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INTEGRAL INEQUALITIES FOR HARMONICALLY s-GODUNOVA-LEVIN FUNCTIONS

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Abstract. In this paper, some new classes of harmonically convex functions are introduced and investigated. We derive several Hermite-Hadamard inequalities for these new classes of harmonically convex functions. The ideas and techniques of this paper may be extended for other classes of convex functions.
Keywords: Harmonically convex functions; Hermite-Hadamard inequalities; Godunova-Levin functions.

1. Introduction

Convex functions and their generalized forms are used for solving different problems, see [1, 20] and the references therein. It is also known that convex functions are closely related to the theory of inequalities and many inequalities are consequences of the applications of convex functions, see [20]. One of the most extensively studied inequalities by researchers is the well-known Hermite-Hadamard inequality, which is a necessary and sufficient condition for a function to be convex.

For more details on Hermite-Hadamard type of integral inequalities, see [2-6,8-12,14-24].

Godunova and Levin [7] introduced a new class of convex functions, which is called as Godunova-Levin functions or Q class of functions. Iscan [8] introduced and studied another class of convex functions which is known as harmonically convex functions. Noor et al. [12] and Zhang et al. [21] have extended the concept of harmonically convex functions and obtained some results.
Motivated and inspired by the ongoing research in the field, we introduce and consider some new classes of harmonically Godunova-Levin functions. For these new classes of Godunova-Levin functions, we derive several new Hermite-Hadamard inequalities. This is the main motivation of this paper. It is an interesting problem

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to explore the application of harmonically Godunova-Levin functions, in various fields of pure and applied sciences.

2. Preliminaries

In this section, we discuss some previously known concepts and define new classes of Godunova-Levin functions. From now onward I is an interval included in $\mathbb{R}^* = (-\infty, 0) \bigcup (0, +\infty)$ unless otherwise specified.

Definition 2.1. [7] A function $f : I \to \mathbb{R}$ is said to be Godunova-Levin function, if f is nonnegative and

$$f((1-t)x + ty) \leq \frac{1}{1-t} f(x) + \frac{1}{t} f(y), \quad \forall x, y \in I, t \in (0,1). \tag{2.1}$$

Definition 2.2. [8] A function $f : I \to \mathbb{R}$ is said to be harmonically convex function, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq (1-t) f(x) + t f(y), \quad \forall x, y \in I, t \in [0,1]. \tag{2.2}$$

For some recent investigations, extensions and generalizations of harmonically convex function interested readers are referred to [9, 10, 16].
We now define some new classes of harmonically convex functions.

Definition 2.3. A function $f : I \to \mathbb{R}$ is said to be harmonically s-Godunova-Levin function of the first kind, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq \frac{1}{1-t^s} f(x) + \frac{1}{t^s} f(y), \ \forall x, y \in I, t \in (0,1), s \in (0,1]. \tag{2.3}$$

Definition 2.4. A function $f : I \to \mathbb{R}$ is said to be harmonically s-Godunova-Levin function of the second kind, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq \frac{1}{(1-t)^s} f(x) + \frac{1}{t^s} f(y), \ \forall x, y \in I, t \in (0,1), s \in [0,1]. \tag{2.4}$$

Definition 2.5. A function $f : I \to (0, \infty)$ is said to be harmonically $\log -s$-Godunova-Levin function of the first kind, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq f(x)^{\frac{1}{1-t^s}} f(y)^{\frac{1}{t^s}}, \quad \forall x, y \in I, t \in (0,1), s \in (0,1]. \tag{2.5}$$

From above inequality it follows that

$$\log f\left(\frac{xy}{tx + (1-t)y}\right) \leq \frac{1}{1-t^s} \log f(x) + \frac{1}{t^s} \log f(y) = \log \left[f(x)^{\frac{1}{1-t^s}} f(y)^{\frac{1}{t^s}} \right].$$

Definition 2.6. A function $f : I \to (0,\infty)$ is said to be harmonically $\log -s$-Godunova-Levin function of the second kind, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le f(x)^{\frac{1}{(1-t)^s}} f(y)^{\frac{1}{t^s}}, \quad \forall x,y \in I, t \in (0,1), s \in [0,1]. \tag{2.6}$$

From which it follows that

$$\log f\left(\frac{xy}{tx+(1-t)y}\right) \le \frac{1}{(1-t)^s}\log f(x) + \frac{1}{t^s}\log f(y) = \log\left[f(x)^{\frac{1}{(1-t)^s}} f(y)^{\frac{1}{t^s}}\right].$$

We now give the definition of hypergeometric series which will be used in the obtaining some integrals.

Definition 2.7. [13] For the real or complex numbers a, b, c, other than $0, -1, -2, \ldots$, the hypergeometric series is defined by

$${}_2F_1[a,b,c;z] = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \ldots = \sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m}\frac{z^m}{m!}.$$

Here $(\phi)_m$ is the Pochhammer symbol, which is defined by

$$(\phi)_m = \begin{cases} 1 & m = 0, \\ \phi(\phi+1)\ldots(\phi+m-1), & m > 0, \end{cases}$$

which has the integral form:

$${}_2F_1[a,b;c;z] = \frac{1}{\mathrm{B}(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,\mathrm{d}t,$$

where $|z| < 1, c > b > 0$ and

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t,$$

is Euler function Beta with

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

We need the following result, which is due to Iscan [8]. This result plays an important role in obtaining the main results of this paper.

Lemma 2.1. *[8] Let $f : I \to \mathbb{R}$ be a differentiable function on I^0 (interior of I) and $a, b \in I$ with $a < b$ and $ab > 0$. If $f' \in L[a,b]$, then*

$$\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x = \frac{ab(b-a)}{2}\int_0^1 \frac{1-2t}{(tb+(1-t)a)^2} f'\left(\frac{ab}{tb+(1-t)a}\right)\mathrm{d}t$$

3. Main Results

In this section, we establish our main results.

Theorem 3.1. *Let $f : I \to \mathbb{R}$ where $a, b \in I$, $a < b$ and $ab > 0$ be harmonically s-Godunova-Levin function of second kind. Then*

$$\frac{1}{2^{s+1}} f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \mathrm{d}x \leq \frac{f(a)+f(b)}{1-s}. \tag{3.1}$$

Proof. Since f is harmonically s-Godunova-Levin function of second kind, so, we have

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq \frac{1}{(1-t)^s} f(x) + \frac{1}{t^s} f(y).$$

For $t = \frac{1}{2}$, $x = \frac{ab}{ta+(1-t)b}$ and $y = \frac{ab}{tb+(1-t)a}$, we have

$$\frac{1}{2^s} f\left(\frac{2ab}{a+b}\right) \leq \left[f\left(\frac{ab}{ta+(1-t)b}\right) + f\left(\frac{ab}{tb+(1-t)a}\right)\right].$$

Integrating both sides of above inequality with respect to t on [0,1], we have

$$\frac{1}{2^{s+1}} f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \mathrm{d}x. \tag{3.2}$$

Also

$$f\left(\frac{ab}{ta+(1-t)b}\right) \leq \frac{1}{(1-t)^s} f(a) + \frac{1}{t^s} f(b).$$

Integrating both sides of above inequality with respect to t on [0,1], we have

$$\frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \mathrm{d}x \leq \frac{f(a)+f(b)}{1-s}. \tag{3.3}$$

Combining inequalities (3.2) and (3.3) completes the proof. □

Now using Lemma 2.1, we prove our next results.

Theorem 3.2. *Let $f : I \to \mathbb{R}$ be a differentiable function on I^0 where $a, b \in I^0$ with $a < b$, $ab > 0$ and $f' \in L[a, b]$. If $|f'|^q$ is harmonically s-Godunova-Levin function of second kind on I, then for $q \geq 1$, we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \mathrm{d}x\right| \leq \frac{ab(b-a)}{2} \mathcal{K}_1^{1-\frac{1}{q}} \left(\mathcal{K}_2 |f'(a)|^q + \mathcal{K}_3 |f'(b)|^q\right)^{\frac{1}{q}}, \tag{3.4}$$

where

$$\mathcal{K}_1 = \frac{1}{ab} - \frac{2}{(b-a)^2}\log\left(\frac{(a+b)^2}{4ab}\right), \tag{3.5}$$

$$\mathcal{K}_2 = \int_0^1 \frac{|1-2t|t^{-s}}{(tb+(1-t)a)^2}\,\mathrm{d}t, \tag{3.6}$$

and

$$\mathcal{K}_3 = \int_0^1 \frac{|1-2t|(1-t)^{-s}}{(tb+(1-t)a)^2}\,\mathrm{d}t, \tag{3.7}$$

respectively.

Proof. Using Lemma 2.1, power mean inequality and the fact that $|f'|^q$ is harmonically s-Godunova-Levin function of second kind, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x\right| \\
&\le \frac{ab(b-a)}{2}\int_0^1 \left|\frac{1-2t}{(tb+(1-t)a)^2}\right|\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|\mathrm{d}t \\
&\le \frac{ab(b-a)}{2}\left(\int_0^1 \left|\frac{1-2t}{(tb+(1-t)a)^2}\right|\mathrm{d}t\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\left|\frac{1-2t}{(tb+(1-t)a)^2}\right|\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q \mathrm{d}t\right)^{\frac{1}{q}} \\
&\le \frac{ab(b-a)}{2}\left(\int_0^1 \left|\frac{1-2t}{(tb+(1-t)a)^2}\right|\mathrm{d}t\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\frac{|1-2t|}{(tb+(1-t)a)^2}\left[\frac{1}{t^s}|f'(a)|^q + \frac{1}{(1-t)^s}|f'(b)|^q\right]\mathrm{d}t\right)^{\frac{1}{q}} \\
&= \frac{ab(b-a)}{2}\mathcal{K}_1^{1-\frac{1}{q}}\left(\mathcal{K}_2|f'(a)|^q + \mathcal{K}_3|f'(b)|^q\right)^{\frac{1}{q}},
\end{aligned}$$

where

$$\mathcal{K}_1 = \int_0^1 \frac{|1-2t|}{(tb+(1-t)a)^2}\,\mathrm{d}t = \frac{1}{ab} - \frac{2}{(b-a)^2}\log\left(\frac{(a+b)^2}{4ab}\right).$$

This completes the proof. □

Corollary 3.1. *Under the assumptions of Theorem 3.2 and for $q = 1$, we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right| \le \frac{ab(b-a)}{2}\left(\mathcal{K}_2|f'(a)| + \mathcal{K}_3|f'(b)|\right),$$

where $\mathcal{K}_2, \mathcal{K}_3$ are given by (3.6) *and* (3.7) *respectively.*

Theorem 3.3. *Let $f : I \to \mathbb{R}$ be a differentiable function on I^0 where $a, b \in I^0$ with $a < b$, $ab > 0$ and $f' \in L[a, b]$, if $|f'|^q$ is harmonically s-Godunova-Levin function of second kind on I where $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right| \le \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\mathcal{K}_4|f'(a)|^q + \mathcal{K}_5|f'(b)|^q\right)^{\frac{1}{q}},$$

where

$$\mathcal{K}_4 = \int_0^1 \frac{t^{-s}}{(tb+(1-t)a)^{2q}}dt = \frac{b^{-2q}}{1-s}\,{}_2F_1\left[2q, 1; 2-s; 1-\frac{a}{b}\right], \tag{3.8}$$

and

$$\mathcal{K}_5 = \int_0^1 \frac{(1-t)^{-s}}{(tb+(1-t)a)^{2q}}dt = \frac{b^{-2q}}{1-s}\,{}_2F_1\left[2q, 1-s; 2-s; 1-\frac{a}{b}\right], \tag{3.9}$$

respectively.

Proof. Using Lemma 2.1, Holder's inequality and the fact that $|f'|^q$ is harmonically s-Godunova-Levin function of second kind, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right| \\
&\le \frac{ab(b-a)}{2}\int_0^1 \left|\frac{1-2t}{(tb+(1-t)a)^2}\right|\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|\mathrm{d}t \\
&\le \frac{ab(b-a)}{2}\left(\int_0^1 |1-2t|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1 \frac{1}{(tb+(1-t)a)^{2q}}\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&\le \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 \frac{1}{(tb+(1-t)a)^{2q}}\left\{\frac{1}{t^s}|f'(a)|^q + \frac{1}{(1-t)^s}|f'(b)|^q\right\}\mathrm{d}t\right)^{\frac{1}{q}} \\
&= \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\mathcal{K}_4|f'(a)|^q + \mathcal{K}_5|f'(b)|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

Theorem 3.4. *Let* $f : I = [a,b] \subset \mathbb{R} \to (0,\infty)$ *be harmonically* $\log -s$*-Godunova-Levin function of second kind, where* $a < b$ *and* $ab > 0$*. Then*

$$f\left(\frac{2ab}{a+b}\right)^{\frac{1}{2^{s+1}}} \le \exp\left[\frac{ab}{b-a}\int_a^b \frac{\log f(x)}{x^2}\mathrm{d}x\right] \le (f(a)f(b))^{\frac{1}{1-s}}.$$

Proof. Since f is harmonically $\log -s$-Godunova-Levin function of second kind, so, we have

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le f(x)^{\frac{1}{(1-t)^s}} f(y)^{\frac{1}{t^s}}.$$

This implies that

$$\log f\left(\frac{xy}{tx+(1-t)y}\right) \le \frac{1}{(1-t)^s}\log f(x) + \frac{1}{t^s}\log f(y).$$

For $t = \frac{1}{2}$, $x = \frac{ab}{ta+(1-t)b}$ and $y = \frac{ab}{tb+(1-t)a}$, we have

$$\frac{1}{2^s}\log f\left(\frac{2ab}{a+b}\right) \le \left[\log f\left(\frac{ab}{ta+(1-t)b}\right) + \log f\left(\frac{ab}{tb+(1-t)a}\right)\right].$$

Integrating both sides of above inequality with respect to t on [0,1], we have

$$\frac{1}{2^{s+1}}\log f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a}\int_a^b \frac{\log f(x)}{x^2}\mathrm{d}x. \tag{3.10}$$

Also

$$\log f\left(\frac{ab}{ta+(1-t)b}\right) \le \frac{1}{(1-t)^s}\log f(a) + \frac{1}{t^s}\log f(b).$$

Integrating both sides of above inequality with respect to t on [0,1], we have

$$\frac{ab}{b-a}\int_a^b \frac{\log f(x)}{x^2}\mathrm{d}x \le \frac{\log(f(a)f(b))}{1-s}. \tag{3.11}$$

Combining inequalities (3.10) and (3.11), we have

$$\log f\left(\frac{2ab}{a+b}\right)^{\frac{1}{2^{s+1}}} \le \frac{ab}{b-a}\int_a^b \frac{\log f(x)}{x^2}\mathrm{d}x \le \log(f(a)f(b))^{\frac{1}{1-s}}.$$

Taking antilog on each side of the above inequality, we have

$$f\left(\frac{2ab}{a+b}\right)^{\frac{1}{2^{s+1}}} \leq \exp\left[\frac{ab}{b-a}\int_a^b \frac{\log f(x)}{x^2}\mathrm{d}x\right] \leq (f(a)f(b))^{\frac{1}{1-s}}.$$

This completes the proof. □

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