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A COMMON RANDOM FIXED POINT THEOREM OF RATIONAL INEQUALITY IN POLISH SPACES WITH APPLICATION

Rashwan A. Rashwan and Hasanen A. Hammad

Abstract. In this paper, we prove a new common random fixed point theorem for a pair of random operators satisfying random F-contraction of rational inequality in polish spaces. An application to a system of random nonlinear integral equations is discussed. Finally, we give some examples to verify our results.
Keywords: random fixed point, F-contraction, polish spaces, random nonlinear integral equations.

1. Introduction

Throughout this paper, we will refer to $\mathbb{R}$ by the set of all real numbers, $\mathbb{R}^+$ by the set of all positive real numbers and $\mathbb{N}$ by the set of all natural numbers.

The very famous Banach contraction principle [5] can be stated as follows:

Theorem 1.1. *Let (X, d) be a complete metric space and T be a mapping of X into itself satisfying:*

$$d(Tx, Ty) \leq kd(x, y) \quad \forall x, y \in X, \tag{1.1}$$

where k is a constant in [0,1). Then T has a unique fixed point $x \in X$.

In the literature, there is a great number of generalization of the Banach contraction principle see for instance [4, 9, 15, 23]. One of the extension was due to Wardowski [24].

Wardowski [24] has introduced the concept of an F-contraction as follows:

Definition 1.1. [24] Let $\digamma$ be the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ such that
(F_1) F is strictly increasing i.e., for all $a, b \in \mathbb{R}^+$ such that $a < b$, $F(a) < F(b)$;
(F_2) for every sequence $\{a_n\}_{n\in N}$ of positive numbers $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$;
(F_3) there exists $\lambda \in (0, 1)$ such that $\lim_{a\to 0^+} a^\lambda F(a) = 0$.

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Definition 1.2. [24] Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be an F-contraction on (X,d) if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\forall x, y \in X, d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y)). \tag{1.2}$$

From (F_1) and (1.2) it is easy to conclude that every F-contraction T is a contractive mapping and hence necessarily continuous.

Example 1.1. The following functions $F : \mathbb{R}^+ \to \mathbb{R}$ are the elements of $\mathcal{F}$

$$\begin{array}{ll} (i)\ \ F(\alpha) = \ln(\alpha) & (ii)\ F(\alpha) = \ln(\alpha) + \alpha \\ (iii)\ F(\alpha) = \frac{-1}{\sqrt{\alpha}} & (iv)\ F(\alpha) = \ln(\alpha^2 + \alpha). \end{array}$$

Remark 1.1. Consider $F(t) = \frac{-1}{t^{\frac{1}{p}}}$, $p > 1$, $t > 0$, then $F \in \mathcal{F}$.

Proof. Since $F^{'}(t) = \frac{1}{p.t^{1+\frac{1}{p}}} > 0$ then, F satisfies (F_1) and it is clear that the condition (F_2) hold. Since $p > 1$, $\frac{1}{p} < 1$, we take $\frac{1}{p} < \lambda < 1$ and then, $\lim_{t \to 0^+} t^\lambda F(t) = \lim_{t \to 0^+}(-t^{\lambda - \frac{1}{p}}) = 0$. So F satisfies (F_3). This gives us $F \in \mathcal{F}$. $\square$

Wardowski [24] stated a modified version of Banach contraction principle as follows:

Theorem 1.2. *Let (X,d) be a complete metric space and $T : X \to X$ be an F-contraction. Then T has a unique fixed point $x \in X$.*

Abbas et al. [1] generalized the notion of F -contraction and proved certain fixed point results. Batra et al. [6, 7] extended the concept of F-contraction on graphs and altered distances. Recently, Cosentino and Vetro [10] followed the approach of F-contraction and obtained some fixed point theorems for Hardy-Rogers-type self mappings in complete metric and ordered metric spaces. Recently some authors extended the notion of F -contraction and proved some fixed point theorems under suitable conditions (see [2, 3, 14]).

Probabilistic functional analysis is an important mathematical area of research due to its applications to probabilistic models in applied problems that appear in approximation theory, game and potential theory, theory of integral and differential equations and others. Random operator theory is needed for the study of various classes of random equations. Important contributions to the study of random equations have been presented in [8, 21] among others. The study of random fixed point problems was initiated by the Prague school of probability research. The first results were studied in 1955-1956 by Špaček and Hanš in the context of Fredholm integral equations with random kernel. In a separable metric space, Random fixed point theorems for random contraction mappings were proved by Hanš [11] and Špaček [22].

Bharucha-Reid [8] attracted the attention of several mathematicians and gave wings to this theory. Itoh [12] extended the results of Špaček and Hanš in multi-valued contractive mappings and obtained random fixed point theorems with an application to random differential equations in Banach spaces. Now, it has became a full fledged research area and a vast amount of mathematical activities have been carried out in this direction (see [19, 20]).

Recently, some authors [16, 17, 18] applied a random fixed point theorem to prove the existence of a solution in a separable Banach space of a random nonlinear integral equations.

In this paper we find a new common random fixed point result for a pair of stochastic mappings satisfying F-contraction of rational type in polish spaces. Finally we apply our result to discuss the existence of a unique solution to the random nonlinear integral equations.

2. Preliminary Notes

Let (X, β_X) be a polish space, that is a separable complete metric space, where β_X is a $\sigma-$algebra of Borel subsets of X and let (Ω, β, μ) denote a complete probability measure space with a non-empty set Ω, measure μ and β be a $\sigma-$algebra of subsets of Ω.

Definition 2.1. [13] A measurable mapping $x : \Omega \to X$ is called:

(i) $X-$valued random variable, if the inverse image under the mapping x of every Borel set B of X belongs to β, that is, $x^{-1}(B) \in \beta$ for all $B \in \beta$.

(ii) finitely valued random variable, if it is constant on each of a finite number of disjoint sets $A_i \in \beta$ and is equal to 0 on $\Omega - \left(\bigcup_{i=1}^{n} A_i\right)$.

(iii) simple random variable if it is finitely valued and $\mu\{\omega : \|x(\omega)\| > 0\} < \infty$.

(iv) strong random variable, if there exists a sequence $\{x_n(\omega)\}$ of simple random variables which converges to $x(\omega)$ almost surely, i.e., there exists a set $A_0 \in \beta$ with $\mu(A_0) = 0$ such that $\lim_{n\to\infty} x_n(\omega) = x(\omega)$, $\omega \in \Omega - A_0$.

(v) weak random variable, if the function $x^*(x(\omega))$ is a real valued random variables for each $x^* \in X^*$, the space X^* denoting the dual space of X.

Definition 2.2. [13] Let Y be a Banach space.

(i) A measurable mapping $f : \Omega \times X \to Y$ is said to be a random mapping if $f(\omega, x) = Y(\omega)$ is a $Y-$valued random variable for every $x \in X$.

(ii) A measurable mapping $f : \Omega \times X \to Y$ is said to be a continuous random mapping if the set of all $\omega \in \Omega$ for which $f(\omega, x)$ is a continuous function of x has measure one.

(iii) An equation of the type $f(\omega, x(\omega)) = x(\omega)$, where $f : \Omega \times X \to X$ is a random mapping is called a random fixed point equation.

(iv) Any measurable mapping $x : \Omega \to X$ which satisfies the random fixed point equation $f(\omega, x(\omega)) = x(\omega)$ almost surely is said to be a wide sense solution of the fixed point equation.

(v) Any $X-$valued random variable $x(\omega)$ which satisfies $\mu\{\omega : f(\omega, x(\omega)) = x(\omega)\} = 1$ is said to be a random solution of the fixed point equation or a random fixed point of f.

(vi) A measurable mapping $x : \Omega \to X$ is called:

(a) a random fixed point of a random operator $f : \Omega \times X \to X$ if $f(\omega, x(\omega)) = x(\omega)$ for every $\omega \in \Omega$.

(b) a random coincidence of random operators $T, f : \Omega \times X \to X$ if $T(\omega, x(\omega)) = f(\omega, x(\omega))$ for every $\omega \in \Omega$.

(c) a common random fixed point of random mappings $T, f : \Omega \times X \to X$ if $T(\omega, x(\omega)) = f(\omega, x(\omega)) = x(\omega)$ for every $\omega \in \Omega$.

Example 2.1. [13] Let $X = \mathbb{R}$ and C be a non-measurable subset of X. Consider $f : \Omega \times X \to Y$ is a random mapping defined as $f(\omega, x(\omega)) = x^2(\omega) + x(\omega) - 1$ for all $\omega \in \Omega$. It's clearly that, the real-valued function $x(\omega) = 1$ is a random fixed point of f. However, the real-valued function $y(\omega) = \begin{cases} -1, & \omega \notin C \\ 1, & \omega \in C \end{cases}$ is a wide sense solution of the fixed point equation $f(\omega, x(\omega)) = x(\omega)$ without being a random fixed point of f. Therefore, a random solution is a wide sense solution of the fixed point equation but the converse is not necessarily true.

3. Main Result

We begin with the following definition.

Definition 3.1. Let (X, d) be a polish space and (Ω, β) be a measurable space. The random mappings $T, S : \Omega \times X \to X$ are called a pair of rational type F-contraction if for all $x, y \in X$ and $\omega \in \Omega$, we have

$$\tau + F(d(T(\omega, x), S(\omega, y))) \leq F(N(x(\omega), y(\omega))), \tag{3.1}$$

where $F \in \digamma$, $\tau > 0$ and

$$N(x(\omega), y(\omega)) = \max\left\{ d(x(\omega), y(\omega)), \frac{d(y(\omega), S(\omega, y))[1 + d(x(\omega), T(\omega, x))]}{1 + d(x(\omega), y(\omega))}, \frac{d(x(\omega), T(\omega, x))d(y(\omega), S(\omega, y))}{1 + d(T(\omega, x), S(\omega, y))} \right\}.$$

Now, we shall prove a common random fixed point theorem under generalized contractive condition (3.1).

Theorem 3.1. *Let (X, d) be a polish space and (Ω, β, μ) is a complete probability measure space. Suppose that $T, S : \Omega \times X \to X$ are random mappings such that* *(i) (S,T) is a pair of measurable and continuous mappings for all $x, y \in X$ and $\omega \in \Omega$,*

(ii) (S,T) is a pair of rational type F-contractions.
Then there exists a common random fixed point $p(\omega)$ of S and T.

Proof. Consider the function $\xi_\circ(\omega) : \Omega \to X$ be an arbitrary measurable mapping. We can define a sequence of measurable mappings $\{\xi_n(\omega)\}$ from Ω to X as following:

$$\xi_{2n+1}(\omega) = T(\omega, \xi_{2n}(\omega))$$

and

$$\xi_{2n+2}(\omega) = S(\omega, \xi_{2n+1}(\omega))$$

for all $\omega \in \Omega$ and $n = 0, 1, 2, ...$
Applying the contractive condition (3.1), we get

$$\begin{aligned} &F(d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))) \\ &= F(d(T(\omega, \xi_{2n}(\omega)), S(\omega, \xi_{2n+1}(\omega)))) \leq F(N(\xi_{2n}(\omega), \xi_{2n+1}(\omega))) - \tau, \end{aligned} \tag{3.2}$$

for all $n \in \mathbb{N} \cup \{0\}$, where

$$\begin{aligned} &N(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) \\ &= \max\left\{ \begin{array}{c} d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)), \frac{d(\xi_{2n+1}(\omega), S(\omega, \xi_{2n+1}(\omega)))[1+d(\xi_{2n}(\omega), T(\omega, \xi_{2n}(\omega)))]}{1+d(\xi_{2n}(\omega), \xi_{2n+1}(\omega))}, \\ \frac{d(\xi_{2n}(\omega), T(\omega, \xi_{2n}(\omega)))d(\xi_{2n+1}(\omega), S(\omega, \xi_{2n+1}(\omega)))}{1+d(T(\omega, \xi_{2n}(\omega)), S(\omega, \xi_{2n+1}(\omega)))} \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)), \frac{d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))[1+d(\xi_{2n}(\omega), \xi_{2n+1}(\omega))]}{1+d(\xi_{2n}(\omega), \xi_{2n+1}(\omega))}, \\ \frac{d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)).d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))}{1+d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))} \end{array} \right\} \\ &\leq \max\left\{ d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)), d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)) \right\}. \end{aligned}$$

If

$$\max\left\{ d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)), d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)) \right\} = d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)),$$

then

$$N(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) = d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)).$$

From (3.2), we get

$$F(d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))) \leq F(d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))) - \tau,$$

which is a contradiction due to (F_1), therefore

$$F(d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))) \leq F(d(\xi_{2n}(\omega), \xi_{2n+1}(\omega))) - \tau,$$

for all $n \in \mathbb{N} \cup \{0\}$, $\omega \in \Omega$. Hence

$$F(d(\xi_{n+1}(\omega), \xi_{n+2}(\omega))) \leq F(d(\xi_n(\omega), \xi_{n+1}(\omega))) - \tau. \tag{3.3}$$

By (3.3), we obtain that

$$F(d(\xi_n(\omega),\xi_{n+1}(\omega))) \leq F(d(\xi_{n-2}(\omega),\xi_{n-1}(\omega))) - 2\tau.$$

Repeating these steps, we can write

$$F(d(\xi_n(\omega),\xi_{n+1}(\omega))) \leq F(d(\xi_\circ(\omega),\xi_1(\omega))) - n\tau. \tag{3.4}$$

Taking the limit in (3.4), we obtain $\lim_{n\to\infty} F(d(\xi_n(\omega),\xi_{n+1}(\omega))) = -\infty$. Since $F \in \digamma$,

$$\lim_{n\to\infty} d(\xi_n(\omega),\xi_{n+1}(\omega)) = 0. \tag{3.5}$$

From the axiom (F_3) of F-contraction, there exists $\lambda \in (0,1)$ such that

$$\lim_{n\to\infty} (d(\xi_n(\omega),\xi_{n+1}(\omega)))^\lambda F(d(\xi_n(\omega),\xi_{n+1}(\omega))) = 0. \tag{3.6}$$

By (3.4), for all $n \in \mathbb{N}$, yields

$$\begin{aligned}&(d(\xi_n(\omega),\xi_{n+1}(\omega)))^\lambda.[F(d(\xi_n(\omega),\xi_{n+1}(\omega))) - F(d(\xi_\circ(\omega),\xi_1(\omega)))]\\ &\leq -(d(\xi_n(\omega),\xi_{n+1}(\omega)))^\lambda.n\tau \leq 0.\end{aligned} \tag{3.7}$$

Considering (3.5), (3.6) and taking $n \to \infty$ in (3.7), we observe that

$$\lim_{n\to\infty} (n(d(\xi_n(\omega),\xi_{n+1}(\omega)))^\lambda) = 0. \tag{3.8}$$

By (3.8), there exists $n_1 \in \mathbb{N}$, such that $n(d(\xi_n(\omega),\xi_{n+1}(\omega)))^\lambda \leq 1$ for all $n \geq n_1$, or, for all $n \geq n_1$

$$d(\xi_n(\omega),\xi_{n+1}(\omega)) \leq \frac{1}{n^{\frac{1}{\lambda}}}. \tag{3.9}$$

Using (3.9), for $m > n \geq n_1$, we have

$$\begin{aligned}d(\xi_n(\omega),\xi_m(\omega)) &\leq d(\xi_n(\omega),\xi_{n+1}(\omega)) + d(\xi_{n+1}(\omega),\xi_{n+2}(\omega)) + \ldots + d(\xi_{m-1}(\omega),\xi_m(\omega))\\ &= \sum_{i=1}^{m-1} d(\xi_i(\omega),\xi_{i+1}(\omega)) \leq \sum_{i=1}^{\infty} d(\xi_i(\omega),\xi_{i+1}(\omega)) \leq \sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{\lambda}}}.\end{aligned}$$

The convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{\lambda}}}$ leads to $\lim_{n\to\infty} d(\xi_n(\omega),\xi_m(\omega)) = 0$. Therefore $\{\xi_n(\omega)\}$ is a Cauchy sequence in a polish space (X,d). Since (X,d) is complete, there exists measurable function $p(\omega) : \Omega \to X$ such that $\lim_{n\to\infty} \xi_n(\omega) = p(\omega)$, moreover,

$$\lim_{n\to\infty} \xi_{2n+1}(\omega) = \lim_{n\to\infty} \xi_{2n+2}(\omega) = p(\omega).$$

The continuity of S yields,

$$p(\omega) = \lim_{n\to\infty} \xi_{2n+2}(\omega) = \lim_{n\to\infty} S(\omega,\xi_{2n+1}(\omega)) = S(\omega, \lim_{n\to\infty} \xi_{2n+1}(\omega)) = S(\omega,p(\omega)).$$

Similarly, $p(\omega) = T(\omega, p(\omega))$. Thus we get $p(\omega) = T(\omega, p(\omega)) = S(\omega, p(\omega))$. Hence the pair (S, T) has a common random fixed point.

Now we show that $p(\omega)$ is a unique common fixed point. Assume the contrary, that is there exists $v(\omega) \in \Omega \times X$ such that $v(\omega) = T(\omega, v(\omega)) = S(\omega, v(\omega))$. From the contractive condition (3.1), we obtain that

$$\tau + F(d(T(\omega, p), S(\omega, v))) \leq F(N(p(\omega), v(\omega))), \tag{3.10}$$

where

$$\begin{aligned} &N(p(\omega), v(\omega)) \\ &= \max\left\{ d(p(\omega), v(\omega)), \tfrac{d(v(\omega),S(\omega,v))[1+d(p(\omega),T(\omega,p))]}{1+d(p(\omega),v(\omega))}, \tfrac{d(p(\omega),T(\omega,p))d(v(\omega),S(\omega,v))}{1+d(T(\omega,p),S(\omega,v))} \right\} \\ &= \max\{d(p(\omega), v(\omega)), 0, 0\} = d(p(\omega), v(\omega)). \end{aligned}$$

Using (3.10), we get

$$\tau + F(d(p(\omega), v(\omega))) \leq F(d(p(\omega), v(\omega))),$$

which implies that $d(p(\omega), v(\omega)) < d(p(\omega), v(\omega))$, which is a contradiction. Hence $p(\omega) = v(\omega)$. This complete the proof. $\square$

The following example justify Theorem 3.1.

Example 3.1. Let (Ω, β) be a measurable space and $C = \{6, 7, 8, 9\} \subset \mathbb{R}$ with the usual metric $d(x(\omega), y(\omega)) = |x(\omega) - y(\omega)|$. Consider $\Omega = C$ and β be the sigma algebra of Lebesgue's measurable subset of Ω. Define random mappings $T, S : \Omega \times C \to C$ for all $\omega \in \Omega$ by

$$T(\omega, x) = \begin{cases} 9 & \text{if } x = 6 \\ \\ 7 & \text{otherwise} \end{cases} \quad \text{and} \quad S(\omega, x) = \begin{cases} 8 & \text{if } x = 6 \\ \\ 7 & \text{otherwise} \end{cases}.$$

Define the function $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(t) = \ln(t)$ for all $t > 0$. Then the contractive condition (3.1) is satisfied. Indeed for all $x, y \in C$ and $\omega \in \Omega$, the following inequality

$$\tau + \ln(d(T(\omega, x), S(\omega, y))) \leq \ln(N(x(\omega), y(\omega))),$$

holds for all $\tau > 0$. Particularly, for $x(\omega) = 6$ and $y(\omega) = 9$, one can write

$$\begin{aligned} N(6,9) &= \max\left\{d(6,9), \frac{d(9, S(\omega, 9)).[1 + d(6, T(\omega, 6))]}{1 + d(6, 9)}, \frac{d(6, T(\omega, 6)).d(9, S(\omega, 9))}{1 + d(T(\omega, 6), S(\omega, 9))}\right\} \\ &= \max\{3, 2, 2\} = 3, \end{aligned}$$

and

$$d(T(\omega, x), S(\omega, y)) = d(T(\omega, 6), S(\omega, 9)) = 2.$$

Thus

$$\tau + \ln(d(T(\omega, 6), S(\omega, 9))) = \tau + \ln(2) \leq \ln(N(6, 9)) = \ln(3),$$

which implies that

$$\tau + \ln(d(T(\omega, x), S(\omega, y))) \leq \ln(N(x(\omega), y(\omega))).$$

Therefore, (S, T) is a pair of rational type F-contractions. Hence all axioms of Theorem 3.1 are satisfied and $x(\omega) = 7$ is a unique common random fixed point of T and S.

The proof of the following corollary is obtained by setting $S = T$ in Theorem 3.1.

Corollary 3.1. *Let (X, d) be a polish space and (Ω, β, μ) is a complete probability measure space. Suppose that $T : \Omega \times X \to X$ be a random mapping such that*

(i) T is measurable and continuous for all $x, y \in X$ and $\omega \in \Omega$,

(ii) T is a rational type F-contraction.
Then T has a unique random fixed point $p(\omega)$.

4. Application to the random of system nonlinear integral equations

Our plan is to apply Theorem 3.1 to the existence of a unique solution to the following system:

$$\left\{ \begin{array}{l} x(t,\omega) = h(t,\omega) + \int\limits_0^t K_1(\omega, t, s, x(\omega, s))ds \\ y(t,\omega) = h(t,\omega) + \int\limits_0^t K_2(\omega, t, s, y(\omega, s))ds \end{array} \right. , \tag{4.1}$$

where,

(i) $\omega \in \Omega$ is a supporting set of the probability measure space (Ω, β, μ),

(ii) $x(t, \omega)$ and $y(t, \omega)$ are unknown vector-valued random variables for each $t \in [0, a]$, $a > 0$,

(iii) $h(t, \omega)$ is the stochastic free term defined for $t \in [0, a]$,

(iv) $k_1(t, s, \omega)$ and $k_2(t, s, \omega)$ is real stochastic kernels defined for $t, s \in [0, a]$ and both measurable in t on $[0, a]$.

The integral equations (4.1) in stochastic version is a similar to Volterra integral equation of the second kind in deterministic case.

Let $C([0, a], \mathbb{R})$ be the space of all continuous functions defined on $[0, a]$, define a supremum norm as

$$\|x(\omega)\|_\tau = \sup_{t\in[0,a]} \{x(\omega, t)e^{-t\tau}\}, \tau > 0, \omega \in \Omega.$$

It's obvious that $C([0, a], \mathbb{R}, \|.\|_\tau)$ is a polish space under the distance

$$d(x(\omega), y(\omega)) = \sup_{t\in[0,a]} \left\| |x(\omega, t) - y(\omega, t)| \, e^{-t\tau} \right\|_\tau ,$$

for all $x(\omega), y(\omega) \in C([0, a], \mathbb{R})$.

Next, we consider a system (4.1) under the following axioms:

(R_1) $h(t, \omega) \in C([0, a], \mathbb{R})$,

(R_2) $K_1, K_2 : \Omega \times [0,a] \times [0,a] \times \mathbb{R} \to \mathbb{R}$ is random continuous function satisfying

$$|K_1(\omega,t,s,u(\omega)) - K_2(\omega,t,s,v(\omega))| \leq \tau e^{-\tau} N(u(\omega),v(\omega)),\ \tau > 0,$$

for all $t, s \in [0,a]$ and $u(\omega), v(\omega) \in C([0,a],\mathbb{R})$, where

$$\begin{aligned} &N(u(\omega),v(\omega)) \\ &= \max\left\{\ d(u(\omega),v(\omega)), \tfrac{d(v(\omega),S(\omega,v))[1+d(u(\omega),T(\omega,u))]}{1+d(u(\omega),v(\omega))}, \tfrac{d(u(\omega),T(\omega,u))d(v(\omega),S(\omega,v))}{1+d(T(\omega,u),S(\omega,v))}\ \right\}. \end{aligned}$$

Now, our theorem concerned with the existence solution of random integral system (4.1) become affordable.

Theorem 4.1. *Let (Ω,β,μ) be probability measure space and $\mathbb{R}$ is a polish space, then the system (4.1) under assumptions (R_1) and (R_2) has a unique random solution.*

Proof. For $x(\omega), y(\omega) \in C([0,a],\mathbb{R})$, $\omega \in \Omega$ and $t \in [0,a]$, we define the random operators $S, T : \Omega \times [0,a] \to \mathbb{R}$ by

$$\begin{aligned} T(x)(\omega,t) &= h(t,\omega) + \int_0^t K_1(\omega,t,s,x(\omega,s))ds \\ S(y)(\omega,t) &= h(t,\omega) + \int_0^t K_2(\omega,t,s,y(\omega,s))ds. \end{aligned}$$

By this, we have

$$|T(x)(\omega,t) - S(y)(\omega,t)| = \int_0^t |K_1(\omega,t,s,x(\omega,s)) - K_2(\omega,t,s,y(\omega,s))|\, ds$$

$$\begin{aligned} &\leq \int_0^t \tau e^{-\tau}(|N(u(\omega),v(\omega))|)ds \\ &= \int_0^t \tau e^{-\tau}(N(u(\omega),v(\omega))e^{\tau s})e^{-\tau s}ds \quad (\text{since } N(u(\omega),v(\omega)) \geq 0) \\ &\leq \int_0^t \tau e^{-\tau} \|N(u(\omega),v(\omega))\|_\tau e^{\tau s}ds \\ &= \tau e^{-\tau} \|N(u(\omega),v(\omega))\|_\tau \int_0^t e^{\tau s}ds = \tau e^{-\tau} \|N(u(\omega),v(\omega))\|_\tau \frac{1}{\tau}(e^{\tau t} - 1) \\ &\leq \tau e^{-\tau} \|N(u(\omega),v(\omega))\|_\tau \frac{1}{\tau} e^{\tau t} = e^{-\tau} \|N(u(\omega),v(\omega))\|_\tau e^{\tau t}. \end{aligned}$$

This gives

$$|T(x)(\omega,t)-S(y)(\omega,t)|\,.e^{-\tau t}\leq e^{-\tau}\left\|N(u(\omega),v(\omega))\right\|_{\tau},$$

or, equivalently

$$\|T(x)(\omega,t)-S(y)(\omega,t)\|\leq e^{-\tau}\left\|N(u(\omega),v(\omega))\right\|_{\tau},$$

or,

$$\tau+\ln(\|T(x)(\omega,t)-S(y)(\omega,t)\|)\leq \ln(\|N(u(\omega),v(\omega))\|_{\tau}).$$

Hence, the F-contraction (3.1) holds by taking $F(t)=ln(t)$ for every $t>0$. So, all conditions of Theorem 3.1 are satisfied. Therefore, the system of random integral equations (4.1) has a unique random solution. □

The following example support Theorem 4.1.

Example 4.1. Let (Ω,β) by a measurable space where β is a $\sigma-$algebra subsets of Ω. Consider $\Omega=[0,1]$ and the following system of random nonlinear integral equations for $t,s\in[0,a]$ and $\omega\in\Omega$,

$$\begin{aligned} x(t,\omega)&=e^{4t\omega}+\int_0^t\left(\frac{e^{-\frac{1}{4}}}{4(t+\frac{s}{1+s}+x(s,\omega))}\right)ds \\ y(t,\omega)&=e^{4t\omega}+\int_0^t\left(\frac{e^{-\frac{1}{4}}}{4(t+\frac{s}{1+s}+y(s,\omega))}\right)ds. \end{aligned}\tag{4.2}$$

System (4.2) is a particular case of system (4.1), where $h(t,\omega)=e^{\frac{t\omega}{\tau}}=e^{4t\omega}$ and

$$K_i(\omega,t,s,u(\omega,s))=\frac{e^{-\frac{1}{4}}}{4(t+\frac{s}{1+s}+u(s,\omega))},i=1,2.$$

It is clear that (R_1) is satisfied. For (R_2), we have

$$\begin{aligned} |K_1(\omega,t,s,x(\omega,s)-K_2(\omega,t,s,y(\omega,s)| &= \left|\frac{e^{-\frac{1}{4}}}{4(t+\frac{s}{1+s}+x(s,\omega))}-\frac{e^{-\frac{1}{4}}}{4(t+\frac{s}{1-s}+y(s,\omega))}\right| \\ &= \frac{1}{4}e^{-\frac{1}{4}}\left|\frac{x(s,\omega)-y(s,\omega)}{(t+\frac{s}{1-s}+x(s,\omega)).(t+\frac{s}{1-s}+y(s,\omega))}\right| \\ &\leq \frac{1}{4}e^{-\frac{1}{4}}\left|x(s,\omega)-y(s,\omega)\right|=\tau e^{-\tau}N(x(\omega),y(\omega)). \end{aligned}$$

Therefore (H_2) is hold with $\tau=\frac{1}{4}>0$ and $N(x(\omega),y(\omega))=|x(s,\omega)-y(s,\omega)|\,.$
By Theorem 4.1, the system (4.2) has a unique random solution.

5. Conclusion

In this work, we first suggest the new concept of random F-contraction mappings of rational type. We also prove the existence and uniqueness of a common random fixed point theorem in polish spaces. Our results improve and extend some fixed point results for deterministic mappings in various spaces. The solution of random Volterra integral equations of the second type is presented and if we put $h(t,\omega)=0$ in system (4.1), we have the first type. To support this work, we give two numerical examples. Finally, we can close Theorem 3.1 by taking various values of $N(x(\omega),y(\omega))$ as the following:

$$N(x(\omega),y(\omega)) = \max\left\{d(x(\omega),y(\omega)), \frac{d(y(\omega),S(\omega,y))[1+d(x(\omega),T(\omega,x))]}{1+d(x(\omega),y(\omega))}\right\},$$

$$N(x(\omega),y(\omega))=\max\left\{d(x(\omega),y(\omega)), \frac{d(x(\omega),T(\omega,x)).d(y(\omega),S(\omega,y))]}{1+d(x(\omega),y(\omega))}, \frac{d(x(\omega),T(\omega,x))d(y(\omega),S(\omega,y))]}{1+d(T(\omega,x),S(\omega,y))}\right\},$$

$$N(x(\omega),y(\omega)) = max\{d(x(\omega),y(\omega)), d(x(\omega),T(\omega,x)), d(y(\omega),S(\omega,y))\}.$$

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Rashwan A. Rashwan
Department of Mathematics
Faculty of Science, Assuit University
Assuit 71516, Egypt.
rr_rashwan54@yahoo.com

Hasanen A. Hammad
Department of Mathematics
Faculty of Science, Sohag University
Sohag 71516, Egypt.
h_elmagd89@yahoo.com