FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 29, No 3 (2014), 233–242

NEW RESULTS FOR A WEIGHTED NONLINEAR SYSTEM OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. This paper studies the existence of solutions for a weighted system of nonlinear fractional integro-differential equations. New existence and uniqueness results are established using the Banach fixed point theorem. Other existence results are obtained using the Schaefer fixed point theorem. Some concrete examples are also presented to illustrate the possible application of the established analytical results.

Keywords: Fractional Integro-Differential Equations; Weighted Nonlinear System; Riemann-Liouville fractional integral; Caputo fractional derivative

1. Introduction

The differential equations of fractional order arise in many scientific disciplines, such as physics, chemistry, control theory, signal processing and biophysics. For more details, we refer the reader to [3, 5, 7, 8, 9, 13, 16] and the references therein. Recently, there has been a significant progress in the investigation of these equations (see [4, 6, 10, 15, 17]). On the other hand, the study of coupled systems of fractional differential equations is also of great importance. Such systems occur in various problems of applied science. For some recent results on the fractional systems, we refer the reader to ([11, 14, 19].

In this paper, we discuss the existence and uniqueness of solutions for the following coupled system of fractional integro-differential equations:

$$(1.1)\qquad \begin{cases} D^{\alpha}u(t) = \varphi_1(t)\, f_1(t,u(t),v(t)) + \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\sigma)} f_1(s,u(s),v(s))\,ds, \\ D^{\beta}v(t) = \varphi_2(t)\, f_2(t,u(t),v(t)) + \int_0^t \frac{(t-s)^{\theta-1}}{\Gamma(\theta)} f_2(s,u(s),v(s))\,ds, \\ u(0)=a,\, v(0)=b,\, t\in[0,1], \end{cases}$$

Received October 15, 2013.; Accepted March 12, 2014.
2010 *Mathematics Subject Classification.* 34A34, 34B10.

where D^{α}, D^{β} denote the Caputo fractional derivatives, $0 < \alpha < 1$, $0 < \beta < 1$, σ and θ are non-negative real numbers, φ_1, φ_2 are two continuous functions, $a > 0, b > 0$, f_1 and f_2 are two functions that will be specified later.

The paper is organized as follows: In section 2, we present some preliminaries and lemmas. Section 3 is devoted to the existence and uniqueness of solutions of problem (1.1). In the last section, some examples are presented to illustrate our results.

2. Preliminaries

The following notations, definitions and preliminary facts will be used throughout this paper [15, 18].

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, for $f \in L^1([a,b],\mathbb{R})$ is defined by:

$$I^{\alpha} f(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)\, d\tau, \qquad a \le t \le b, \tag{2.1}$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u} u^{\alpha-1} du$.

Definition 2.2. The fractional derivative of $f \in C^n([a,b],\mathbb{R})$, $n \in N^*$, in the sense of Caputo, of order α, $n-1 < \alpha < n$ is defined by:

$$D^{\alpha} f(t) = \int_a^t \frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} f^{(n)}(\tau)\, d\tau, \quad t \in [a,b]. \tag{2.2}$$

The following lemmas give some properties of Riemann-Liouville fractional integral and Caputo fractional derivative [12, 13]:

Lemma 2.1. *Given $f \in L^1([a,b],\mathbb{R})$, then for all $t \in [a,b]$ we have*

$I^r I^s f(t) = I^{r+s} f(t)$, *for* $r, s > 0$.

$D^s I^s f(t) = f(t)$, *for* $s > 0$.

$D^r I^s f(t) = I^{s-r} f(t)$, *for* $s > r > 0$.

To study the coupled system (1.1), we need the following two lemmas [12, 13]:

Lemma 2.2. *For $n-1 < \alpha < n$, where $n \in \mathbb{N}^*$, the general solution of the equation $D^{\alpha} x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, \tag{2.3}$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, \ldots, n-1$.

Lemma 2.3. *Let $n-1<\alpha<n$, where $n\in\mathbb{N}^*$. Then, for $x\in C^n([0,1],\mathbb{R})$ we have*

$$I^\alpha D^\alpha x(t)=x(t)+c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1}, \tag{2.4}$$

for some $c_i\in\mathbb{R}, i=0,1,2,\ldots,n-1, n=[\alpha]+1$.

We also need the following auxiliary lemma:

Lemma 2.4. *Let $f\in C([0,1],\mathbb{R})$. The solution of the problem*

$$D^\alpha x(t)=(\varphi f)(t)+\int_0^t\frac{(t-s)^{\sigma-1}}{\Gamma(\sigma)}f(s)\,ds, 0<\alpha<1,\quad \sigma>0 \tag{2.5}$$

subject to the boundary condition,

$$x(0)=x_0^*,$$

is given by

$$x(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(\varphi f)(s)\,ds+\int_0^t\frac{(t-s)^{\alpha+\sigma-1}}{\Gamma(\alpha+\sigma)}f(s)\,ds+x_0^*. \tag{2.6}$$

Proof. Setting

$$y(t)=x(t)-I^\alpha(\varphi f)(t)-I^{\alpha+\sigma}f(t), \tag{2.7}$$

we get

$$D^\alpha y(t)=D^\alpha x(t)-D^\alpha I^\alpha(\varphi f)(t)-D^\alpha I^{\alpha+\sigma}f(t). \tag{2.8}$$

Then, by lemma 2.1,

$$D^\alpha y(t)=D^\alpha x(t)-(\varphi f)(t)-I^\sigma f(t). \tag{2.9}$$

Thus, (2.5) is equivalent to $D^\alpha y(t)=0$.

Finally, thanks to lemma 2.2, we obtain that $y(t)$ is constant, i.e., $y(t)=y(0)=x(0)=x_0^*$, and the proof of lemma 2.4 is achieved. □

3. Main Results

We introduce in this paragraph the following assumptions concerning the functions f_1 and f_2 introduced in (1.1):

$(H1)$: There exist non-negative real numbers $m_i, n_i, (i = 1, 2)$, such that for all $t \in [0, 1]$ and $(u_1, v_1), (u_2, v_2) \in \mathbb{R}^2$, we have

$$\begin{aligned} \left| f_1(t, u_2, v_2) - f_1(t, u_1, v_1) \right| &\leq m_1 |u_2 - u_1| + m_2 |v_2 - v_1|, \\ \left| f_2(t, u_2, v_2) - f_2(t, u_1, v_1) \right| &\leq n_1 |u_2 - u_1| + n_2 |v_2 - v_1|. \end{aligned}$$

$(H2)$: The functions f_1 and $f_2 : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous.

$(H3)$: There exist two positive numbers L_1 and L_2, such that

$$\left| f_1(t, u, v) \right| \leq L_1, \left| f_2(t, u, v) \right| \leq L_2, t \in [0, 1], (u, v) \in \mathbb{R}^2.$$

Our first result is given by:

Theorem 3.1. *Assume that* $(H1)$ *holds. Setting*

$$\begin{aligned} M_1 &:= \frac{\left\| \varphi_1 \right\|_\infty}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha + \sigma + 1)}, \\ M_2 &:= \frac{\left\| \varphi_2 \right\|_\infty}{\Gamma(\beta + 1)} + \frac{1}{\Gamma(\beta + \theta + 1)}. \end{aligned}$$

Then if

$$(M_1 + M_2)(m_1 + m_2 + n_1 + n_2) < 1, \tag{3.1}$$

the fractional system (1.1) *has exactly one solution on* $[0, 1]$.

Proof. Let us consider

$$X := C([0, 1], \mathbb{R}).$$

This space, equipped with the norm $\|.\|_X = \|.\|_\infty$ defined by

$$\|f\|_\infty = sup\{|f(x)|, x \in [0, 1]\},$$

is a Banach space. Also, the product space $(X \times X, \|(u, v)\|_{X \times X})$ is a Banach space, with norm $\|(u, v)\|_{X \times X} = \|u\|_X + \|v\|_X$.

Consider now the operator $\phi : X \times X \to X \times X$ defined by

$$\phi(u, v)(t) = \left(\phi_1(u, v)(t), \phi_2(u, v)(t) \right), \tag{3.2}$$

where,

$$\phi_1(u,v)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_1(s)\, f_1(s,u(s),v(s))\,ds$$

$$+\int_0^t \frac{(t-s)^{\alpha+\sigma-1}}{\Gamma(\alpha+\sigma)} f_1(s,u(s),v(s))\,ds + a, \tag{3.3}$$

and

$$\phi_2(u,v)(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\varphi_2(s)\, f_2(s,u(s),v(s))\,ds$$

$$+\int_0^t \frac{(t-s)^{\beta+\theta-1}}{\Gamma(\beta+\theta)} f_2(s,u(s),v(s))\,ds + b. \tag{3.4}$$

We shall show that T is a contraction.
Let $(u_1,v_1),(u_2,v_2)\in X\times X$. Then, for each $t\in[0,1]$, we have

$$\left|\phi_1(u_2,v_2)(t)-\phi_1(u_1,v_1)(t)\right| \le$$

$$\left(\int_0^t \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sup_{0\le s\le 1}\left|\varphi_1(s)\right|ds + \int_0^t \tfrac{(t-s)^{\alpha+\sigma-1}}{\Gamma(\alpha+\sigma)}ds\right)$$
$$\times \sup_{0\le s\le 1}\left|f_1(s,u_2(s),v_2(s)) - f_1(s,u_1(s),v_1(s))\right|. \tag{3.5}$$

For all $t\in[0,1]$, we get

$$\left|\phi_1(u_2,v_2)(t)-\phi_1(u_1,v_1)(t)\right| \le \left(\frac{\left\|\varphi_1\right\|_\infty}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+\sigma+1)}\right)$$

$$\times \sup_{0\le s\le 1}\left|f_1(s,u_2(s),v_2(s)) - f_1(s,u_1(s),v_1(s))\right|. \tag{3.6}$$

Using $(H1)$, we can write:

$$\left|\phi_1(u_2,v_2)(t)-\phi_1(u_1,v_1)(t)\right| \le M_1\left(m_1\left|u_2(t)-u_1(t)\right| + m_2\left|v_2(t)-v_1(t)\right|\right). \tag{3.7}$$

Thus,

$$\left|\phi_1(u_2,v_2)(t)-\phi_1(u_1,v_1)(t)\right| \le M_1(m_1+m_2)\left(\|u_2-u_1\|_X + \|v_2-v_1\|_X\right). \tag{3.8}$$

Consequently,

$$\left\|\phi_1(u_2,v_2)-\phi_1(u_1,v_1)\right\|_X \le M_1(m_1+m_2)\|(u_2-u_1,v_2-v_1)\|_{X\times X}. \tag{3.9}$$

With the same arguments as before, we get

$$\left\|\phi_2(u_2,v_2)-\phi_2(u_1,v_1)\right\|_X \le M_2(n_1+n_2)\|(u_2-u_1,v_2-v_1)\|_{X\times X}. \tag{3.10}$$

Finally, using (3.9) and (3.10), we deduce that

$$\left\|\phi(u_2,v_2)-\phi(u_1,v_1)\right\|_{X\times X} \le (M_1+M_2)(m_1+m_2+n_1+n_2)\|(u_2-u_1,v_2-v_1)\|_{X\times X}. \tag{3.11}$$

Thanks to (3.1), we conclude that T is a contraction mapping. Hence, by the Banach fixed point theorem, there exists a unique fixed point which is a solution of (1.1). □

The second result is the following theorem:

Theorem 3.2. *Assume that* $(H2)$ *and* $(H3)$ *are satisfied. Then the problem* (1.1) *has at least one solution on* $[0,1]$.

Proof. First of all, we show that the operator T is completely continuous. (Note that T is continuous on $X\times X$ in view of the continuity of f_1 and f_2).

Step 1: Let us take $\gamma>0$ and $B_\gamma := \{(u,v)\in X\times X; \|(u,v)\|_{X\times X}\le\gamma\}$, and assume that $(H3)$ holds. Then for $(u,v)\in B_\gamma$, we have

$$\begin{aligned}|T_1(u,v)(t)| \le{}& \frac{t^\alpha \sup_{0\le t\le1}|\varphi_1(t)|}{\Gamma(\alpha+1)}\sup_{0\le t\le1}\left|f_1(t,u(t),v(t))\right| \\ &+\frac{t^{\alpha+\sigma}}{\Gamma(\alpha+\sigma+1)}\sup_{0\le t\le1}\left|f_1(t,u(t),v(t))\right|+a.\end{aligned} \tag{3.12}$$

For all $t\in[0,1]$, and by $(H3)$, we obtain

$$\|T_1(u,v)\|_X \le L_1M_1+a<+\infty. \tag{3.13}$$

We have also

$$\|T_2(u,v)\|_X \le L_2M_2+b<+\infty. \tag{3.14}$$

Then, by (3.13) and (3.14), $\|T(u,v)\|_{X\times X}$ is bounded by C, where

$$C := L_1M_1+L_2M_2+a+b. \tag{3.15}$$

Step 2: The equi-continuity of T: Let $t_1,t_2\in[0,1], t_1<t_2$ and $(u,v)\in B_\gamma$. Since $0<\alpha<1$, then we can write

$$\begin{aligned}&|\,T_1(u,v)(t_2)-T_1(u,v)(t_1)\,|\le \\ &|\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_1(s)\,f_1(s,u(s),v(s))\,ds-\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_1(s)\,f_1(s,u(s),v(s))\,ds \\ &+\int_0^{t_2}\frac{(t_2-s)^{\alpha+\sigma-1}}{\Gamma(\alpha+\sigma)}f_1(s,u(s),v(s))\,ds-\int_0^{t_1}\frac{(t_1-s)^{\alpha+\sigma-1}}{\Gamma(\alpha+\sigma)}f_1(s,u(s),v(s))\,ds\,|\,.\end{aligned} \tag{3.16}$$

Using $(H3)$, we can write

$$| T_1(u,v)(t_2) - T_1(u,v)(t_1) | \leq \frac{L_1 \left\|\varphi_1\right\|_\infty \left(t_2^\alpha - t_1^\alpha + (t_2-t_1)^\alpha\right)}{\Gamma(\alpha+1)}$$

$$+ \frac{L_1 \left(t_2^{\alpha+\sigma} - t_1^{\alpha+\sigma} + (t_2-t_1)^{\alpha+\sigma}\right)}{\Gamma(\alpha+\sigma+1)}. \tag{3.17}$$

Analogously, we can obtain

$$|T_2(u,v)(t_2) - T_2(u,v)(t_1)| \leq \frac{L_1 \left\|\varphi_2\right\|_\infty \left(t_2^\beta - t_1^\beta + (t_2-t_1)^\beta\right)}{\Gamma(\beta+1)}$$

$$+ \frac{L_1 \left(t_2^{\beta+\theta} - t_1^{\beta+\theta} + (t_2-t_1)^{\beta+\theta}\right)}{\Gamma(\beta+\theta+1)}. \tag{3.18}$$

Thanks to (3.17) and (3.18), yields

$$|T(u,v)(t_2) - T(u,v)(t_1)| \leq \frac{L_1 \left\|\varphi_1\right\|_\infty \left(t_2^\alpha - t_1^\alpha + (t_2-t_1)^\alpha\right)}{\Gamma(\alpha+1)}$$

$$+ \frac{L_1 \left(t_2^{\alpha+\sigma} - t_1^{\alpha+\sigma} + (t_2-t_1)^{\alpha+\sigma}\right)}{\Gamma(\alpha+\sigma+1)}$$

$$+ \frac{L_1 \left\|\varphi_2\right\|_\infty \left(t_2^\beta - t_1^\beta + (t_2-t_1)^\beta\right)}{\Gamma(\beta+1)}$$

$$+ \frac{L_1 \left(t_2^{\beta+\theta} - t_1^{\beta+\theta} + (t_2-t_1)^{\beta+\theta}\right)}{\Gamma(\beta+\theta+1)}. \tag{3.19}$$

As $t_2 \to t_1$, the right-hand side of (3.19) tends to zero. Then, as a consequence of Steps $1,2$ and by Arzela-Ascoli theorem, we conclude that T is completely continuous.

Next, we consider the set

$$\Omega = \{(u,v) \in X \times X / (u,v) = \lambda T(u,v), 0 < \lambda < 1\}, \tag{3.20}$$

and show that it is bounded.
Let $(u,v) \in \Omega$, then $(u,v) = \lambda T(u,v)$, for some $0 < \lambda < 1$. Hence, for $t \in [0,1]$, we have:

$$u(t) = \lambda T_1(u,v)(t),\ v(t) = \lambda T_2(u,v)(t). \tag{3.21}$$

Thus,

$$\|(u,v)\|_{X\times X} = \lambda \|T(u,v)\|_{X\times X}. \tag{3.22}$$

Thanks to (H_3), we get

$$\|(u,v)\|_{X\times X} \le \lambda C, \tag{3.23}$$

where C is defined by (3.15). We obtain that Ω is bounded.

As a conclusion of the Schaefer fixed point theorem, we deduce that T has at least one fixed point, which is a solution of (1.1). □

4. Examples

To illustrate our results, we will present two examples.

Example 4.1. Consider the following fractional differential system:

$$\begin{cases} D^{\frac{1}{2}}u(t) = \frac{\exp(-t)}{32\sqrt{1+t}}\left(\frac{\sin(u(t)+v(t))}{18(\ln(t+1)+1)}+1\right) + \int_0^t \frac{(t-s)^{\frac{5}{2}}}{\Gamma\left(\frac{7}{2}\right)}\left(\frac{\sin(u(s)+v(s))}{18(\ln(s+1)+1)}+1\right)ds, t\in[0,1],\\ D^{\frac{1}{2}}v(t) = \frac{\exp\left(-t^2\right)}{32\sqrt{1+t^2}}\frac{\sin u(s)+\sin v(s)}{16(t\exp(t^2)+1)} + \int_0^t \frac{(t-s)^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)}\frac{\sin u(s)+\sin v(s)}{16(s\exp(s^2)+1)}ds, t\in[0,1],\\ u(0) = \sqrt{3}, v(0) = \sqrt{2}, \end{cases} \tag{4.1}$$

where, $\alpha = \beta = \frac{1}{2}, \sigma = \frac{7}{2}$, and $\theta = \frac{5}{2}, a = \sqrt{3}, b = \sqrt{2}$, $f_1(t,u,v) = \frac{\sin(u+v)}{32(\ln(t+1)+1)}+1$, $f_2(t,u,v) = \frac{\sin u+\sin v}{16(t\exp(t^2)+1)}$, $\varphi_1(t) = \frac{\exp(-t)}{32\sqrt{1+t}}$ and $\varphi_2(t) = \frac{\exp\left(-t^2\right)}{32\sqrt{1+t^2}}$.
For $(u_1,v_1),(u_2,v_2)\in\mathbb{R}^2, t\in[0,1]$, we have

$$\begin{aligned} \left|f_1(t,u_2,v_2) - f_1(t,u_1,v_1)\right| &\le \frac{1}{18}\left(|u_2-u_1|+|v_2-v_1|\right),\\ \left|f_2(t,u_2,v_2) - f_2(t,u_1,v_1)\right| &\le \frac{1}{16}\left(|u_2-u_1|+|v_2-v_1|\right). \end{aligned}$$

Then,

$$\begin{aligned} M_1 &= 0.076,\quad M_2 = 0.201,\\ m_1 &= m_2 = \frac{1}{18},\quad n_1 = n_2 = \frac{1}{16}. \end{aligned}$$

Hence,

$$(M_1+M_2)\sum_{i=1}^{2}(m_i+n_i) = 0.065 < 1.$$

The conditions of the Theorem 3.1 hold. Therefore, the problem (4.1) has a unique solution on $[0,1]$.

Example 4.2. Consider the following problem:

$$(4.2)\qquad \begin{cases} D^{\frac{3}{4}}u(t) = \cosh(1-\pi t^2)\cos(u(t)+v(t)) + \ln(t+4) \\ \quad + \int_0^t \frac{(t-s)^{\sqrt{11}-1}}{\Gamma(\sqrt{11})}\left[\cos(u(s)+v(s)) + \ln(s+4)\right] ds, t \in [0,1], \\ D^{\frac{5}{7}}v(t) = \sinh(1-\pi t^2)\, t \exp(-|u(t)| - |v(t)|) \\ \quad + \int_0^t \frac{(t-s)^{\sqrt{7}-1}}{\Gamma(\sqrt{7})} s \exp(-|u(s)| - |v(s)|) ds, t \in [0,1], \\ u(0) = 2, v(0) = \sqrt{5}. \end{cases}$$

For this example, we have $\alpha = \frac{3}{4}, \beta = \frac{5}{7}, \sigma = \sqrt{11}, \theta = \sqrt{7}, a = 2, b = \sqrt{5}$, and for all $t \in [0,1]$, we have $\varphi_1(t) = \cosh(1-\pi t^2), \varphi_2(t) = \sinh(1-\pi t^2)$, and for each $(u,v) \in \mathbb{R}^2$

$$\begin{aligned} f_1(t,u,v) &= \cos(u+v) + \ln(t+4), \\ f_2(t,u,v) &= t\exp(-|u| - |v|). \end{aligned}$$

It's clear that f_1 and f_2 are continuous and bounded functions. Thus the conditions of Theorem 3.2 hold, then the problem (4.2) has at least one solution on $[0,1]$.

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