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ON CERTAIN HESSENBERG MATRICES RELATED WITH LINEAR RECURRENCES

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Abstract. In this paper, we present various results for permanents and determinants of some Hessenberg matrices. Also, some special cases for permanents are given.

Keywords: Hessenberg matrices, permanents, determinants.

1. Introduction

Matrix methods are useful tools deriving some properties of linear recurrences. Some authors obtained many connections between certain sequences and permanents of Hessenberg matrices in the literature [1]-[4],[6],[10]-[12].

The permanent of an n -square matrix $\mathbf{A}_n = [a_{ij}]$ is defined by

$$\text{per} \mathbf{A}_n = \sum_{\sigma \in D_n} \prod_{i=1}^n a_{i\sigma(i)},$$

where the summation extends over all permutations σ of the symmetric group D_n .

In [9], Minc defined the super-diagonal matrix and showed that the permanent of the matrix equals the order k -Fibonacci number.

In [5], Kılıç derived recurrence relations and generating matrices for the sums of usual tribonacci numbers and $4n$ subscripted tribonacci sequences $\{T_{4n}\}$, and their sums. Also, the relationships between these sequences and permanents of certain matrices are obtained.

In [6], Kılıç and Taşcı found the relationships between the sums of the Fibonacci and Lucas numbers and 1-factors of bipartite graphs.

In [7], Kılıç and Taşcı defined the $n \times n$ tridiagonal Toeplitz $(0, -1, 1)$ -matrix $\mathbf{M}_n = [m_{i,j}]$ with $m_{i,i} = -1$ for $1 \leq i \leq n$, $m_{i,i+1} = m_{i+1,i} = 1$ for $1 \leq i \leq n - 1$

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and 0 otherwise, and the $n \times n$ tridiagonal Toeplitz $(0, -1, 1)$ -matrix $\mathbf{L}_n = [l_{i,j}]$ with $l_{i,i} = -1$ for $2 \leq i \leq n$, $l_{i,i+1} = l_{i+1,i} = 1$ for $1 \leq i \leq n - 1$, $l_{1,1} = -\frac{1}{2}$ and 0 otherwise. They showed $\text{per}\mathbf{M}_n = F_{-(n+1)}$ and $\text{per}\mathbf{L}_n = \frac{L_{-n}}{2}$, where F_n and L_n is the n th Fibonacci and Lucas numbers, respectively.

In [8], Li showed new Fibonacci-Hessenberg matrices and gave another proof of the well-known results relative to the Pell and Perrin numbers.

In [3], Kalman showed that the $(n+k)$ -th term of a sequence is defined recursively as a linear combination of the preceding k terms:

$$(1.1) \quad u_{n+k} = c_0u_n + c_1u_{n+1} + \dots + c_{k-1}u_{n+k-1}$$

in which the initial terms $u_0 = \dots = u_{k-2} = 0, u_{k-1} = 1$ and c_0, c_1, \dots, c_{k-1} are constants.

In [10], considering the generalized Fibonacci-Narayana sequence $\{G_n(a, c, r)\}$, Ramírez derived some relations between this sequence and a permanent of one type of the upper Hessenberg matrix. For example,

$$\text{per} \begin{bmatrix} a & c & c & \cdots & c & & & 0 \\ 1 & a & 0 & 0 & \cdots & c & & \\ & \ddots & \ddots & \ddots & \ddots & & \ddots & \\ & & 1 & a & 0 & 0 & \cdots & c \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & 1 & a & 0 & 0 \\ & & & & & 1 & a & 0 \\ & & & & & & 1 & a \end{bmatrix} = G_{n+r-1}(a, c, r),$$

where the generalized Fibonacci-Narayana sequence $\{G_n(a, c, r)\}_{n \in \mathbb{N}}$ is defined as follows:

$$G_n(a, c, r) = aG_{n-1}(a, c, r) + cG_{n-r}(a, c, r), \quad 2 \leq r \leq n,$$

with the initial conditions $G_0(a, c, r) = 0, G_i(a, c, r) = 1$, for $i = 1, 2, \dots, r - 1$.

In [12], Trojovský defined tridiagonal matrices $\mathbf{B}_n^\delta = [b_{ij}^\delta]$ in the form

$$\begin{cases} 1 & \text{if } i = j \text{ or } i = j - 1, \\ (-1)^{j+\delta} & \text{if } i = j + 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\delta \in \{0, 1\}$ and showed

$$\det \mathbf{B}_n^\delta = \begin{cases} F_{(n+4-6\delta)/2} & \text{if } n \equiv 0 \pmod{2}, \\ F_{(n+1)/2} & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

2. Some results

In this section, we define the sequence $\{R_n(a, b, c, d)\}$ and determine some relationships between the terms of this sequence and permanents of certain upper Hessen-

berg matrices. A sequence $\{R_n(a, b, c, d)\}$ is defined by for $3 \leq d \leq n$,

$$(2.1) \quad R_n(a, b, c, d) = aR_{n-1}(a, b, c, d) + bR_{n-2}(a, b, c, d) + cR_{n-d}(a, b, c, d),$$

in which $R_0(a, b, c, d) = 0, R_1(a, b, c, d) = R_2(a, b, c, d) = \dots = R_{d-2}(a, b, c, d) = 1$ and $R_{d-1}(a, b, c, d) = a$. The sequence $\{R_n(a, b, c, d)\}$ is a generalization of the tribonacci sequence. When $a = b = c = 1$ and $d = 3, R_n(1, 1, 1, 3) = T_n$ (the n th tribonacci number). If $c = 0$ and $d = 3$, the generalized Fibonacci sequence $\{U_n(a, b)\}$ is obtained. If $a = b = 1, c = 0$ and $d = 3$, the Fibonacci sequence $\{F_n\}$ is obtained and if $a = c = 1, b = 0$ and $d = 3$, the Narayana sequence is obtained.

The generating function $R(z)$ of $R_n(a, b, c, d)$ is given by

$$R(z) = \frac{(a - 1 + bz)z^{d-1} - bz^3 - az^2 + z}{(1 - z)(1 - az - bz^2 - cz^d)}.$$

Now we give relationships between terms of the sequence $\{R_n(a, b, c, d)\}$ and the permanents of certain matrices.

For $n \geq 1$, define a $n \times n$ matrix $\mathbf{H}_n(a, b, c, d, k, t) = [h_{i,j}]$ with $h_{i+1,i} = 1$ for $1 \leq i \leq n - 2, h_{i,i} = a$ for $1 \leq i \leq n - 1, h_{i,i+1} = b$ for $1 \leq i \leq n - 1, h_{1,i} = c$ for $3 \leq i \leq d, h_{i,d+i-1} = c$ for $2 \leq i \leq n - d + 1, h_{n,n-1} = k, h_{n,n} = t$, and 0 otherwise, i.e.,

$$(2.2) \quad \mathbf{H}_n(a, b, c, d, k, t) = \begin{bmatrix} a & b & c & \dots & c & 0 & \dots & 0 \\ 1 & a & b & 0 & \dots & c & \dots & \vdots \\ & & \ddots & \ddots & \ddots & \dots & \ddots & 0 \\ & & & 1 & a & b & 0 & \dots & c \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & & & \ddots & \ddots & \ddots & 0 \\ & & & & & & 1 & a & b \\ 0 & & & & & & & k & t \end{bmatrix}.$$

Then we give the following Theorem.

Theorem 2.1. *Let $\mathbf{H}_n(a, b, c, d, k, t)$ be the matrix defined in (2.2). Then, for $n \geq 1$ and $d \geq 3$,*

$$(2.3) \quad \text{per}\mathbf{H}_n(a, b, c, d, k, t) = kR_{n+d-2}(a, b, c, d) - (ka - t)R_{n+d-3}(a, b, c, d),$$

where the real numbers k and t .

Proof. (Induction on n) If $n = 1$, then we have

$$\text{per}\mathbf{H}_1(a, b, c, d, k, t) = t = kR_{d-1}(a, b, c, d) - (ka - t)R_{d-2}(a, b, c, d).$$

Suppose that the equation holds for $n - 1$. Then we show that the equation holds for n . Expanding the $\text{per}\mathbf{H}_n$ with respect to the last column d times, we write

$$\begin{aligned} & \text{per}\mathbf{H}_n(a, b, c, d, k, t) \\ &= a\text{per}\mathbf{H}_{n-1}(a, b, c, d, k, t) + b\text{per}\mathbf{H}_{n-2}(a, b, c, d, k, t) + c\text{per}\mathbf{H}_{n-d}(a, b, c, d, k, t). \end{aligned}$$

By our assumption, we have

$$\begin{aligned} \text{per}\mathbf{H}_n(a, b, c, d, k, t) &= a(kR_{n+d-3}(a, b, c, d) - (ka - t)R_{n+d-4}(a, b, c, d)) \\ &\quad + b(kR_{n+d-4}(a, b, c, d) - (ka - t)R_{n+d-5}(a, b, c, d)) \\ &\quad + c(kR_{n-2}(a, b, c, d) - (ka - t)R_{n-3}(a, b, c, d)) \\ &= kR_{n+d-2}(a, b, c, d) - (ka - t)R_{n+d-3}(a, b, c, d). \end{aligned}$$

Thus, the proof is complete. \square

When $t = a$ and $k = 1$ in (2.3), we have $\text{per}\mathbf{H}_n(a, b, c, d, 1, a) = R_{n+d-2}(a, b, c, d)$.

For $n \geq 1$, define a $n \times n$ matrix $\mathbf{E}_n(a, b, c, d, k, t) = [e_{i,j}]$ with $e_{i+1,i} = -1$ for $1 \leq i \leq n - 2$, $e_{i,i} = a$ for $1 \leq i \leq n - 1$, $e_{i,i+1} = b$ for $1 \leq i \leq n - 1$, $e_{1,i} = c$ for $3 \leq i \leq d$, $e_{i,d+i-1} = c$ for $2 \leq i \leq n - d + 1$, $e_{n,n-1} = -k$, $e_{n,n} = t$, and 0 otherwise, i.e.,

$$\mathbf{E}_n(a, b, c, d, k, t) = \begin{bmatrix} a & b & c & \cdots & c & 0 & \cdots & 0 \\ -1 & a & b & 0 & \cdots & c & \cdots & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \cdots & \ddots & 0 \\ & & -1 & a & b & 0 & \cdots & c \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots & \ddots & 0 \\ & & & & & -1 & a & b \\ 0 & & & & & & -k & t \end{bmatrix}.$$

It is clearly showed from [2] that

$$\det \mathbf{E}_n(a, b, c, d, k, t) = \text{per}\mathbf{H}_n(a, b, c, d, k, t).$$

Now, we take the $n \times n$ matrix $\mathbf{H}_n(a, b, c, 3, k, t)$ by the following form:

$$\mathbf{H}_n(a, b, c, 3, k, t) = \begin{bmatrix} a & b & c & & & & & 0 \\ 1 & a & b & c & & & & \\ & \ddots & \ddots & \ddots & \ddots & & & \\ & & 1 & a & b & c & & \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & \ddots & c \\ & & & & & 1 & a & b \\ 0 & & & & & & k & t \end{bmatrix}.$$

Then, we have

$$(2.4) \quad \text{per}\mathbf{H}_n(a, b, c, 3, k, t) = kR_{n+1}(a, b, c, 3) - (ka - t) R_n(a, b, c, 3).$$

For example, from [9], for $a = b = c = t = k = 1$ in (2.4), we have that

$$\text{per}\mathbf{H}_n(1, 1, 1, 3, 1, 1) = T_{n+1} = \text{per}\mathbf{F}(n, 3),$$

where T_n is the n th tribonacci number.

For $n > 1$; we define an $n \times n$ matrix $\mathbf{W}_n(a, b, c, 3, k, t)$ as in the compact form, by the definition of $\mathbf{H}_n(a, b, c, 3, k, t)$;

$$(2.5) \quad \mathbf{W}_n(a, b, c, 3, k, t) = \begin{bmatrix} 1 & 1 & & \dots & & 1 \\ 1 & & & & & \\ 0 & & & & & \\ \vdots & & \mathbf{H}_{n-1}(a, b, c, 3, k, t) & & & \\ 0 & & & & & \end{bmatrix}.$$

Now, we have the following theorem:

Theorem 2.2. *Let $\mathbf{W}_n(a, b, c, 3, k, t)$ be the matrix defined in (2.5). Then, for $n > 2$*

$$\text{per}\mathbf{W}_n(a, b, c, 3, k, t) = k \sum_{i=1}^n R_i(a, b, c, 3) - (ka - t) \sum_{i=1}^n R_{i-1}(a, b, c, 3).$$

Proof. (Induction on n) If $n = 3$, we write

$$\begin{aligned} & \text{per}\mathbf{W}_3(a, b, c, 3, k, t) \\ &= k + t + at + bk = k \sum_{i=1}^3 R_i(a, b, c, 3) - (ka - t) \sum_{i=1}^3 R_{i-1}(a, b, c, 3). \end{aligned}$$

Suppose that the equation holds for n . Then we show that the equation holds for $n + 1$. From the definitions of matrices $\mathbf{H}_n(a, b, c, 3, k, t)$ and $\mathbf{W}_n(a, b, c, 3, k, t)$, expanding the $\text{per}\mathbf{W}_{n+1}(a, b, c, 3, k, t)$ with respect to the first column gives us

$$\text{per}\mathbf{W}_{n+1}(a, b, c, 3, k, t) = \text{per}\mathbf{H}_n(a, b, c, 3, k, t) + \text{per}\mathbf{W}_n(a, b, c, 3, k, t).$$

By our assumption and (2.4), we have

$$\begin{aligned} & \text{per}\mathbf{W}_{n+1}(a, b, c, 3, k, t) \\ &= kR_{n+1}(a, b, c, 3) - (ka - t) R_n(a, b, c, 3) \\ & \quad + k \sum_{i=1}^n R_i(a, b, c, 3) - (ka - t) \sum_{i=1}^n R_{i-1}(a, b, c, 3) \\ &= k \sum_{i=1}^{n+1} R_i(a, b, c, 3) - (ka - t) \sum_{i=1}^{n+1} R_{i-1}(a, b, c, 3). \end{aligned}$$

Thus the proof is obtained. \square

When $a = b = c = 1$ in (2.1), the sequence $\{R_n\}$, special case of the sequence $\{R_n(a, b, c, d)\}$, is defined by the recurrence

$$(2.6) \quad R_n = R_{n-1} + R_{n-2} + R_{n-d}, \quad 3 \leq d \leq n$$

in which $R_0 = 0, R_1 = R_2 = \dots = R_{d-2} = R_{d-1} = 1$ and especially from (1.1), the sequence $\{S_n\}$ is defined by

$$(2.7) \quad S_n = S_{n-1} + S_{n-2} + S_{n-d}, \quad 3 \leq d \leq n$$

in which $S_0 = S_1 = S_2 = \dots = S_{d-3} = 0$ and $S_{d-2} = S_{d-1} = 1$. For $d = 3$ in (2.6) and (2.7), the sequences $\{R_n\}$ and $\{S_n\}$ coincide as the tribonacci sequence.

For $n > 1$, define an $n \times n$ matrix $\mathbf{Z}_n = [z_{i,j}]$ with $z_{i+1,i} = 1$ for $1 \leq i \leq n - 1$, $z_{i,i} = 1$ for $1 \leq i \leq n$, $z_{1,2} = 0$, $z_{i,i+1} = 1$ for $2 \leq i \leq n - 1$, $z_{1,i} = 1$ for $3 \leq i \leq d$, $z_{i,d+i-1} = 1$ for $2 \leq i \leq n - d + 1$ and 0 otherwise, i.e.,

$$(2.8) \quad \mathbf{Z}_n = \begin{bmatrix} 1 & 0 & 1 & \cdots & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 1 & \dots & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \dots & \ddots & 0 \\ & & & 1 & 1 & 1 & 0 & \cdots & 1 \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & & & \ddots & \ddots & \ddots & 0 \\ & & & & & & 1 & 1 & 1 \\ 0 & & & & & & & 1 & 1 \end{bmatrix}.$$

Then we give the following Theorem.

Theorem 2.3. *Let \mathbf{Z}_n be the matrix defined in (2.8). Then, for $n \geq 5$ and $d \geq 3$,*

$$per \mathbf{Z}_n = R_{n+d-3} + R_{n+d-4} - R_{n+d-5} + S_{n-5}.$$

Proof. We prove this by induction on n . For $n = 5$, we write

$$per \mathbf{Z}_5 = 2R_{d+1} + 1 = R_{d+2} + R_{d+1} - R_d + S_0.$$

The claim is true for $n = 5$. Assume that the claim is true for $n - 1$. Thus we show that the claim is true for n . Expanding the $per \mathbf{Z}_n$ according to the last column d times, we have

$$per \mathbf{Z}_n = per \mathbf{Z}_{n-1} + per \mathbf{Z}_{n-2} + per \mathbf{Z}_{n-d}.$$

By our assumption, we have

$$\begin{aligned} per \mathbf{Z}_n &= R_{n+d-4} + R_{n+d-5} - R_{n+d-6} + S_{n-6} \\ &\quad + R_{n+d-5} + R_{n+d-6} - R_{n+d-7} + S_{n-7} \\ &\quad + R_{n-3} + R_{n-4} - R_{n-5} + S_{n-d-5}. \end{aligned}$$

From the sequences $\{R_n\}$ and $\{S_n\}$, we write

$$\text{per} \mathbf{Z}_n = R_{n+d-3} + R_{n+d-4} - R_{n+d-5} + S_{n-5}.$$

So the proof is complete. \square

For $n \geq 1$, define the $n \times n$ matrix $\mathbf{V}_n^\delta = [v_{i,j}]$ with $v_{i+1,i} = 1$ for $1 \leq i \leq n-2$, $v_{i,i} = 1$ for $1 \leq i \leq n-1$, $v_{1,i+1} = v_{i,i+1} = (-1)^{i-\delta}$ for $1 \leq i \leq n$ and 0 otherwise, where $\delta \in \{0, 1\}$. i.e.,

$$(2.9) \quad \mathbf{V}_n^\delta = \begin{bmatrix} 1 & (-1)^{1-\delta} & (-1)^{2-\delta} & \dots & & (-1)^{n-2-\delta} & (-1)^{n-1-\delta} \\ 1 & 1 & (-1)^{2-\delta} & 0 & \dots & & \dots \\ 0 & \ddots & \ddots & \ddots & \ddots & \dots & \ddots \\ & & 1 & 1 & & 0 & \dots \\ & & & \ddots & \ddots & (-1)^{n-2-\delta} & \ddots \\ & & & & \ddots & 1 & (-1)^{n-1-\delta} \\ 0 & & & & & 1 & 1 \end{bmatrix}.$$

Theorem 2.4. *Let \mathbf{V}_n^δ be the matrix defined in (2.9). Then, for $n \geq 1$,*

$$\det \mathbf{V}_n^\delta = \begin{cases} (-1)^\delta \left(F_{(n+5+3(-1)^\delta)/2} - 2 - (-1)^\delta \right) & \text{if } n \equiv 0 \pmod{2}, \\ (-1)^\delta \left(F_{(n+5)/2} - 2 + (-1)^\delta \right) & \text{if } n \equiv 1 \pmod{2}, \end{cases}$$

where F_n is the n th Fibonacci number.

Proof. For $n = 1$, $\det \mathbf{V}_1^0 = 1 = (F_3 - 1)$, $\det \mathbf{V}_1^1 = 1 = -(F_3 - 3)$ and for $n = 2$, $\det \mathbf{V}_2^1 = 0 = -(F_2 - 1)$, $\det \mathbf{V}_2^0 = 2 = (F_5 - 3)$.

We show that the claim is true for $n - 1$. Using expansion on the first column of $\det \mathbf{V}_n^1$, we get as follows

$$\det \mathbf{V}_n^1 = \det \mathbf{B}_{n-1}^0 - \det \mathbf{V}_{n-1}^0.$$

From (1.2) and the induction hypothesis, we have

$$\begin{aligned} \det \mathbf{V}_n^1 &= \begin{cases} F_{n/2} - (F_{(n+4)/2} - 1) & \text{if } n \equiv 0 \pmod{2}, \\ F_{(n+3)/2} - (F_{(n+7)/2} - 3) & \text{if } n \equiv 1 \pmod{2}, \end{cases} \\ &= \begin{cases} -(F_{(n+2)/2} - 1) & \text{if } n \equiv 0 \pmod{2}, \\ -(F_{(n+5)/2} - 3) & \text{if } n \equiv 1 \pmod{2}. \end{cases} \end{aligned}$$

Similarly, using $\det \mathbf{V}_n^0 = \det \mathbf{B}_{n-1}^1 + \det \mathbf{B}_{n-2}^0 + \det \mathbf{V}_{n-2}^0$, the desired result is given. We have the proof. \square

Theorem 2.5. For $n \geq 1$, we have

$$\text{per}\mathbf{V}_n^1 = \begin{cases} L_{(n+2)/2} - 1 & \text{if } n \equiv 0 \pmod{2}, \\ L_{(n-1)/2} - 1 & \text{if } n \equiv 1 \pmod{2}, \end{cases}$$

and

$$\text{per}\mathbf{V}_n^0 = \begin{cases} F_{(n+2)/2} - 1 & \text{if } n \equiv 0 \pmod{2}, \\ F_{(n+5)/2} - 1 & \text{if } n \equiv 1 \pmod{2}, \end{cases}$$

where F_n is the n th Fibonacci number and L_n is the n th Lucas number.

Proof. Considering $\text{per}\mathbf{B}_n^\delta = \begin{cases} F_{(n-2+6\delta)/2} & \text{if } n \equiv 0 \pmod{2}, \\ F_{(n+1)/2} & \text{if } n \equiv 1 \pmod{2}, \end{cases}$ for $\delta \in \{0, 1\}$ and the equalities

$$\text{per}\mathbf{V}_n^1 = \text{per}\mathbf{B}_{n-1}^0 + \text{per}\mathbf{V}_{n-1}^0 \quad \text{and} \quad \text{per}\mathbf{V}_n^0 = \text{per}\mathbf{B}_{n-1}^1 - \text{per}\mathbf{B}_{n-2}^0 + \text{per}\mathbf{V}_{n-2}^0,$$

we have the proof from induction on n . \square

3. Some special cases

In this section, we give some special cases of the above theorems:

- For $b = 1$, $c = 0$ and $d = 3$, the generalized Fibonacci sequence $\{U_n(a, 1)\}$,

$$\begin{aligned} & \text{per}\mathbf{H}_n(a, 1, 0, 3, k, t) \\ = & \text{per} \begin{bmatrix} a & 1 & 0 & \cdots & & \\ 1 & a & 1 & 0 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & a & 1 \\ & & & & k & t \end{bmatrix} = tU_n(a, 1) + kU_{n-1}(a, 1), \end{aligned}$$

$$\begin{aligned} & \text{per}\mathbf{W}_n(a, 1, 0, 3, k, t) \\ = & \text{per} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & & & & \\ 0 & & H_{n-1} & & \\ \vdots & & & & \\ & & & & \end{bmatrix} \\ = & \frac{1}{a} (tU_n(a, 1) + (k+t)U_{n-1}(a, 1) + kU_{n-2}(a, 1) + k(a-1) - t), \end{aligned}$$

and then,

$$\text{per}\mathbf{W}_n(a, 1, 0, 3, k, t) = \text{per}\mathbf{H}_n(a, 1, 0, 3, k, t) + \text{per}\mathbf{H}_{n-1}(a, 1, 0, 3, k, t) + k(a-1) + t.$$

- For $a = 1, b = 2, c = 0$ and $d = 3$ in (2.1), $\{J_n\}$ is the Jacobsthal sequence and for $n \geq 2$,

$$\begin{aligned} & \text{per} \mathbf{H}_n(1, 2, 0, 3, k, t) \\ = & \text{per} \begin{bmatrix} 1 & 2 & 0 & & & & \\ 1 & 1 & 2 & 0 & & & \\ & \ddots & \ddots & \ddots & \ddots & \cdots & \\ & & & & 1 & 1 & 2 \\ & & & & & k & t \end{bmatrix} = tJ_n + 2kJ_{n-1}. \end{aligned}$$

- For $a = b = k = 1, c = 0$ and $d = 3$ in (2.1), $\{F_n\}$ is the Fibonacci sequence,

$$\begin{aligned} & \text{per} \mathbf{H}_n(1, 1, 0, 3, 1, t) \\ = & \text{per} \begin{bmatrix} 1 & 1 & 0 & & & & \\ 1 & 1 & 1 & 0 & & & \\ & \ddots & \ddots & \ddots & \ddots & \cdots & \\ & & & & 1 & 1 & 1 \\ & & & & & 1 & t \end{bmatrix} = tF_n + F_{n-1} \end{aligned}$$

and

$$\text{per} \mathbf{W}_n(1, 1, 0, 3, 1, t) = \text{per} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & & & & \\ 0 & & H_{n-1} & & \\ \vdots & & & & \\ & & & & \end{bmatrix} = t(F_{n+1} - 1) + F_n.$$

- For $d = 3$ in (2.6) and (2.7), $\{T_n\}$ is the tribonacci sequence and

$$\text{per} \mathbf{Z}_n = \text{per} \begin{bmatrix} 1 & 0 & 1 & & & & \\ 1 & 1 & 1 & 1 & & & \\ & \ddots & \ddots & \ddots & \ddots & \cdots & \\ & & & & 1 & 1 & 1 \\ & & & & & 1 & 1 \end{bmatrix} = 2T_{n-1} + T_{n-3} + T_{n-5}.$$

REFERENCES

1. R. A. BRUALDI and P. M. GIBSON: *Convex Polyhedra of Doubly Stochastic Matrices I: Applications of the Permanents Function*. J. Combin. Theory, Series A **22** (1977), 194-230.

2. N. D. CAHILL, J. R. D'ERRICO, D. A. NARAYAN and J. Y. NARAYAN: *Fibonacci determinants*. The College Math. Journal **33** (2002), no. 3, 221–225.
3. D. KALMAN: *Generalized Fibonacci numbers by matrix methods*. Fibonacci Quart. **20** (1982), no.1, 73–76.
4. E. KILIÇ and D. TAŞCI: *On the permanents of some tridiagonal matrices with applications to the Fibonacci and Lucas numbers*. Rocky Mountain J. Math. **37** (2007), no. 6, 203–219.
5. E. KILIÇ: *Tribonacci sequences with certain indices and their sums*. Ars Combin. **86** (2008), 13–22.
6. E. KILIÇ and D. TAŞCI: *On families of bipartite graphs associated with sums of Fibonacci and Lucas numbers*. Ars Combin. **89** (2008), 31–40.
7. E. KILIÇ and D. TAŞCI: *Negatively subscripted Fibonacci and Lucas numbers and their complex factorizations*. Ars Combin. **96** (2010), 275–288
8. H-C. LI: *On Fibonacci-Hessenberg matrices and the Pell and Perrin numbers*. Appl. Math. Comput. **218** (2012), 8353–8358.
9. H. MINC: *Permanents of $(0,1)$ -circulants*. Canadian Math. Bulletin **7** (1964), 253–263.
10. J. L. RAMÍREZ: *Hessenberg matrices and the generalized Fibonacci-Narayana sequence*. Filomat **29** (2015), no. 7, 1557–1563.
11. S. KOPARAL, N. ÖMÜR AND C. D. ÇOLAK : *Some permanents of Hessenberg matrices*. Accepted in Filomat.
12. P. TROJOVSKÝ: *On determinants of tridiagonal matrices with $(-1,1)$ -diagonal or superdiagonal in relation to Fibonacci numbers*. Global Journal of Pure and Applied Mathematics **12** (2016), no. 2, 1885–1892.

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