cases are possible.

Case 1: If $x, y \in [0, 1)$, then

$$\begin{aligned}4d(fx, fy) &= 4\Big(\frac{1}{8}x^2 + \frac{1}{8}y^2\Big)^2 = \frac{1}{16}\Big(x^2 + y^2\Big)^2 \\ &\leq \frac{1}{4}(x + y)^2 = \frac{1}{4}d(x, y) \\ &\leq \frac{1}{4}M_s(x, y) = \beta\Big(M_s(x, y)\Big)M_s(x, y).\end{aligned}$$

Case 2: If $x, y \in [1, 2)$, then

$$\begin{aligned}4d(fx, fy) &= 4\Big(\frac{1}{8}x + \frac{1}{8}y\Big)^2 = \frac{1}{16}(x + y)^2 \\ &\leq \frac{1}{4}(x + y)^2 = \frac{1}{4}d(x, y) \\ &\leq \frac{1}{4}M_s(x, y) = \beta\Big(M_s(x, y)\Big)M_s(x, y).\end{aligned}$$

Case 3: If $x, y \in [2, +\infty)$, then

$$\begin{aligned}4d(fx, fy) &= 4\Big(\frac{1}{4} + \frac{1}{4}\Big)^2 = 1 = \frac{1}{4}(1 + 1)^2 \\ &\leq \frac{1}{4}(x + y)^2 = \frac{1}{4}d(x, y) \\ &\leq \frac{1}{4}M_s(x, y) = \beta\Big(M_s(x, y)\Big)M_s(x, y).\end{aligned}$$

Case 4: If $x \in [0,1)$ and $y \in [1,2)$, then

$$\begin{aligned} 4d(fx,fy) &= 4\Big(\frac{1}{8}x^2+\frac{1}{8}y\Big)^2 \le 4\Big(\frac{1}{8}x+\frac{1}{8}y\Big)^2 = \frac{1}{16}\Big(x^2+y^2\Big)^2 \\ &\le \frac{1}{4}(x+y)^2 = \frac{1}{4}d(x,y) \\ &\le \frac{1}{4}M_s(x,y) = \beta\Big(M_s(x,y)\Big)M_s(x,y). \end{aligned}$$

Case 5: If $x \in [0,1)$ and $y \in [2,+\infty)$, then

$$\begin{aligned} 4d(fx,fy) &= 4\Big(\frac{1}{8}x^2+\frac{1}{4}\Big)^2 \le 4\Big(\frac{1}{8}x+\frac{1}{8}y\Big)^2 = \frac{1}{16}\Big(x+y\Big)^2 \\ &\le \frac{1}{4}(x+y)^2 = \frac{1}{4}d(x,y) \\ &\le \frac{1}{4}M_s(x,y) = \beta\Big(M_s(x,y)\Big)M_s(x,y). \end{aligned}$$

Case 6: If $x \in [1,2)$ and $y \in [2,+\infty)$, then

$$\begin{aligned} 4d(fx,fy) &= 4\Big(\frac{1}{8}x+\frac{1}{4}\Big)^2 \le 4\Big(\frac{1}{8}x+\frac{1}{8}y\Big)^2 = \frac{1}{16}(x+y)^2 \\ &\le \frac{1}{4}(x+y)^2 = \frac{1}{4}d(x,y) \\ &\le \frac{1}{4}M_s(x,y) = \beta\Big(M_s(x,y)\Big)M_s(x,y). \end{aligned}$$

Therefore, all the hypotheses Corollary 3.5 are satisfied and hence f has a unique fixed point. Indeed, 0 is the unique fixed point of f.

We now present an example showing that there are situations where our results can be used to conclude about the existence of (common) fixed points, while some other known results cannot be applied.

Example 4.3. Let $X = \{0,1,2,3,4\}$ be equipped with the b-metric defined by

$$d(x,y) = \begin{cases} (x+y)^2, & x \neq y, \\ 0, & x = y, \end{cases}$$

for all $x, y \in X$. Obviously, (X,d) is a b-complete b-metric space with $s = \frac{49}{25}$. Define the partial order " $\preceq$ " by

$$\preceq := \Big\{(0,0),(1,1),(1,2),(2,2),(3,3),(4,2),(4,4)\Big\}.$$

Consider self-maps f and g as

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 1 & 2 \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 1 & 1 \end{pmatrix}.$$

It is easy to see that f and g are weakly increasing mappings with respect to $\preceq$, and that f and g are continuous.

Take $\psi, \phi \in \Psi$ given by $\psi(t) = \sqrt{t}$ and $\phi(t) = \frac{t}{300}$. In order to check the contractive condition 3.1, only the case $x = 2$, $y = 4$ is nontrivial (when x and y are comparable and the left-hand side of the contractive condition 3.1 is positive). Then

$$\psi\Big(s^2 d(f2, g4)\Big) = \sqrt{s^2 \cdot 3^2} = \frac{147}{25} = \sqrt{36} - \frac{36}{300} = \psi\Big(M_s(2,4)\Big) - \phi\Big(M_s(2,4)\Big).$$

Thus, all the conditions of Theorem 3.1 are satisfied and hence f and g have a common fixed point. Indeed, 0 and 2 are two common fixed points of f and g. (Note that the ordered set $(\{0,2\}, \preceq)$ is not well ordered.

However, take $x = 1$ and $y = 4$ (which are not comparable). Then

$$\begin{aligned}\psi\Big(s^2 d(f1, g4)\Big) &= \sqrt{s^2 \cdot 3^2} = \frac{147}{25} \\ &> \frac{59}{12} = \sqrt{25} - \frac{25}{300} = \psi\Big(M_s(1,4)\Big) - \phi\Big(M_s(1,4)\Big).\end{aligned}$$

Hence, this result cannot be applied in the context of b-metric spaces without order.

5. An application to integral equations

In this section, we present an application of our results to establish the existence of a solution for a class of nonlinear quadratic integral equations.

Consider the nonlinear quadratic integral equation

$$x(t) = g(t) + \lambda \int_0^1 k(t,s) f(s, x(s))ds, \quad \text{for all t} \in [0,1],\ \lambda \geq 0. \tag{5.1}$$

Let Γ denote the class of those functions $\gamma : [0,+\infty) \to [0,+\infty)$ which satisfy the following conditions:

(a) γ is nondecreasing and $(\gamma(t))^p \leq \gamma(t^p)$ for al $p \geq 1$.

(b) There exists $\phi \in \Psi$ such that $\gamma(t) = t - \phi(t)$ for all $t \in [0,+\infty)$.

For example, $\gamma_1(t) = kt$, where $0 \leq k < 1$ and $\gamma_2(t) = \frac{t}{t+1}$ are in Γ.

Let $X = C([0,1], \mathbb{R})$ be the set of all real continuous functions defined on $[0,1]$ with the standard metric given by

$$\rho(x,y) = \sup_{t\in[0,1]} \big|x(t) - y(t)\big| \qquad \text{for all x, y} \in \text{X}.$$

Now, for $p \geq 1$, we define

$$d(x,y) = \big(\rho(x,y)\big)^p = \Big(\sup_{t\in[0,1]} \big|x(t) - y(t)\big|\Big)^p = \sup_{t\in[0,1]} \big|x(t) - y(t)\big|^p$$

for all $x, y \in X$. Obviously, (X, d) is a b-complete b-metric space with $s = 2^{p-1}$. We endow X with the partial order " $\preceq$ " given by

$$x, y \in X, \quad x \preceq y \quad \Longleftrightarrow \quad x(t) \leq y(t) \quad \text{for all t} \in [0,1].$$

Moreover, as in [29], it is proved that $(X, \preceq, d)$ is regular.

We will analyze the integral equation 5.1 under the following assumptions:

(i) $g : [0,1] \to \mathbb{R}$ is a continuous function.

(ii) $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $f(t,x) \geq 0$ and there exist constant $0 \leq L < 1$ and $\gamma \in \Gamma$ such that for all $x, y \in \mathbb{R}$,

$$\big|f(t,x) - f(t,y)\big| \leq L\gamma\big(|x-y|\big).$$

(iii) $k : [0,1] \times [0,1] \to \mathbb{R}$ is continuous at $t \in [0,1]$ for every $s \in [0,1]$ and measurable at $s \in [0,1]$ for all $t \in [0,1]$ such that $k(t,s) \geq 0$ and $\int_0^1 k(t,s)ds \leq K$.

(iv) $\lambda^p K^p L^p \leq \frac{1}{2^{2p-2}}$.

(v) There exists $x_0 \in X$ such that $x_0(t) \leq g(t) + \lambda \int_0^1 k(t,s) f(s, x_0(s))ds$ for all $t \in [0,1]$.

Now, we have the following result of existence of solutions for nonlinear quadratic integral equations.

Theorem 5.1. *Under the assumptions* (i)-(v)*, the equation 5.1 has a unique solution in $X = C([0,1], \mathbb{R})$.*

Proof. We consider the self-map $T : X \to X$ defined by

$$Tx(t) = g(t) + \lambda \int_0^1 k(t,s) f\big(s, x(s)\big)ds \qquad \text{for all t} \in [0,1].$$

By virtue of our assumptions, T is well defined, that is, if $x \in X$, then $Tx \in X$. Note that $u^* \in X$ is a solution of 5.1 if and only if u^* is a fixed point of T.

Now, let $x, y \in X$ with $x \preceq y$. By applying the condition (ii), we conclude that $0 \leq f(s, y(s)) - f(s, x(s))$ for all $s \in [0,1]$. On the other hand, by definition of T, we have

$$\begin{aligned} Ty(t) - Tx(t) &= g(t) + \lambda \int_0^1 k(t,s) f\big(s, y(s)\big)ds - g(t) - \lambda \int_0^1 k(t,s) f\big(s, x(s)\big)ds \\ &= \lambda \int_0^1 k(t,s)\big[f\big(s, y(s)\big) - f\big(s, x(s)\big)\big]ds \geq 0 \end{aligned}$$

for all $t \in [0,1]$. This implies that $Ty(t) \geq Tx(t)$ for all $t \in [0,1]$ and hence $Tx \preceq Ty$. Therefore, T is a nondecreasing mapping with respect to $\preceq$.

Now, suppose that $x, y \in X$ with $x \preceq y$. Hence, by using condition (ii) and the definition of T, for all $t \in [0,1]$, we have

$$\begin{aligned}
\big|Tx(t) - Ty(t)\big| &= \bigg|g(t) + \lambda \int_0^1 k(t,s) f\big(s, x(s)\big) ds \\
&\quad - g(t) - \lambda \int_0^1 k(t,s) f\big(s, y(s)\big) ds\bigg| \\
&\leq \lambda \int_0^1 k(t,s) \big| f\big(s, x(s)\big) - f\big(s, y(s)\big)\big| ds \\
&\leq \lambda \int_0^1 k(t,s) \lambda\big(|x(s) - y(s)|\big) ds.
\end{aligned}$$

Since the function γ is nondecreasing, it follows that

$$\gamma\big(\big|x(s) - y(s)\big|\big) \leq \gamma\Big(\sup_{t\in[0,1]} \big|x(s) - y(s)\big|\Big) = \gamma\big(\rho(x,y)\big).$$

Hence, we obtain

$$\big|Tx(t) - Ty(t)\big| \leq \lambda K L \gamma\big(\rho(x,y)\big).$$

Then, we have

$$\begin{aligned}
d(Tx, Ty) &= \sup_{t\in[0,1]} \big|Tx(t) - Ty(t)\big|^p \\
&\leq \big\{\lambda K L \gamma\big(\rho(x,y)\big)\big\}^p \\
&\leq \lambda^p K^p L^p \gamma\big(d(x,y)\big) \\
&\leq \lambda^p K^p L^p \gamma\big(M_s(x,y)\big) \\
&\leq \lambda^p K^p L^p \big[M_s(x,y) - \phi\big(M_s(x,y)\big)\big] \\
&\leq \frac{1}{2^{2p-2}} \big[M_s(x,y) - \phi\big(M_s(x,y)\big)\big].
\end{aligned}$$

Now, by applying condition (v) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$. Consequently, all the required hypotheses of Corollary 3.6 are satisfied and hence T has a unique fixed point $u^* \in C([0,1], \mathbb{R})$, that is, u^* is the unique solution to 5.1. This completes the proof. $\square$

Example 5.1. Consider the functional integral equation

$$x(t) = \frac{t^2}{1+t^4} + \frac{1}{27} \int_0^1 \frac{s \cos t}{18(1+t)} \frac{|x(s)|}{1+|x(s)|} ds \qquad \text{for all t} \in [0,1]. \tag{5.2}$$

Observe that this equation is a special case of the equation 5.1 with

$$g(t) = \frac{t^2}{1+t^4}, \qquad k(t,s) = \frac{s}{1+t}$$

and

$$f(t,x)=\frac{\cos t}{18}\frac{|x|}{1+|x|}.$$

Put

$$Tx(t)=\frac{t^2}{1+t^4}+\frac{1}{27}\int_0^1 \frac{s\cos t}{18(1+t)}\frac{|x(s)|}{1+|x(s)|}ds$$

for all $t\in[0,1]$ and for all $x\in C([0,1],\mathbb{R})$. Then

(i) $g:[0,1]\to\mathbb{R}$ is a continuous function.

(ii) $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function and $f(t,x)\geq 0$. Indeed, by using $\gamma(t)=\frac{1}{3}t$, we see that $\gamma\in\Gamma$ and

$$\Big(\gamma(t)\Big)^p=\left(\frac{1}{3}\right)^p=\frac{1}{3^p}t^p\leq\frac{1}{3}t^p=\gamma(t^p)$$

for all $p\geq 1$. Further, for arbitrarily fixed $x,\in\mathbb{R}$ such that $x\geq y$ and for all $t\in[0,1]$, we obtain

$$\begin{aligned}\Big|f(t,x)-f(t,y)\Big| &= \left|\tfrac{\cos t}{18}\tfrac{|x|}{1+|x|}-\tfrac{\cos t}{18}\tfrac{|y|}{1+|y|}\right|\\ &\leq \tfrac{1}{18}|x-y|=\tfrac{1}{6}\gamma\Big(|x-y|\Big).\end{aligned}$$

Thus, $L=\frac{1}{6}$.

(iii) $k:[0,1]\times[0,1]\to\mathbb{R}$ is continuous at $t\in[0,1]$ for every $s\in[0,1]$ and measurable at $s\in[0,1]$ for all $t\in[0,1]$ and $k(t,s)\geq 0$. Moreover, we have

$$\begin{aligned}\int_0^1 k(t,s)ds &= \textstyle\int_0^1 \frac{s}{1+t}ds=\frac{1}{2(1+t)}\\ &\leq \tfrac{1}{2}=K.\end{aligned}$$

(iv) By taking $L=\frac{1}{6}$, $K=\frac{1}{2}$ and $\lambda=\frac{1}{27}$, then inequality $\lambda^pK^pL^p\leq\frac{1}{2^{2p-2}}$ appearing in assumption (iv) has the following form:

$$\frac{1}{27^p}\times\frac{1}{2^p}\times\frac{1}{6^p}\leq\frac{1}{2^{2p-2}}.$$

It is easily seen that each number $p\geq 1$ satisfies the above inequality.

(v) By choosing $x_0(t)=t$ for all $t\in[0,1]$, we have $Tx_0(t)=t$ for all $t\in[0,1]$. Hence, $x_0(t)\leq Tx_0(t)$ for all $t\in[0,1]$. Therefore, $x_0\preceq Tx_0$.

Consequently, all required assumptions of Theorem 5.1 are satisfied. Hence the integral equation 5.2 has a unique solution in $C([0,1],\mathbb{R})$.

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Mina Dinarvand
Faculty of Mathematics
K. N. Toosi University of Technology
P.O. Box 16315-1618
Tehran, Iran
dinarvand_mina@yahoo.com