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QUASI STATISTICAL CONVERGENCE IN CONE METRIC SPACES

Nihan Turan, Emrah Evren Kara and Merve İlkhan

Abstract. The main purpose of this paper is to define a new type of statistical convergence of sequences in a cone metric space and investigate the relations of these sequences with some other sequences.

Keywords: Cone metric, statistical convergence, statistical boundedness.

1. Introduction and Preliminaries

The study of statistical convergence apparently goes back to Steinhaus [19] and Fast [7]. This concept has been studied under different names in spaces such as topological spaces, cone metric spaces etc. (see, for example [5],[8],[9],[12],[13],[14],[18]). Long-Guang and Xian [11] suggested the idea of a cone metric space. The main difference with a metric is that a cone metric is valued in an ordering Banach space. Later, several authors studied cone metric spaces and applied different names. This concept takes a vital role in computer science, statistics and some other research areas as well as general topology (see, for example [2],[2],[7],[11],[16]). The definition of statistical convergence and statistical boundedness of a sequence in a cone metric space was studied by Kedian, Shou and Ying [13]. In [10], the authors defined the concept of a quasi-statistical filter. Also it is known that statistical convergence is related to Cesaro summability and strong-Cesaro summability (see, for example [4],[3],[18]). Recently, Sakaoğlu and Yurdakadim [15] defined the notions of quasi-statistical convergence and strongly-Cesaro summability by relying on [4], [3], [10] and [18], and they found some inclusion theorems between these concepts. In the present paper, we introduce the quasi-statistical convergence and quasi-statistical boundedness of a sequence on a cone metric space, and obtain some theorems related to quasi-statistically convergent sequences. Later, we give the definition of strongly-quasi summable sequences in a cone metric space and we also investigate some theorems related to quasi-statistically convergent sequences and

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strongly–quasi summable sequences. Finally, we present some results related to these theorems.

Throughout this paper, by $\mathbb{N}$ and $\mathbb{R}$ we denote the set of all positive integers and the set of all real numbers, respectively. For a subset S of $\mathbb{N}$, $|S|$ stands for the cardinality of S.

Definition 1.1. ([7]) Let $S \subset \mathbb{N}$ and $S(m) = \{i \in S : i \leq m\}$ for each $m \in \mathbb{N}$. If the following limit exists, then

$$\delta(S) = \lim_{m\to\infty} \frac{|S(m)|}{m}$$

is called the asymptotic (or natural) density of S. It is clear that $\delta(S) \in [0,1]$. Also, if $\delta(S) = 1$, then S is said to be statistically dense. It can be easily obtained that $\delta(\mathbb{N} - S) = 1 - \delta(S)$ for each $S \subset \mathbb{N}$.

Definition 1.2. ([8]) A sequence (x_m) in $\mathbb{R}$ is said to be statistically convergent to a point $x \in \mathbb{R}$ if for each $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{m} \{i \leq m : |x_i - x| \geq \varepsilon\} = 0$$

or equivalently

$$\lim_{m\to\infty} \frac{1}{m} \{i \leq m : |x_i - x| < \varepsilon\} = 1.$$

Definition 1.3. ([1]) Let E be a real Banach space. A subset P of E is called a cone if it satisfies the following conditions:

(1) $P \neq \emptyset$, $P \neq \{0\}$ and P is closed.

(2) $ax + by \in P$ for all $x, y \in P$ and $a, b \in \mathbb{R}$ with $a, b \geq 0$.

(3) If $x \in P$ and $-x \in P$, then $x = 0$ for all $x, y \in P$.

A partial ordering " $\preceq$ " with respect to P is defined by $x \preceq y \Leftrightarrow y - x \in P$. Also, we mean $x \prec y \Leftrightarrow x \preceq y$, $x \neq y$ and $x \prec\prec y \Leftrightarrow y - x \in E^+$, where E^+ denotes the interior of P; that is $E^+ = \{c \in E : 0 \prec\prec c\}$. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $0 \preceq x \preceq y$ implies $\|x\| \leq K \|y\|$. The least positive number satisfying this inequality is called the normal constant of P.

In this study, we always suppose that E is a Banach space, P is a cone in E with $E^+ \neq \emptyset$ and " $\preceq$ " is a partial ordering with respect to P.

Definition 1.4. ([17]) Let X be a non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies

1. $0 \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,

2. $d(x,y) = d(y,x)$ for all $x, y \in X$,

3. $d(x,y) \preceq d(x,z) + d(y,z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, (X,d) is called a cone metric space.

Definition 1.5. ([11]) A sequence (x_n) in a cone metric space (X,d) is said to be convergent to $x \in X$ if for every $c \in E^+$ there exists a natural number N such that $d(x_n, x) \prec\prec c$ for all $n > N$.

Definition 1.6. ([13]) A sequence (x_n) in a cone metric space (X,d) is said to be statistically convergent to $x \in X$ if for every $c \in E^+$

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : d(x_k, x) \prec\prec c\}\right| = 1.$$

It is denoted by st-$\lim\limits_{n\to\infty} x_n = x$.

Definition 1.7. ([13]) A sequence (x_n) in a cone metric space (X,d) is said to be statistically bounded if there exist $\alpha \in X$ and $c \in E^+$ such that

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : d(x_k, \alpha) \preceq c\}\right| = 1.$$

Definition 1.8. ([15]) Let $s = (s_n)$ be a sequence of positive real numbers such that

$$\lim_n s_n = \infty \quad \text{and} \quad \limsup_n \frac{s_n}{n} < \infty. \tag{1.1}$$

The quasi density of a subset $K \subset \mathbb{N}$ with respect to the sequence $s = (s_n)$ is defined by

$$\delta_s(K) = \lim_{n\to\infty} \frac{1}{s_n} \left|\{k \le n : k \in K\}\right|.$$

A sequence (x_n) in $\mathbb{R}$ is called quasi-statistical convergent to x provided that for every $\varepsilon > 0$ the set $K_\varepsilon = \{k \in \mathbb{N} : |x_k - x| \ge \varepsilon\}$ has quasi-density zero. It is denoted by st_q-$\lim_{n\to\infty} x_n = x$.

Throughout the study, we assume that $s = (s_n)$ and $t = (t_n)$ are sequences of positive real numbers satisfying the conditions in (1.1).

Definition 1.9. ([15]) A sequence (x_n) in $\mathbb{R}$ is said to be strongly quasi-summable to $x \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{k=1}^{n} |x_k - x| = 0.$$

2. Main Results

In this section, we first define the quasi-statistical convergence of a sequence in a cone metric space. Later, we give some results related to this concept.

Definition 2.1. A sequence (x_n) in a cone metric space (X, d) is said to be quasi-statistical convergent to a point $x \in X$ if for every $c \in E^+$ we have

$$\lim_{n\to\infty} \frac{1}{s_n} \left|\{k \leq n : d(x_k, x) \prec\prec c\}\right| = 1$$

or equivalently

$$\lim_{n\to\infty} \frac{1}{s_n} \left|\{k \leq n : c \preceq d(x_k, x)\}\right| = 0.$$

We denote it by $st_q - \lim_{n\to\infty} x_n = x$. If we take $(s_n) = (n)$, then we obtain that (x_n) is statistical convergent.

Theorem 2.1. *Let (x_n) be a sequence in a cone metric space (X, d). If (x_n) is convergent to $x \in X$, then it is quasi-statistical convergent to x.*

Proof. Let $\lim_{n\to\infty} x_n = x$. Then, for every $c \in E^+$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) \prec\prec c$ for every $n > n_0$. It follows that

$$\frac{1}{s_n} \left|\{k \leq n : c \preceq d(x_k, x)\}\right| \leq \frac{n_0}{s_n}$$

which means $\lim_{n\to\infty} \frac{1}{s_n} \left|\{k \leq n : c \preceq d(x_k, x)\}\right| = 0$. Hence, (x_n) is quasi-statistical convergent to x. □.

The converse of the previous theorem does not hold which can be seen from the following example.

Example 2.1. Let $E = \mathbb{R}$, $P = [0, \infty)$ and $X = \mathbb{R}$. Consider X with usual metric $d(x, y) = |x - y|$. Let $s_n = n^{3/4}$. Define a sequence (x_n) as follows:

$$x_n = \begin{cases} 0, & n \neq m^2 \text{ for all } m \in \mathbb{N} \\ n, & n = m^2 \text{ for some } m \in \mathbb{N} \end{cases}$$

It is obvious that (x_n) is not convergent. On the other hand, it is quasi-statistical convergent to 0. Indeed, given any $c \in E^+$, we obtain the inclusion

$$\{n : c \preceq d(x_n, 0)\} \subset \{n : n = m^2, m \in \mathbb{N}\}.$$

Hence we conclude that

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{s_n} \left|\{k \leq n : c \preceq d(x_k, 0)\}\right| &\leq \lim_{n\to\infty} \frac{1}{s_n} \left|\left\{k \leq n : k = m^2, m \in \mathbb{N}\right\}\right| \\ &= \lim_{n\to\infty} \frac{1}{s_n} \left[|\sqrt{n}|\right] = 0. \end{aligned}$$

Theorem 2.2. *Let (x_n) be a sequence in a cone metric space (X, d). If (x_n) is quasi-statistical convergent to $x \in X$, then it is statistical convergent to x.*

Proof. Let $st_q\text{-}\lim_{n\to\infty} x_n = x$ and $M = \sup_n \frac{s_n}{n}$. Then, for every $c \in E^+$, we have $\lim_{n\to\infty} \frac{1}{s_n} |\{k \le n : c \preceq d(x_k, x)\}| = 0$. The statistical convergence of the sequence (x_n) follows from the following inequality

$$\frac{1}{n} |\{k \le n : c \preceq d(x_k, x)\}| \le \frac{M}{s_n} |\{k \le n : c \preceq d(x_k, x)\}| .$$

□.

The converse of the previous theorem does not hold which can be seen from the following example.

Example 2.2. Let $X = \mathbb{R}$, $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \ge 0\}$, $X = \mathbb{R}$ and $d : X \times X \to E$ be the cone metric defined by $d(x, y) = (|x - y|, \alpha |x - y|)$, where $\alpha > 0$ is a constant. Assume that the sequence (s_n) satisfies $\lim_n \frac{\sqrt{n}}{s_n} = \infty$. We can choose a subsequence (s_{n_p}) such that $s_{n_p} > 1$ for each $p \in \mathbb{N}$. Consider the sequence (x_n) defined by

$$x_n = \begin{cases} s_n, & n = m^2 \text{and } s_n \in \{s_{n_p} : p \in \mathbb{N}\} \\ 1, & n = m^2 \text{ and } s_n \notin \{s_{n_p} : p \in \mathbb{N}\} \quad (m \in \mathbb{N}) \\ 0, & \text{otherwise.} \end{cases}$$

Then, we have

$$d(x_n, 0) = \begin{cases} (s_n, \alpha s_n), & n = m^2 \text{ and } s_n \in \{s_{n_p} : p \in \mathbb{N}\} \\ (1, \alpha), & n = m^2 \text{ and } s_n \notin \{s_{n_p} : p \in \mathbb{N}\} \quad (m \in \mathbb{N}) \\ (0, 0), & \text{otherwise.} \end{cases}$$

It is easy to see that (x_n) is statistical convergent to zero. Now, we show that (x_n) is not quasi-statistical convergent to zero; that is,

$$\lim_{n\to\infty} \frac{1}{s_n} |\{k \le n : c \preceq d(x_k, 0)\}| \ne 0.$$

For $c = (1, \alpha) \in E^+$ and $n \in \mathbb{N}$, we have

$$|\{k \le n : c \preceq d(x_k, 0)\}| = \left|\{k \le n : k = m^2, m \in \mathbb{N}\}\right|$$

and

$$\frac{1}{s_n} |\{k \le n : c \preceq d(x_k, 0)\}| = \frac{1}{s_n} \left(\sqrt{n} - r_n\right), \tag{2.1}$$

where $0 \le r_n < 1$. If we take the limit of (2.1) as $n \to \infty$, we conclude that (x_n) is not quasi-statistical convergent to zero.

Consequently, we have the following diagram:

convergent ⇒ quasi-statistical convergent ⇒ statistical convergent

Theorem 2.3. *Assume that*

$$h = \inf_n \frac{s_n}{n} > 0. \tag{2.2}$$

If a sequence (x_n) in a cone metric space (X, d) is statistical convergent to $x \in X$, then it is quasi-statistical convergent to x.

Proof. The proof follows from the inequality

$$\frac{1}{n} \left|\{k \leq n : c \preceq d(x_k, L)\}\right| \geq h \frac{1}{s_n} \left|\{k \leq n : c \preceq d(x_k, L)\}\right|.$$

□.

Corollary 2.1. *Assume that the sequence (s_n) satisfies (2.2). Then, (x_n) is statistical convergent to x if and only if (x_n) is quasi-statistical convergent to x.*

Theorem 2.4. *If (x_n) is quasi-statistical convergent to x in a cone metric space (X, d), then there is a sequence (y_n) which is convergent to x and quasi-statistical null sequence (z_n) such that $x_n = y_n + z_n$ for all $n \in \mathbb{N}$.*

Proof. Let $st_q - \lim_{n\to\infty} x_n = x$. If the terms of the sequence (x_n) is constant after a certain stage, then the proof is trivial. Otherwise given any $c \in E^+$, we can find an increasing sequence of positive integers (N_j) such that $N_0 = 0$ and $\frac{1}{s_n} \left|\left\{k \leq n : \frac{e}{j} \preceq d(x_k, x)\right\}\right| < \frac{1}{j}$ for all $n > N_j$ $(j = 1, 2, ...)$. Let us define (y_k) and (z_k) as follows:

$$\begin{array}{llll} z_k = 0 & \text{and} \quad y_k = x_k; & \text{if } N_0 < k \leq N_1, & \\ z_k = 0 & \text{and} \quad y_k = x_k; & \text{if } d(x_k, x) \prec\prec \frac{e}{j}, & N_j < k \leq N_{j+1}, \\ z_k = x_k - x & \text{and} \quad y_k = x; & \text{if } \frac{e}{j} \preceq d(x_k, x), & N_j < k \leq N_{j+1}. \end{array}$$

It is easy to see that $x_k = y_k + z_k$ for all $k \in \mathbb{N}$. Now, we show that (y_k) is convergent to x. Given any $c \in E^+$, choose $j \in \mathbb{N}$ such that $\frac{e}{j} \prec\prec c$.

$$\text{If } \frac{e}{j} \preceq d(x_k, x) \text{ for } k > N_j, \text{ then } d(y_k, x) = d(x, x) = 0.$$

If $d(x_k, x) \prec\prec \frac{e}{j}$ for $k > N_j$, then $d(y_k, x) = d(x_k, x) \prec\prec \frac{e}{j} \prec\prec c$. Hence, it follows that $\lim_{k\to\infty} y_k = x$.

To show that (z_k) is quasi-statistical null sequence; it is enough to prove that

$$\lim_{n\to\infty} \frac{1}{s_n} \left|\{k \leq n : z_k \neq 0\}\right| = 0.$$

For $c \in E^+$, it is clear that the inclusion

$$\{k \leq n : c \preceq d(z_k, 0)\} \subseteq \{k \leq n : z_k \neq 0\}$$

holds for all $n \in \mathbb{N}$. Thus, we have

$$|\{k \leq n : c \preceq d(z_k, 0)\}| \leq |\{k \leq n : z_k \neq 0\}|.$$

Given any $\delta > 0$ there is a $j \in \mathbb{N}$ such that $\frac{1}{j} < \delta$. If $N_j < k \leq N_{j+1}$, we have

$$|\{k \leq n : z_k \neq 0\}| = \left|\left\{k \leq n : \frac{e}{j} \preceq d(x_k, x)\right\}\right|.$$

Thus, we have

$$\frac{1}{s_n}|\{k \leq n : z_k \neq 0\}| \leq \frac{1}{s_n}\left|\left\{k \leq n : \frac{e}{v} \preceq d(x_k, x)\right\}\right| < \frac{1}{v} < \frac{1}{j} < \delta$$

for $N_v < k \leq N_{v+1}$ and $v > j$ which concludes the proof. □.

The following result is an immediate consequence of the previous theorem.

Corollary 2.2. *If (x_n) is quasi-statistical convergent to x, then it has a subsequence (y_n) which is convergent to x.*

Definition 2.2. A sequence (x_n) in a cone metric space (X, d) is said to be quasi-statistical Cauchy if for every $c \in E^+$ there exists $n_0 \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{1}{s_n}|\{k \leq n : d(x_k, x_{n_0}) \prec\prec c\}| = 1$$

or equivalently

$$\lim_{n\to\infty} \frac{1}{s_n}|\{k \leq n : c \preceq d(x_k, x_{n_0})\}| = 0.$$

Theorem 2.5. *Let (x_n) be a sequence in a cone metric space (X, d). If (x_n) is a Cauchy sequence, then it is a quasi-statistical Cauchy sequence.*

Proof. Let (x_n) be a Cauchy sequence. Then, for every $c \in E^+$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \prec\prec c$ for every $n, m \geq n_0$. It follows that

$$\frac{1}{s_n}|\{k \leq n : c \preceq d(x_k, x_{n_0})\}| \leq \frac{n_0}{s_n}$$

which means $\lim_{n\to\infty} \frac{1}{s_n}|\{k \leq n : c \preceq d(x_k, x_{n_0})\}| = 0$. Hence, (x_n) is quasi-statistical Cauchy. □.

The sequence given in Example 2.1 is also a quasi-statistical Cauchy sequence which is not a Cauchy sequence.

Theorem 2.6. *Let (x_n) be a sequence in a cone metric space (X, d). If (x_n) is a quasi-statistical Cauchy sequence, then it is a statistical Cauchy sequence.*

Proof. Let (x_n) be a quasi-statistical Cauchy sequence. Then, for every $c \in E^+$ there exists $n_0 \in \mathbb{N}$ such that $\lim_{n\to\infty} \frac{1}{s_n} |\{k \leq n : c \preceq d(x_k, x_{n_0})\}| = 0$. Thus we have

$$\begin{aligned} \frac{1}{n} |\{k \leq n : c \preceq d(x_k, x_{n_0})\}| &= \frac{s_n}{n} \frac{1}{s_n} |\{k \leq n : c \preceq d(x_k, x_{n_0})\}| \\ &\leq K \frac{1}{s_n} |\{k \leq n : c \preceq d(x_k, x_{n_0})\}|, \end{aligned}$$

where $K = \sup_n \frac{s_n}{n}$. This implies that (x_n) is a statistical Cauchy sequence in X. □.

Consequently, we have the following diagram:

Cauchy $\Rightarrow$ quasi-statistical Cauchy $\Rightarrow$ statistical Cauchy

Definition 2.3. A sequence (x_n) in a cone metric space (X, d) is said to be quasi-statistical bounded if there exist $\alpha \in X$ and $c \in E^+$ such that

$$\lim_{n\to\infty} \frac{1}{s_n} |\{k \leq n : c \preceq d(x_k, \alpha)\}| = 0.$$

Theorem 2.7. *If (x_n) is quasi-statistical bounded sequence in a cone metric space (X, d), then it is statistical bounded.*

Proof. Let (x_n) be a quasi-statistical bounded sequence, $\alpha \in X$ and $H = \sup_n \frac{s_n}{n}$. Since the inequality

$$\frac{1}{n} |\{k \leq n : c \preceq d(x_k, \alpha)\}| \leq \frac{H}{s_n} |\{k \leq n : c \preceq d(x_k, \alpha)\}|$$

holds, the proof follows immediately. □.

Lemma 2.1. *Let P be a normal cone with normal constant K. The following statements hold for sequences (x_n) and (y_n) in a cone metric space (X, d).*

1. $st_q - \lim_{n\to\infty} x_n = x \Leftrightarrow st_q - \lim_{n\to\infty} d(x_n, x) = 0$
2. *If* $st_q - \lim_{n\to\infty} x_n = x$ *and* $st_q - \lim_{n\to\infty} y_n = y$, *then* $st_q - \lim_{n\to\infty} d(x_n, y_n) = d(x, y)$.

Proof. (1) Suppose that $st_q - \lim_{n\to\infty} x_n = x$. Then, for every $c \in E^+$, we have

$$\lim_{n\to\infty} \frac{1}{s_n} |\{k \leq n : d(x_k, x) \prec\prec c\}| = 1.$$

Given any $\varepsilon > 0$, choose $c \in E^+$ such that $K \|c\| < \varepsilon$. Suppose that $k \in \mathbb{N}$ satisfies $d(x_k, x) \prec\prec c$. Since P is a normal cone with normal constant K, we can write

$$\|d(x_k, x)\| \leq K \|c\| < \varepsilon.$$

Consequently, we obtain

$$\frac{1}{s_n}\left|\{k \le n : d(x_k, x) \prec\prec c\}\right| \le \frac{1}{s_n}\left|\{k \le n : \|d(x_k, x)\| < \varepsilon\}\right|.$$

Hence, we conclude that

$$\lim_{n\to\infty}\frac{1}{s_n}\left|\{k \le n : \|d(x_k, x)\| < \varepsilon\}\right| = 1$$

which means $st_q - \lim_{n\to\infty} d(x_n, x) = 0$.

Conversely, suppose that $st_q - \lim_{n\to\infty} d(x_n, x) = 0$. Then for every $\varepsilon > 0$, we have

$$\lim_{n\to\infty}\frac{1}{s_n}\left|\{k \le n : \|d(x_k, x)\| < \varepsilon\}\right| = 1.$$

Given any $c \in E^+$, we can find an $\varepsilon > 0$ such that $c - a \in E^+$ for all $a \in E$ with $\|a\| < \varepsilon$. Hence, if we choose $k \in \mathbb{N}$ such that $\|d(x_k, x)\| < \varepsilon$, then we obtain $d(x_k, x) \prec\prec c$ which implies that the inclusion $\{k : \|d(x_k, x)\| < \varepsilon\} \subset \{k : d(x_k, x) \prec\prec c\}$ holds. It follows that

$$\frac{1}{s_n}\left|\{k \le n : \|d(x_k, x)\| < \varepsilon\}\right| \le \frac{1}{s_n}\left|\{k \le n : d(x_k, x) \prec\prec c\}\right|.$$

Thus, we conclude that $\lim_{n\to\infty}\frac{1}{s_n}\left|\{k \le n : d(x_k, x) \prec\prec c\}\right| = 1$ and so $st_q - \lim_{n\to\infty} x_n = x$.

(2) Suppose $st_q - \lim_{n\to\infty} x_n = x$ and $st_q - \lim_{n\to\infty} y_n = y$. Given any $\varepsilon > 0$, choose $c \in E^+$ such that $\|c\| < \frac{\varepsilon}{4K+2}$. For $k \in \mathbb{N}$ with $d(x_k, x) \prec\prec c$ and $d(y_k, y) \prec\prec c$, we have $\|d(x_k, y_k) - d(x, y)\| < \varepsilon$ from the proof of Lemma 5 in [11]. Hence, the inclusion

$$\{k : \|d(x_k, y_k) - d(x, y)\| \ge \varepsilon\} \subset \{k : c \preceq d(x_k, x)\} \cup \{k : c \preceq d(y_k, y)\}$$

holds. It follows that

$$\lim_{n\to\infty}\frac{1}{s_n}\left|\{k \le n : \|d(x_k, y_k) - d(x, y)\| \ge \varepsilon\}\right| = 0$$

which means that $st_q - \lim_{n\to\infty} d(x_n, y_n) = d(x, y)$. □.

Remark 2.1. Note that P does not need to be a normal cone to prove the sufficiency condition in 1 of Lemma 2.1. That is; if $st_q - \lim_{n\to\infty} d(x_n, x) = 0$ in a cone metric space (X, d), then we have $st_q - \lim_{n\to\infty} x_n = x$.

Theorem 2.8. *Let (x_n) and (y_n) be two sequences in a cone metric space (X, d). If $st_q - \lim_{n\to\infty} y_n = y$ and $d(x_n, y) \preceq d(y_n, y)$ for every $n \in \mathbb{N}$, then $st_q - \lim_{n\to\infty} x_n = y$.*

Proof. Suppose that $st_q - \lim_{n\to\infty} y_n = y$ and $d(x_n, y) \preceq d(y_n, y)$ for every $n \in \mathbb{N}$. The proof follows from the fact that

$$\frac{1}{s_n} \left|\{k \leq n : d(y_k, y) \preceq c\}\right| \leq \frac{1}{s_n} \left|\{k \leq n : d(x_k, y) \preceq c\}\right|.$$

□.

Definition 2.4. A sequence (x_n) in a cone metric space (X, d) is said to be strongly quasi-summable to x, if

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{k=1}^{n} \|d(x_k, x)\| = 0$$

holds.

We will use N_q^s and S_q^s for the set of all strongly quasi-summable sequences and all quasi-statistical convergent sequences, respectively. That is,

$$N_q^s = \left\{(x_n) : \lim_{n\to\infty} \frac{1}{s_n} \sum_{k=1}^{n} \|d(x_k, x)\| = 0 \text{ for some } x\right\}$$

and

$$S_q^s = \left\{(x_n) : \lim_{n\to\infty} \frac{1}{s_n} \left|\{k \leq n : c \preceq d(x_k, x)\}\right| = 0 \text{ for some } x \in \mathbb{R} \text{ and for all } c \in E^+\right\}$$

If we take $t = (t_n)$ instead of $s = (s_n)$, we will write N_q^t and S_q^t instead of N_q^s and S_q^s, respectively.

Theorem 2.9. *Let $s_n \leq t_n$ for every $n \in \mathbb{N}$. If a sequence (x_n) in a cone metric space (X, d) is quasi-statistical convergent to x with respect to $s = (s_n)$, then (x_n) sequence is quasi-statistical convergent to x with respect to $t = (t_n)$.*

Proof. Suppose that for every $c \in E^+$ we have $\lim_{n\to\infty} \frac{1}{s_n} \left|\{k \leq n : c \preceq d(x_k, x)\}\right| = 0$. Since $s_n \leq t_n$ holds for every $n \in \mathbb{N}$, we have the inequality

$$\frac{1}{s_n} \left|\{k \leq n : c \preceq d(x_k, x)\}\right| \geq \frac{1}{t_n} \left|\{k \leq n : c \preceq d(x_k, x)\}\right|.$$

Letting $n \to \infty$ in both sides of the above inequality, we obtain that the sequence (x_n) is quasi-statistical convergent to x with respect to $t = (t_n)$. □.

Now, we consider the sequence (x_n) in Example 2.2 and if we take $t_n = n$ and $s_n = n^{1/4}$, then we observe that the sequence (x_n) is quasi-statistical convergent to zero with respect to the sequence $t = (t_n)$ but the sequence (x_n) is not quasi-statistical convergent to zero with respect to the sequence $s = (s_n)$. Thus, the following result can given as a consequence of this theorem.

Corollary 2.3. *Let $s_n \leq t_n$ for every $n \in \mathbb{N}$. Then, the inclusion $S_q^s \subset S_q^t$ strictly holds.*

Theorem 2.10. *Let $s_n \leq t_n$ for every $n \in \mathbb{N}$. If a sequence (x_n) in a cone metric space (X, d) is strongly quasi-summable to x with respect to $s = (s_n)$, then the sequence (x_n) is quasi-statistical convergent to x with respect to $t = (t_n)$.*

Proof. Let $\lim\limits_{n\to\infty} \frac{1}{s_n} \sum\limits_{k=1} \|d(x_k, x)\| = 0$. By using the fact that

$$\sum_{k=1}^{n} \|d(x_k, x)\| = \sum_{\substack{k=1 \\ \|d(x_k,x)\| \geq \varepsilon}}^{n} \|d(x_k, x)\| + \sum_{\substack{k=1 \\ \|d(x_k,x)\| < \varepsilon}}^{n} \|d(x_k, x)\| \geq \varepsilon \left|\{k \leq n : \|d(x_k, x)\| \geq \varepsilon\}\right|$$

and $s_n \leq t_n$ for every $n \in \mathbb{N}$, we obtain

$$\frac{1}{\varepsilon} \frac{1}{s_n} \sum_{k=1}^{n} \|d(x_k, x)\| \geq \frac{1}{t_n} \left|\{k \leq n : \|d(x_k, x)\| \geq \varepsilon\}\right|.$$

Since the limit of the left side equals to zero, we have $st_q - \lim\limits_{n\to\infty} d(x_n, x) = 0$ with respect to $t = (t_n)$. From Remark 2.1, we conclude that $st_q - \lim\limits_{n\to\infty} x_n = x$ with respect to $t = (t_n)$. □.

The converse of this theorem is not always true.

Example 2.3. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = \mathbb{R}$ and $d : X \times X \to E$ be the cone metric defined by $d(x, y) = (|x - y|, |x - y|)$. Consider the sequence (x_n) defined by

$$x_n = \begin{cases} 1, & n = m^2 \\ 0, & n \neq m^2 \end{cases} \quad m \in \mathbb{N}$$

Let $(s_n) = \left(n^{\frac{1}{4}}\right)$ and $(t_n) = (n)$. We have

$$d(x_n, 0) = \begin{cases} (1,1), & n = m^2 \\ (0,0), & n \neq m^2 \end{cases} \quad m \in \mathbb{N}$$

Hence, given any $c \in E^+$ and $n \in \mathbb{N}$, we obtain

$$\frac{1}{t_n} \left|\{k \leq n : d(x_k, 0) \prec\prec c\}\right| \geq \frac{1}{t_n} \left|\{k \leq n : n \neq m^2\}\right|.$$

Since the limit of the right side equals 1, we conclude that the sequence (x_n) is quasi-statistical convergent to zero with respect to $t = (t_n)$.

Now, we will show that the sequence (x_n) is not strongly quasi-summable to zero with respect to $s = (s_n)$. It is clear that

$$\|d(x_k, 0)\| = \begin{cases} \sqrt{2}, & k = m^2 \\ 0, & k \neq m^2 \end{cases} \quad m \in \mathbb{N}$$

Then, we obtain that

$$\begin{aligned}\sum_{k=1}^{n}\left\|d\left(x_k,0\right)\right\| &= 0\left|\left\{k\leq n: k\neq m^2 \text{ for all } m\in\mathbb{N}\right\}\right| \\ &+ \sqrt{2}\left|\left\{k\leq n: k=m^2 \text{ for some } m\in\mathbb{N}\right\}\right| \\ &= 0.\left(n-\left[\left|\sqrt{n}\right|\right]\right)+\sqrt{2}\left(\left[\left|\sqrt{n}\right|\right]\right).\end{aligned}$$

and so

$$\lim_{n\to\infty}\frac{1}{s_n}\sum_{k=1}^{n}\left\|d\left(x_k,0\right)\right\| = \lim_{n\to\infty}\frac{1}{s_n}\sqrt{2}\left(\left[\left|\sqrt{n}\right|\right]\right)=\infty.$$

Consequently, we find that

$$\lim_{n\to\infty}\frac{1}{s_n}\sum_{k=1}^{n}\left\|d\left(x_k,0\right)\right\|\neq 0.$$

which means the sequence (x_n) is not strongly quasi-summable to zero with respect to $s=(s_n)$.

Corollary 2.4. *Let $s_n\leq t_n$ for every $n\in\mathbb{N}$. The inclusion $N_q^s\subset S_q^t$, strictly holds.*

Theorem 2.11. *Let $s_n\leq t_n$ for every $n\in\mathbb{N}$. If a sequence (x_n) in a cone metric space (X,d) is strongly quasi-summable to x with respect to $s=(s_n)$, then the sequence (x_n) is strongly quasi-summable sequence to x with respect to $t=(t_n)$.*

Proof. Suppose that the sequence (x_n) is strongly quasi-summable to x with respect to $s=(s_n)$. Then, we have

$$\lim_{n\to\infty}\frac{1}{s_n}\sum_{k=1}^{n}\left\|d\left(x_k,x\right)\right\|=0.$$

From the fact that $s_n\leq t_n$ for every $n\in\mathbb{N}$, we have the following inequality

$$\frac{1}{s_n}\sum_{k=1}^{n}\left\|d\left(x_k,x\right)\right\|\geq\frac{1}{t_n}\sum_{k=1}^{n}\left\|d\left(x_k,x\right)\right\|.$$

Hence, we conclude that $\lim\limits_{n\to\infty}\frac{1}{t_n}\sum\limits_{k=1}^{n}\left\|d\left(x_k,x\right)\right\|=0.$ □.

But the converse of this theorem is not always true. To observe this, consider the sequences (x_n), $s=(s_n)$ and $t=(t_n)$ defined in Example 2.3. It can be shown that $(x_n)\in N_q^t$ and $(x_n)\notin N_q^s$. Thus, the following corollary can be given as a result of this theorem.

Corollary 2.5. *Let $s_n\leq t_n$ for every $n\in\mathbb{N}$. The inclusion $N_q^s\subset N_q^t$, strictly holds.*

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Nihan Turan
Faculty of Science and Arts
Department of Mathematics
Düzce University
Düzce, Turkey
nhn_trn08@hotmail.com

Emrah Evren Kara
Faculty of Science and Arts
Department of Mathematics
Düzce University
Düzce, Turkey
karaeevren@gmail.com

Merve İlkhan
Faculty of Science and Arts
Department of Mathematics
Düzce University
Düzce, Turkey
merveilkhan@gmail.com