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COMMON FIXED POINTS OF A PAIR OF SELFMAPS SATISFYING CERTAIN WEAKLY CONTRACTIVE INEQUALITY INVOLVING RATIONAL TYPE EXPRESSIONS VIA TWO AUXILIARY FUNCTIONS IN PARTIALLY ORDERED METRIC SPACES

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Abstract. In this paper, we prove the existence of coincidence and common fixed points of a pair of selfmaps satisfying a certain weakly contractive inequality with two auxiliary functions involving rational type expressions in partially ordered metric spaces. These results extend some of the known existing results in the literature from a single selfmap to a pair of selfmaps. Examples are provided in support of our results.
Keywords: common fixed points, partially ordered metric spaces, rational type contraction mappings, auxiliary functions

1. Introduction

The Banach contraction principle is one of the pivotal results in fixed point theory. It is a very popular tool for solving existence problems in many different fields of mathematics. Ran and Reurings [15] extended the Banach contraction principle in partially ordered sets. For more work on the existence of fixed points in partially ordered metric spaces, we refer the reader to [1, 3, 7, 8, 9, 13, 16].

In 1975, Dass and Gupta [6] extended the Banach contraction principle through rational expression as follows.

Theorem 1.1. (Dass and Gupta [6]). Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that there exist $\alpha,\ \beta \geq 0$ with $\alpha + \beta < 1$ satisfying

$$d(Tx, Ty) \leq \alpha \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y) \tag{1.1}$$

for all $x,\ y\ \in X$.
Then T has a unique fixed point.

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Definition 1.1. Let $(X, \preceq)$ be a partially ordered set. A mapping $T : X \to X$ is said to be non-decreasing if for any $x, y \in X$, $x \preceq y$ implies that $Tx \preceq Ty$.

In 2013, Cabrera, Harjani and Sadarangani [4] proved the above theorem in the context of partially ordered metric spaces as follows.

Theorem 1.2. (Cabrera, Harjani and Sadarangani [4]) Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and non-decreasing mapping such that (1.1) is satisfied for all $x,\ y\ \in X$ with $x \preceq y$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point.

Theorem 1.3. (Cabrera, Harjani and Sadarangani [4]) Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Assume that if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$ then $x_n \preceq x$ for all $n \in N$. Let $T : X \to X$ be a non-decreasing mapping such that (1.1) is satisfied for all $x,\ y\ \in X$ with $x \preceq y$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$ then T has a fixed point.

Theorem 1.4. (Cabrera, Harjani and Sadarangani [4]) In addition to the hypotheses of Theorem 1.2 (Theorem 1.3), suppose that for every $x,\ y \in X$, there exists $u \in X$ such that $u \preceq x$ and $u \preceq y$. Then T has a unique fixed point.

We write
$\Phi = \{\varphi : [0,\ \infty) \to [0,\ \infty) : \varphi$ is monotonic non-decreasing, continuous and $\varphi(t) = 0 \Leftrightarrow t = 0\}$.
$\Psi = \{\psi : [0,\ \infty) \to [0,\ \infty) :$ for any sequence $\{t_n\}$ in $[0, \infty)$ with $t_n \to t > 0$ implies that $\underline{\lim}\psi(t_n) > 0\}$.

Remark 1.1. If $\psi \in \Psi$ then $\psi(t) > 0$ for $t > 0$.

Remark 1.2. If $t_n \to t$ and $\psi(t_n) \to 0$ implies that $t = 0$.

In 2014, Chandok, Choudhury and Metiya [5] improved Theorem 1.2 and Theorem 1.3 by using the functions of Φ and Ψ.

Theorem 1.5. (Chandok, Choudhury and Metiya [5]) Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and non-decreasing mapping such that for all $x,\ y\ \in X$ with $x \preceq y$,

$$\varphi(d(Tx, Ty)) \leq \varphi(M(x, y)) - \ \psi(N(x, y)) \tag{1.2}$$

for some $\varphi \in \Phi$ and $\psi \in \Psi$, where
$M(x, y) = \max\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)},\ \frac{d(y,Tx)[1+d(x,Ty\})]}{1+d(x,y)}\ ,\ d(x, y)\}$ and
$N(x, y) = \max\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)},\ d(x, y)\}$.
If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point.

Theorem 1.6. (Chandok, Choudhury and Metiya [5]) Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Assume that if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$ then $x_n \preceq x$ for all $n \in N$. Let $T : X \to X$ be a non-decreasing mapping. Suppose that (1.2) holds, where $M(x, y)$, $N(x, y)$ and the conditions upon φ and ψ are the same as in Theorem 1.5. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$ then T has a fixed point.

Theorem 1.7. (Chandok, Choudhury and Metiya [5]) In addition to the hypotheses of Theorem 1.5 (Theorem 1.6), suppose that for every $x,\ y \in X$, there exists $u \in X$ such that $u \preceq x$ and $u \preceq y$. Then T has a unique fixed point.

Recently, Sastry, Babu, Sarma and Krishna [17] improved Theorem 1.5, Theorem 1.6 and Theorem 1.7 by relaxing the continuity of φ and replacing $M(x, y)$ by $M_1(x, y)$ and $N(x, y)$ by $N_1(x, y)$.

Theorem 1.8. (Sastry, Babu, Sarma and Krishna [17]) Let $(X, \preceq)$ be a partially ordered set and (X, d) be a complete metric space. Let $T : X \to X$ be a non-decreasing mapping. Suppose there exists $\varphi : [0, \infty) \to [0, \infty)$ satisfying φ is non-decreasing and $\varphi(t) = 0 \iff t = 0$, and $\psi \in \Psi$ such that
$\varphi(d(Tx, Ty)) \leq \varphi(M_1(x, y)) - \psi(N_1(x, y))$, where
$M_1(x, y) = \max\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \frac{d(x,Tx)[1+d(y,Ty\})]}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Ty\})]}{1+d(x,y)}, d(x, y)\}$
and
$N_1(x, y) = \max\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \frac{d(x,Tx)[1+d(y,Ty\})]}{1+d(x,y)}, d(x, y)\}$, for all $x, y \in X$ with $x \preceq y$.
i.e. $M_1(x, y) = \max\{N_1(x, y), \frac{d(y,Tx)[1+d(x,Ty\})]}{1+d(x,y)}\}$.
If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for $n = 0, 1, 2, ...$ is a Cauchy sequence.

Theorem 1.9. (Sastry, Babu, Sarma and Krishna [17]) In addition to the hypotheses of Theorem 1.8, suppose that T is continuous. Then T has a fixed point.

Theorem 1.10. (Sastry, Babu, Sarma and Krishna [17]) In addition to the hypotheses of Theorem 1.8, assume the following:

(i) $x, y, z \in X$, such that $x < y < z \Rightarrow d(x, y) < d(x, z)$, and $d(y, z) < d(x, z)$

(ii) if $\{x_n\}$ is an increasing sequence in X such that $x_n \to z$, then $x_n \preceq z$ for all $n \in \mathbb{N}$.

Further " for every $u, v \in X$, there exists $z \in X$ which is comparable to both u and v".
Then T has a unique fixed point in X.

In 1986, Jungck [11] defined the concept of compatible mappings.

Definition 1.2. [11] A pair (S,T) of self-mappings of a metric space (X,d) is said to be compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

In 1998, Pant introduced a new continuity condition, known as reciprocal continuity and obtained a common fixed point theorem by using the compatibility in a metric space. The notion of reciprocal continuity is weaker than the continuity of one of the mappings.

Definition 1.3. [14] Two self-mappings S and T of a metric space (X,d) are called reciprocally continuous if $\lim_{n\to\infty} STx_n = Sz$ and $\lim_{n\to\infty} TSx_n = Tz$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z in X.

Definition 1.4. [12] Two self-maps S and T of a metric space (X,d) are said to be weakly compatible if they commute at their coincidence points. i.e. if for any x in X with $Sx = Tx$ then $STx = TSx$.

Definition 1.5. [10] Let $(X,\preceq)$ be a partially ordered set and T and $S : X \to X$ be two selfmaps. T is said to be S-non-decreasing if for all $x, y \in X$, $Sx \preceq Sy$ implies $Tx \preceq Ty$.

In this paper, $(X,\preceq,d)$ denotes a partially ordered metric space, where $(X,\preceq)$ is a partially ordered set, and d is a metric on X. If X is complete with respect to the metric d then we call $(X,\preceq,d)$ a partially ordered complete metric space.

The following lemma is useful in our subsequent discussion.

Lemma 1.1. [2]. Let $(X,\ d)$ be a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_{n+1}, x_n) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ and $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$. For each $k > 0$, corresponding to $n(k)$, we can choose $m(k)$ to be the smallest integer with $m(k) > n(k) > k$ satisfying $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$. Hence for such $m(k)$ and $n(k)$, we have $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$ and $d(x_{m(k)-1}, x_{n(k)}) < \epsilon$.

It can be shown that the following identities are satisfied.

$$(i)\ \lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \epsilon, \qquad (ii)\ \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon,$$

$$(iii)\ \lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \epsilon, \text{ and } (iv)\ \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) = \epsilon.$$

In Section 2, we prove the existence of coincidence and common fixed points of a pair of maps satisfying certain generalized contractive mappings with auxiliary functions $\varphi \in \Phi$ and $\psi \in \Psi$ involving rational type expressions in partially ordered metric spaces. In Section 3, we draw some corollaries from our main results and give examples in support of our results.

2. Main Results

Theorem 2.1. Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let $S,\ T : X \to X$ be self maps of X, and T is S non-decreasing. Suppose that there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$$\varphi(d(Tx, Ty)) \leq \varphi(M(x, y)) - \psi(N(x, y)) \tag{2.1}$$

where

$$M(x, y) = \max\{\frac{d(Sy, Ty)[1 + d(Sx, Tx)]}{1 + d(Sx, Sy)},\ \frac{d(Sx, Tx)[1 + d(Sy, Ty\})]}{1 + d(Sx, Sy)}, \frac{d(Sy, Tx)[1 + d(Sx, Ty\})]}{1 + d(Sx, Sy)}\ ,\ d(Sx, Sy)\}$$

and
$N(x, y)= \max\{\frac{d(Sy,Ty)[1+d(Sx,Tx)]}{1+d(Sx,Sy)}, \frac{d(Sx,Tx)[1+d(Sy,Ty\})]}{1+d(Sx,Sy)}, d(Sx, Sy)\}$
for all $x,\ y \in X$ with $Sx\ \preceq Sy$.
Furthermore, assume that

(i) $T(X) \subseteq S(X)$;

(ii) there exists $x_0 \in X$ such that $Sx_0 \preceq Tx_0$;

(iii) $S(X)$ is a closed subset of X; and

(iv) if any non-decreasing sequence $\{x_n\}$ in X converges to x then $x_n \preceq x$ for all $n = 0, 1, 2, ...$

Then S and T have a coincident point in X.

Proof. By (ii), let $x_0 \in X$ be such that $Sx_0 \preceq Tx_0$. Since $T(X) \subseteq S(X)$, we choose $x_1 \in X$ such that $Tx_0 = Sx_1$. Since $Sx_0 \preceq Tx_0 = Sx_1$, and T is S non-decreasing, we have $Tx_0 \preceq Tx_1$. Again, using $T(X) \subseteq S(X)$, we have $Tx_1 = Sx_2$ for some $x_2 \in X$ so that $Tx_0 \preceq Sx_2$ i.e.$Sx_1 \preceq Sx_2$. By using a similar argument we choose a sequence $\{x_n\}$ in X with $Tx_n = Sx_{n+1}$ and $Sx_n \preceq Sx_{n+1}$ for each $n = 0, 1, 2, ...$.
If $Sx_n = Sx_{n+1}$ for some $n \geq 0$ then $Sx_n = Tx_n$ so that x_n is a coincidence point of S and T.
Hence, with out loss of generality, we assume that $Sx_n \neq Sx_{n+1}$ for each $n \geq 0$.
Since $Sx_{n-1} \preceq Sx_n$, by (2.1) we have,

$$\begin{aligned} \varphi(d(Sx_n, Sx_{n+1})) &= \varphi(d(Tx_{n-1}, Tx_n)) \\ &\leq \varphi(M(x_{n-1}, x_n)) - \psi(N(x_{n-1}, x_n)), \end{aligned} \tag{2.2}$$

where

$$
\begin{aligned}
&M(x_{n-1}, x_n)\\
&= \max\{\frac{d(Sx_n, Tx_n)[1 + d(Sx_{n-1}, Tx_{n-1})]}{1 + d(Sx_{n-1}, Sx_n)}, \frac{d(Sx_{n-1}, Tx_{n-1})[1 + d(Sx_n, Tx_n)]}{1 + d(Sx_{n-1}, Sx_n)},\\
&\quad \frac{d(Sx_n, Tx_{n-1})[1 + d(Sx_{n-1}, Tx_n)]}{1 + d(Sx_{n-1}, Sx_n)}, d(Sx_{n-1}, Sx_n)\}\\
&= \max\{\frac{d(Sx_n, Sx_{n+1})[1 + d(Sx_{n-1}, Sx_n)]}{1 + d(Sx_{n-1}, Sx_n)}, \frac{d(Sx_{n-1}, Sx_n)[1 + d(Sx_n, Sx_{n+1})]}{1 + d(Sx_{n-1}, Sx_n)},\\
&\quad \frac{d(Sx_n, Sx_n)[1 + d(Sx_{n-1}, Sx_{n+1})]}{1 + d(Sx_{n-1}, Sx_n)}, d(Sx_{n-1}, Sx_n)\}\\
&= \max\{d(Sx_n, Sx_{n+1}), \frac{d(Sx_{n-1}, Sx_n)[1 + d(Sx_n, Sx_{n+1})]}{1 + d(Sx_{n-1}, Sx_n)}, d(Sx_{n-1}, Sx_n)\}
\end{aligned}
$$

and

$$
\begin{aligned}
&N(x_{n-1}, x_n)\\
&= \max\{\frac{d(Sx_n, Tx_n)[1 + d(Sx_{n-1}, Tx_{n-1})]}{1 + d(Sx_{n-1}, Sx_n)},\\
&\quad \frac{d(Sx_{n-1}, Tx_{n-1})[1 + d(Sx_n, Tx_n)]}{1 + d(Sx_{n-1}, Sx_n)}, d(Sx_{n-1}, Sx_n)\}\\
&= \max\{\frac{d(Sx_n, Sx_{n+1})[1 + d(Sx_{n-1}, Sx_n)]}{1 + d(Sx_{n-1}, Sx_n)},\\
&\quad \frac{d(Sx_{n-1}, Sx_n)[1 + d(Sx_n, Sx_{n+1})]}{1 + d(Sx_{n-1}, Sx_n)}, d(Sx_{n-1}, Sx_n)\}\\
&= \max\{d(Sx_n, Sx_{n+1}), \frac{d(Sx_{n-1}, Sx_n)[1 + d(Sx_n, Sx_{n+1})]}{1 + d(Sx_{n-1}, Sx_n)}, d(Sx_{n-1}, Sx_n)\}.
\end{aligned}
$$

If $\max\{d(Sx_n, Sx_{n+1}), d(Sx_{n-1}, Sx_n)\} = d(Sx_n, Sx_{n+1})$, then

$$
\begin{aligned}
M(x_{n-1}, x_n) &= \max\{d(Sx_n, Sx_{n+1}), \frac{d(Sx_{n-1}, Sx_n)[1 + d(Sx_n, Sx_{n+1})]}{1 + d(Sx_{n-1}, Sx_n)}\}\\
&= d(Sx_n, Sx_{n+1})
\end{aligned}
$$

and $N(x_{n-1}, x_n) = d(Sx_n, Sx_{n+1})$.
Now from (2.1), we have

$$
\begin{aligned}
\varphi(d(Sx_n, Sx_{n+1})) &\leq \varphi(d(Sx_n, Sx_{n+1})) - \psi(d(Sx_n, Sx_{n+1}))\\
&< \varphi(d(Sx_n, Sx_{n+1})),
\end{aligned}
$$

a contradiction.
Hence $\max\{d(Sx_n, Sx_{n+1}), d(Sx_{n-1}, Sx_n)\} = d(Sx_{n-1}, Sx_n)$. In this case

$$
\begin{aligned}
M(x_{n-1}, x_n) &= \max\{\frac{d(Sx_{n-1}, Sx_n)[1 + d(Sx_n, Sx_{n+1})]}{1 + d(Sx_{n-1}, Sx_n)}, d(Sx_{n-1}, Sx_n)\}\\
&= d(Sx_{n-1}, Sx_n)
\end{aligned}
$$

and $N(x_{n-1}, x_n) = d(Sx_{n-1}, Sx_n)$.
Therefore from (2.2), we have

$$\varphi(d(Sx_n, Sx_{n+1})) \leq \varphi(d(Sx_{n-1}, Sx_n)) - \psi(d(Sx_{n-1}, Sx_n)) \tag{2.3}$$

$$< \varphi(d(Sx_{n-1}, Sx_n)). \tag{2.4}$$

Thus it follows that $\{\varphi(d(Sx_n, Sx_{n+1}))\}$ is a strictly decreasing sequence of positive real numbers and so $\lim\limits_{n\to\infty} \varphi(d(Sx_n, Sx_{n+1}))$ exists and it is r (say).
i.e. $\lim\limits_{n\to\infty} \varphi(d(Sx_n, Sx_{n+1})) = r \geq 0$.
From (2.4), since φ is non-decreasing, it follows that $\{d(Sx_n, Sx_{n+1})\}$ is a strictly decreasing sequence of positive real numbers and so $\lim\limits_{n\to\infty} d(Sx_n, Sx_{n+1})$ exists and it is $r^{'}$ (say). i.e. $\lim\limits_{n\to\infty} d(Sx_n, Sx_{n+1}) = r^{'} \geq 0$.
Suppose that $r^{'} > 0$.
From (2.3), we have
$0 \leq \psi(d(Sx_{n-1}, Sx_n)) \leq \varphi(d(Sx_{n-1}, Sx_n)) - \varphi(d(Sx_n, Sx_{n+1}))$.
On taking limit supremum as $n \to \infty$ on both sides, we have

$$\begin{aligned} 0 \leq \overline{\lim}\psi(d(Sx_{n-1}, Sx_n)) &\leq \overline{\lim}\varphi(d(Sx_{n-1}, Sx_n)) - \underline{\lim}\varphi(d(Sx_n, Sx_{n+1})) \\ &= r - r = 0 \text{ as } n \to \infty \end{aligned}$$

so that $\overline{\lim}\psi(d(Sx_{n-1}, Sx_n)) = 0$. Hence $\underline{\lim}\psi(d(Sx_{n-1}, Sx_n)) = 0$.
Therefore $\lim\limits_{n\to\infty} \psi(d(Sx_{n-1}, Sx_n)) = 0$, which is a contradiction.
Therefore, $r^{'} = 0$. i.e. $\lim\limits_{n\to\infty} d(Sx_n, Sx_{n+1}) = 0$.
We now show that $\{Sx_n\}$ is Cauchy.
Suppose that $\{Sx_n\}$ is not a Cauchy sequence. Then by Lemma 1.1 there exists an $\epsilon > 0$ for which we can find sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that $d(Sx_{m(k)}, Sx_{n(k)}) \geq \epsilon$ and $d(Sx_{m(k)-1}, Sx_{n(k)}) < \epsilon$ and the following identities satisfied.

(i) $\lim\limits_{k\to\infty} d(Sx_{m(k)}, Sx_{n(k)}) = \epsilon$ $\qquad$ (ii) $\lim\limits_{k\to\infty} d(Sx_{m(k)-1}, Sx_{n(k)-1}) = \epsilon$

(iii) $\lim\limits_{k\to\infty} d(Sx_{m(k)-1}, Sx_{n(k)}) = \epsilon$ and (iv) $\lim\limits_{k\to\infty} d(Sx_{n(k)-1}, Sx_{m(k)}) = \epsilon$.

By (2.1), we have

$$\begin{aligned} \varphi(d(Sx_{n(k)}, Sx_{m(k)})) &= \varphi(d(Tx_{n(k)-1}, Tx_{m(k)-1})) \\ &\leq \varphi(M(x_{n(k)-1}, x_{m(k)-1})) - \psi(N(x_{n(k)-1}, x_{m(k)-1})), \end{aligned} \tag{2.5}$$

where

$$\begin{aligned}
&M(x_{n(k)-1}, x_{m(k)-1})\\
&= \max\{\frac{d(Sx_{m(k)-1}, Tx_{m(k)-1})[1 + d(Sx_{n(k)-1}, Tx_{n(k)-1})]}{1 + d(Sx_{n(k)-1}, Sx_{m(k)-1})},\\
&\quad \frac{d(Sx_{n(k)-1}, Tx_{n(k)-1})[1 + d(Sx_{m(k)-1}, Tx_{m(k)-1})]}{1 + d(Sx_{n(k)-1}, Sx_{m(k)-1})},\\
&\quad \frac{d(Sx_{m(k)-1}, Tx_{n(k)-1})[1 + d(Sx_{n(k)-1}, Tx_{m(k)-1})]}{1 + d(Sx_{n(k)-1}, Sx_{m(k)-1})}, d(Sx_{n(k)-1}, Sx_{m(k)-1})\}\\
&= \max\{\frac{d(Sx_{m(k)-1}, Sx_{m(k)})[1 + d(Sx_{n(k)-1}, Sx_{n(k)})]}{1 + d(Sx_{n(k)-1}, Sx_{m(k)-1})},\\
&\quad \frac{d(Sx_{n(k)-1}, Sx_{n(k)})[1 + d(Sx_{m(k)-1}, Sx_{m(k)})]}{1 + d(Sx_{n(k)-1}, Sx_{m(k)-1})},\\
&\quad \frac{d(Sx_{m(k)-1}, Sx_{n(k)})[1 + d(Sx_{n(k)-1}, Sx_{m(k)})]}{1 + d(Sx_{n(k)-1}, Sx_{m(k)-1})}, d(Sx_{n(k)-1}, Sx_{m(k)-1})\},
\end{aligned}$$

and

$$\begin{aligned}
&N(x_{n(k)-1}, x_{m(k)-1})\\
&= \max\{\frac{d(Sx_{m(k)-1}, Tx_{m(k)-1})[1 + d(Sx_{n(k)-1}, Tx_{n(k)-1})]}{1 + d(Sx_{n(k)-1}, Sx_{m(k)-1})},\\
&\quad \frac{d(Sx_{n(k)-1}, Tx_{n(k)-1})[1 + d(Sx_{m(k)-1}, Tx_{m(k)-1})]}{1 + d(Sx_{n(k)-1}, Sx_{m(k)-1})}, d(Sx_{n(k)-1}, Sx_{m(k)-1})\}\\
&= \max\{\frac{d(Sx_{m(k)-1}, Sx_{m(k)})[1 + d(Sx_{n(k)-1}, Sx_{n(k)})]}{1 + d(Sx_{n(k)-1}, Sx_{m(k)-1})},\\
&\quad \frac{d(Sx_{n(k)-1}, Sx_{n(k)})[1 + d(Sx_{m(k)-1}, Sx_{m(k)})]}{1 + d(Sx_{n(k)-1}, Sx_{m(k)-1})}, d(Sx_{n(k)-1}, Sx_{m(k)-1})\}.
\end{aligned}$$

Hence $\lim\limits_{k\to\infty} M(x_{n(k)-1}, x_{m(k)-1}) = \max\{0,\ 0,\ \frac{\epsilon(1+\epsilon)}{1+\epsilon)},\ \epsilon\} = \epsilon$,
$\lim\limits_{k\to\infty} N(x_{n(k)-1}, x_{m(k)-1}) = \max\{0,\ 0, \epsilon\} = \epsilon$.
Since φ is continuous, we have $\overline{\lim}\ \varphi(d(Sx_{n(k)}, Sx_{m(k)})) = \varphi(\epsilon)$.
From (2.5) and taking limit supremum as $n \to \infty$, we have
$\varphi(\epsilon) \leq \varphi(\epsilon) - \underline{\lim}\psi(N(x_{n(k)-1}, x_{m(k)-1}))$, and it implies that
$\underline{\lim}\psi(N(x_{n(k)-1}, x_{m(k)-1})) \leq 0$,
a contradiction.
Therefore $\{Sx_n\}$ is a Cauchy sequence in X.
Since $S(X)$ is complete, there exists $y \in X$ such that $\lim\limits_{n\to\infty} Sx_n = Sy$.

(2.6) Hence $\lim\limits_{n\to\infty} Tx_n = \lim\limits_{n\to\infty} Sx_{n+1} = Sy$ for some $y \in X$.

Now we show that $Sy = Ty$.
Suppose that $Sy \neq Ty$. i.e. $d(Sy, Ty) > 0$.

Since $\{Sx_n\}$ is a non-decreasing sequence, $Sx_n \to Sy$ for some $y \in X$ and by condition (iv), we have $Sx_n \preceq Sy$ for all $n \geq 0$.
Now, from (2.1), we have

$$\varphi(d(Tx_n, Ty)) \leq \varphi(M(x_n, y)) - \psi(N(x_n, y)), \tag{2.7}$$

where

$$\begin{aligned} &M(x_n, y) \\ &= \max\{\frac{d(Sy, Ty)[1 + d(Sx_n, Tx_n)]}{1 + d(Sx_n, Sy)}, \frac{d(Sx_n, Tx_n)[1 + d(Sy, Ty)]}{1 + d(Sx_n, Sy)}, \\ &\qquad \frac{d(Sy, Tx_n)[1 + d(Sx_n, Ty)]}{1 + d(Sx_n, Sy)}, d(Sx_n, Sy)\} \\ &= \max\{\frac{d(Sy, Ty)[1 + d(Sx_n, Sx_{n+1})]}{1 + d(Sx_n, Sy)}, \frac{d(Sx_n, Sx_{n+1})[1 + d(Sy, Ty)]}{1 + d(Sx_n, Sy)}, \\ &\qquad \frac{d(Sy, Sx_{n+1})[1 + d(Sx_n, Ty)]}{1 + d(Sx_n, Sy)}, d(Sx_n, Sy)\} \end{aligned}$$

and

$$\begin{aligned} &N(x_n, y) \\ &= \max\{\frac{d(Sy, Ty)[1 + d(Sx_n, Tx_n)]}{1 + d(Sx_n, Sy)}, \frac{d(Sx_n, Tx_n)[1 + d(Sy, Ty)]}{1 + d(Sx_n, Sy)}, d(Sx_n, Sy)\} \\ &= \max\{\frac{d(Sy, Ty)[1 + d(Sx_n, Sx_{n+1})]}{1 + d(Sx_n, Sy)}, \frac{d(Sx_n, Sx_{n+1})[1 + d(Sy, Sy)]}{1 + d(Sx_n, Sy)}, \\ &\qquad d(Sx_n, Sy)\}. \end{aligned}$$

Also, $\lim_{n\to\infty} M(x_n, y) = d(Sy, Ty)$ and $\lim_{n\to\infty} N(x_n, y) = d(Sy, Ty)$.
Now on taking limit supremum as $n \to \infty$ on both sides of (2.7) we have
$\overline{\lim}\varphi(d(Tx_n, Ty)) \leq \overline{\lim}\varphi(M(x_n, y)) - \underline{\lim}\psi(N(x_n, y))$,
which implies that $\varphi(d(Sy, Ty)) \leq \varphi(d(Sy, Ty)) - \underline{\lim}\psi(N(x_n, y))$
so that $\underline{\lim}\psi(N(x_n, y)) \leq 0$,
a contradiction.
Hence $Ty = Sy$ so that S and T have a coincidence point y. □.

Theorem 2.2. In addition to the hypotheses of Theorem 2.1, assume that

(i) S and T are weakly compatible,

(ii) $Sx = Tx$ implies $Sx \preceq SSx$ for any $x \in X$.

Then T and S have common fixed point in X.

Furthermore, assume the following: Condition(H): there exists $u \in X$ such that $Su \preceq Tu$ and Tu is comparable to Tx and Ty, for all $x, y \in X$.
Then S and T have a unique common fixed point in X.

Proof. From the proof of Theorem 2.1, we have $\{Sx_n\}$ is a non-decreasing sequence that converges to Sy for some $y \in X$ with $Sy = Ty$.
Let $w = Ty = Sy$.
Since S and T are weakly compatible, $Tw = TSy = STy = Sw$.
Suppose that $w \neq Tw$.
By hypothesis (ii) we have $Sy \preceq SSy = STy$.
Therefore, from (2.1), we have

$$\begin{aligned} \varphi(d(w,Tw)) &= \varphi(d(Ty,TTy)) \\ &\leq \varphi(M(y,Ty)) - \psi(N(y,Ty)) \qquad (2.8) \end{aligned}$$

where

$$\begin{aligned} &M(y,Ty) \\ &= \max\{\frac{d(STy,TTy)[1+d(Sy,Ty)]}{1+d(Sy,STy)},\ \frac{d(Sy,Ty)[1+d(STy,TTy)]}{1+d(Sy,STy)}, \\ &\qquad \frac{d(STy,Ty)[1+d(Sy,TTy)]}{1+d(Sy,STy)}\ ,\ d(Sy,STy)\} \\ &= \max\{\frac{d(Sw,TTy)}{1+d(Sy,Sw)},\ 0, \frac{d(Sw,Ty)[1+d(Sy,TTy)]}{1+d(Sy,Sw)}\ ,\ d(Sy,Sw)\} \\ &= \max\{\frac{d(Tw,TTy)}{1+d(w,Tw)},\ 0, \frac{d(Tw,w)[1+d(w,TTy)]}{1+d(w,Tw)}\ ,\ d(w,Tw)\} \\ &= \max\{\frac{d(Tw,Tw)}{1+d(w,Tw)},\ 0, \frac{d(Tw,w)[1+d(w,Tw)]}{1+d(w,Tw)}\ ,\ d(w,Tw)\} \\ &= d(w,Tw), \end{aligned}$$

and

$$\begin{aligned} &N(y,Ty) \\ &= \max\{\frac{d(STy,TTy)[1+d(Sy,Ty)]}{1+d(Sy,STy)},\ \frac{d(Sy,Ty)[1+d(STy,TTy)]}{1+d(Sy,STy)}\ ,\ d(Sy,STy)\} \\ &= \max\{\frac{d(Sw,TTy)}{1+d(Sy,Sw)},\ 0,\ d(Sy,Sw)\} \\ &= \max\{\frac{d(Tw,TTy)}{1+d(w,Tw)},\ 0,\ d(w,Tw)\} \\ &= \max\{\frac{d(Tw,Tw)}{1+d(w,Tw)},\ 0,\ d(w,Tw)\} \\ &= d(w,Tw). \end{aligned}$$

Hence, from (2.8),

$$\begin{aligned} \varphi(d(w,Tw)) &\leq \varphi(d(w,Tw)) - \psi(d(w,Tw)) \\ &< \varphi(d(w,Tw)) \end{aligned}$$

is a contradiction.
Therefore $w = Tw$. Hence $w = Tw = Sw$.
Therefore w is a common fixed point of S and T.
We now prove the uniqueness of common fixed point of S and T.
Let z and w be two common fixed points of S and T. i.e. $Sz = Tz = z$ and $Sw = Tw = w$ with $z \neq w$.
Case (I): z and w are comparable. With out loss of generality we assume that $z \preceq w$. i.e. $Sz \preceq Sw$
From (2.1), we have

$$\begin{aligned} \varphi(d(z,w)) &= \varphi(d(Tz,Tw)) \\ &\leq \varphi(M(z,w)) - \psi(N(z,w)) \qquad (2.9) \end{aligned}$$

where

$$\begin{aligned} &M(z,w) \\ &= \max\{\frac{d(Sw,Tw)[1+d(Sz,Tz)]}{1+d(Sz,Sw)}, \frac{d(Sz,Tz)[1+d(Sw,Tw)]}{1+d(Sz,Sw)}, \\ &\quad \frac{d(Sw,Tz)[1+d(Sz,Tw)]}{1+d(Sz,Sw)}, d(Sz,Sw)\} \\ &= \max\{\frac{d(w,w)}{1+d(z,w)}, 0, \frac{d(w,z)[1+d(z,w)]}{1+d(z,w)}, d(z,w)\} \\ &= \max\{0, 0, d(z,w), d(z,w)\} \\ &= d(z,w). \end{aligned}$$

$$\begin{aligned} N(z,w) &= \max\{\frac{d(Sw,Tw)[1+d(Sz,Tz)]}{1+d(Sz,Sw)}, \frac{d(Sz,Tz)[1+d(Sw,Tw)]}{1+d(Sz,Sw)}, d(Sz,Sw)\} \\ &= \max\{\frac{d(w,w)}{1+d(z,w)}, 0, d(z,w)\} \\ &= \max\{0, 0,, d(z,w)\} \\ &= d(z,w). \end{aligned}$$

Hence, from (2.9), we have

$$\begin{aligned} \varphi(d(z,w)) &\leq \varphi(d(z,w)) - \psi(d(z,w)) \\ &< \varphi(d(z,w)), \end{aligned}$$

a contradiction.
Therefore $z = w$. This shows that S and T have a unique common fixed point in X.

Case (II) : z and w are not comparable.

In this case, by assumption, there exists $u \in X$ such that $Su \preceq Tu$ and Tu is comparable to Tz and Tw.

Subcase (i) : We assume that $Tz \preceq Tu, Tw \preceq Tu$ and $Su \preceq Tu$. Now we set $u = u_0$. Since $T(X) \subseteq S(X)$, there exists $u_1 \in X$ such that

$$Tu_0 = Su_1. \tag{2.10}$$

Since $Tz \preceq Tu$, $Tz = Sz$ and $Tu = Tu_0 = Su_1$, we have

$$Sz \preceq Su_1. \tag{2.11}$$

Since $Su_0 \preceq Tu_0 = Su_1$, we have

$$Su_0 \preceq Su_1. \tag{2.12}$$

Since T is S non-decreasing, from (2.11) and (2.12) we get

$$Tz \preceq Tu_1 \quad \text{and} \tag{2.13}$$

$$Tu_0 \preceq Tu_1. \tag{2.14}$$

Since $T(X) \subseteq S(X)$, there exists $u_2 \in X$ such that

$$Tu_1 = Su_2. \tag{2.15}$$

From (2.10), (2.14) and (2.15) we have

$$Su_1 \preceq Su_2. \tag{2.16}$$

From (2.13)and (2.15), it follows that

$$Sz \preceq Su_2, \qquad \text{since } \; Tz = Sz. \tag{2.17}$$

Since T is S non-decreasing, from (2.16) and (2.17) we get

$$Tu_1 \preceq Tu_2 \quad \text{and} \tag{2.18}$$

$$Tz \preceq Tu_2. \tag{2.19}$$

On continuing this process, we can construct a sequence$\{u_n\}$ in X such that

$$Su_{n+1} = Tu_n, \; Sz \preceq Su_{n+1} \text{ and } Su_n \preceq Su_{n+1} \text{ for } n = 0, 1, 2.... \tag{2.20}$$

(2.21) Also, we can easily see that $Sw \preceq Su_{n+1}$ for $n = 0, 1, 2....$

Since $Su_n \preceq Su_{n+1}$, by using the inequality (2.1), it is easy to see that $\{Su_n\}$ is Cauchy as in the proof of Theorem 2.1. Since $S(X)$ is complete, there exists $v \in X$ such that $Su_n \to Sv$ as $n \to \infty$.

We now show that $Sz = Sv$. Suppose that $Sz \neq Sv$.
Since $Sz \preceq Su_n$, from (2.1) we have

$$\varphi(d(Sz, Su_{n+1})) = \varphi(d(Tz, Tu_n)) \leq \varphi(M(z, u_n)) - \psi(N(z, u_n)) \tag{2.22}$$

where

$$M(z, u_n) = \max\{\frac{d(Su_n, Tu_n)[1 + d(Sz, Tz)]}{1 + d(Sz, Su_n)}, \frac{d(Sz, Tz)[1 + d(Su_n, Tu_n)]}{1 + d(Sz, Su_n)}, \frac{d(Su_n, Tz)[1 + d(Sz, Tu_n)]}{1 + d(Sz, Su_n)}, d(Sz, Su_n)\}$$

and

$$N(z, u_n) = \max\{\frac{d(Su_n, Tu_n)[1 + d(Sz, Tz)]}{1 + d(Sz, Su_n)}, \frac{d(Sz, Tz)[1 + d(Su_n, Tu_n)]}{1 + d(Sz, Su_n)}, d(Sz, Su_n)\}.$$

Hence $\lim_{n\to\infty} M(z, u_n) = \max\{0, 0, d(Sv, Tz), d(Sz, Sv)\} = d(Sv, Sz)$ and
$\lim_{n\to\infty} N(z, u_n) = \max\{0, 0, d(Sz, Sv)\} = d(Sv, Sz)$.
Taking limit supremum on (2.22), we have

$$\varphi(d(Sz, Sv)) \leq \varphi(d(Sz, Sv)) - \underline{\lim}\psi(N(z, u_n)) \tag{2.23}$$

so that $\underline{\lim}\psi(N(z, u_n)) \leq 0$,
a contradiction.
Therefore, $Sz = Sv$.
Similarly we can prove that $Sw = Sv$. Hence $Sz = Sw$, which implies that $z = w$.

Subcase (ii) : We assume that $Tu \preceq Tz, Tu \preceq Tw$ and $Su \preceq Tu$. Now we set $u = u_0$. Since $T(X) \subseteq S(X)$, there exists $u_1 \in X$ such that

$$Tu_0 = Su_1. \tag{2.24}$$

Since $Tu \preceq Tz$, $Tz = Sz$ and $Tu = Tu_0 = Su_1$, we have

$$Su_1 \preceq Sz. \tag{2.25}$$

Since $Su_0 \preceq Tu_0 = Su_1$, we have

$$Su_0 \preceq Su_1. \tag{2.26}$$

Since T is S non-decreasing, from (2.25) and (2.26) we get

$$Tu_1 \preceq Tz \quad \text{and} \tag{2.27}$$

$$Tu_0 \preceq Tu_1. \tag{2.28}$$

Since $T(X) \subseteq S(X)$, there exists $u_2 \in X$ such that

$$Tu_1 = Su_2. \tag{2.29}$$

From (2.24), (2.28) and (2.29) we have

$$Su_1 \preceq Su_2. \tag{2.30}$$

From (2.27)and (2.29), it follows that

$$Su_2 \preceq Sz, \quad \text{since } Tz = Sz. \tag{2.31}$$

Since T is S non-decreasing, from (2.30) and (2.31) we get

$$Tu_1 \preceq Tu_2 \quad \text{and} \tag{2.32}$$

$$Tu_2 \preceq Tz. \tag{2.33}$$

On continuing this process, we can construct a sequence$\{u_n\}$ in X such that

$$Su_{n+1} = Tu_n,\ Su_n \preceq Sz \text{ and } Su_n \preceq Su_{n+1} \text{ for } n = 0, 1, 2.... \tag{2.34}$$

(2.35) Also we can easily see that $Su_n \preceq Sw$ for $n = 0, 1, 2....$

Since $Su_n \preceq Su_{n+1}$, by using the inequality (2.1), it is easy to see that $\{Su_n\}$ is Cauchy as in the proof of Theorem 2.1. Since $S(X)$ is complete, there exists $v \in X$ such that $Su_n \to Sv$ as $n \to \infty$.
We now show that $Sz = Sv$. Suppose that $Sz \neq Sv$.
Since $Su_n \preceq Sz$, from (2.1) we have

$$\varphi(d(Su_{n+1}, Sz)) = \varphi(d(Tu_n, Tz)) \leq \varphi(M(u_n, z)) - \psi(N(u_n, z)) \tag{2.36}$$

where

$$M(u_n, z) = \max\{\frac{d(Sz, Tz)[1 + d(Su_n, Tu_n)]}{1 + d(Su_n, Sz)}, \frac{d(Su_n, Tu_n)[1 + d(Sz, Tz)]}{1 + d(Su_n, Sz)}, \frac{d(Sz, Tu_n)[1 + d(Su_n, Tz)]}{1 + d(Su_n, Sz)}, d(Su_n, Sz)\}$$

and

$$N(u_n, z) = \max\{\frac{d(Sz, Tz)[1+d(Su_n, Tu_n)]}{1 + d(Su_n, Sz)}, \frac{d(Su_n, Tu_n)[1+d(Sz, Tz)]}{1 + d(Su_n, Sz)}, d(Su_n, Sz)\}.$$

Hence $\lim\limits_{n\to\infty} M(u_n, z) = \max\{0, 0, d(Sz, Sv), d(Sv, Sz)\} = d(Sv, Sz)$ and
$\lim\limits_{n\to\infty} N(u_n, z) = \max\{0, 0, d(Sv, Sz)\} = d(Sv, Sz)$.
On taking limit supremum as $n \to \infty$ on (2.36), we have

$$\varphi(d(Sv, Sz)) \leq \varphi(d(Sv, Sz)) - \underline{\lim}\psi(N(u_n, z)) \tag{2.37}$$

so that $\underline{\lim}\psi(N(u_n, z)) \leq 0$,
a contradiction.
Therefore, $Sz = Sv$.
Similarly we can prove that $Sw = Sv$. Hence $Sz = Sw$, which implies that $z = w$.
Subcase (iii) : We assume that $Tu \preceq Tz, Tw \preceq Tu$ and $Su \preceq Tu$.

In this case, $Tw \preceq Tz$ i.e., $w \preceq z$. By case(i)the uniqueness follows.

Subcase (iv) : We assume that $Tz \preceq Tu, Tu \preceq Tw$ and $Su \preceq Tu$.

In this case, $Tz \preceq Tw$ i.e. $z \preceq w$. By case(i) the uniqueness follows.
Hence in either of the two cases S and T have a unique common fixed point. □

Now we relax the closedness of $S(X)$ and condition (iv) of Theorem 2.1, but by imposing the compatible property and reciprocal continuity of a pair of maps and prove the following.

Theorem 2.3. Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let $S,\ T : X \to X$ be self maps of X and T is S non-decreasing. Suppose that there exist $\varphi \in \Phi,\ \psi \in \Psi$ and satisfying the inequality (2.1). Assume that

(i) $T(X) \subseteq S(X)$;

(ii) there exists $x_0 \in X$ such that $Sx_0 \preceq Tx_0$;

(iii) S and T are reciprocally continuous;

(iv) the pair (S, T) is compatible;

(v) $Sz = Tz$ implies $Sz \preceq SSz$ for any $z \in X$.

Then S and T have a common fixed point.

Furthermore, assume that Condition(H) of Theorem 2.2, then S and T have a unique common fixed point in X.

Proof. The sequence $\{x_n\}$ is constructed such that $Sx_{n+1} = Tx_n$ for all $n \geq 0$ and the proof of the Cauchy part of the sequence $\{Sx_n\}$ is the same as that one mentioned in the proof of Theorem 2.1.
Since (X, d) is complete, there exists $z \in X$ such that $\lim_{n\to\infty} Sx_n = z$ and consequently we have $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_{n+1} = z$.
Since S and T are reciprocally continuous, we have
$\lim_{n\to\infty} STx_n = Sz$ and $\lim_{n\to\infty} TSx_n = Tz$.
Again, since S and T are compatible, it follows that
$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, *i.e.*, $d(Sz,\ Tz) = 0$ so that $Sz = Tz$.
Now, since every compatible pair is weakly compatible, by using the compatibility of S and T we have $STz = TSz = TTz$.
Suppose that $Tz \neq TTz$. Now

$\varphi(d(Tz,TTz)) \leq \varphi(M(z,Tz)) - \psi(N(z,Tz))$
where

$$\begin{aligned}M(z,Tz) &= \max\{\frac{d(STz,TTz)[1+d(Sz,Tz)]}{1+d(Sz,STz)},\ \frac{d(Sz,Tz)[1+d(STz,TTz)]}{1+d(Sz,STz)},\\ &\qquad \frac{d(STz,Tz)[1+d(Sz,TTz)]}{1+d(Sz,STz)},\ d(Sz,STz)\}\\ &= \max\{0,0,d(Tz,TTz),d(Tz,TTz)\}\\ &= d(Tz,TTz),\ \text{and in a similar way it is easy to see that } N(z,Tz) = d(Tz,TTz).\end{aligned}$$

Therefore

$$\begin{aligned}\varphi(d(Tz,TTz)) &\leq \varphi(d(Tz,TTz)) - \psi(d(Tz,TTz))\\ &< \varphi(d(Tz,TTz)),\end{aligned}$$

a contradiction.
Hence $Tz = TTz$ so that Tz is a fixed point of T.
Therefore, Tz is a common fixed point of S and T.
We now prove the uniqueness of the common fixed point of S and T.
Let z and w be two common fixed points of S and T. i.e. $Sz = Tz = z$ and $Sw = Tw = w$, with $z \neq w$.
If z and w are comparable then by Case (I) of the proof of Theorem 2.2, the conclusion follows.

We now suppose z and w are not comparable. In this case, by following the line of the Subcase (i) of Case (II) of Theorem 2.2, we reach at (2.20) and (2.21). i.e., there exists a sequence $\{u_n\}$ in X such that
$Su_{n+1} = Tu_n,\ Sz \preceq Su_{n+1}, Sw \preceq Su_{n+1}$ and $Su_n \preceq Su_{n+1}$,
for all $n = 0, 1, 2, ...$
Since $Su_n \preceq Su_{n+1}$, by using the inequality (2.1), it is easy to see that $\{Su_n\}$ is Cauchy as in the proof of Theorem 2.1.
Since X is complete, there exists $v \in X$ such that $Su_n \to v$ as $n \to \infty$.
We now show that $Sz = v$. Suppose that $Sz \neq v$.
Since $Sz \preceq Su_n$, from (2.1) we have

$$\varphi(d(Sz,Su_{n+1})) = \varphi(d(Tz,Tu_n)) \leq \varphi(M(z,u_n)) - \psi(N(z,u_n)) \tag{2.38}$$

where

$$\begin{aligned}M(z,u_n) = \max\{&\frac{d(Su_n,Tu_n)[1+d(Sz,Tz)]}{1+d(Sz,Su_n)}, \frac{d(Sz,Tz)[1+d(Su_n,Tu_n)]}{1+d(Sz,Su_n)},\\ &\frac{d(Su_n,Tz)[1+d(Sz,Tu_n)]}{1+d(Sz,Su_n)}, d(Sz,Su_n)\}\end{aligned}$$

and

$$N(z,u_n) = \max\{\frac{d(Su_n,Tu_n)[1+d(Sz,Tz)]}{1+d(Sz,Su_n)}, \frac{d(Sz,Tz)[1+d(Su_n,Tu_n)]}{1+d(Sz,Su_n)}, d(Sz,Su_n)\}.$$

Hence $\lim_{n\to\infty} M(z,u_n) = \max\{0,0,d(v,Sz),d(Sz,v)\} = d(v,Sz)$ and
$\lim_{n\to\infty} N(z,u_n) = \max\{0,0,d(Sz,v)\} = d(v,Sz)$.
On taking limit supremum as $n\to\infty$ on (2.38), we have

$$\varphi(d(Sz,v)) \le \varphi(d(Sz,v)) - \underline{\lim}\psi(N(z,u_n)) \tag{2.39}$$

so that $\underline{\lim}\psi(N(z,u_n)) \le 0$,
a contradiction.
Therefore, $Sz = v$.
Similarly, we can prove that $Sw = v$. Hence $Sz = Sw$, which implies that $z = w$.
In all other cases we prove the uniqueness of the theorem as in the proof of Theorem 2.2. □

3. Corollaries and Examples

By choosing $S = I_X$ in Theorem 2.1, we have the following corollary.

Corollary 3.1. Let $(X,\preceq,d)$ be a partially ordered complete metric space. Let $T : X \to X$ be a self map of X and T is non-decreasing. Suppose that there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$$\varphi(d(Tx,Ty)) \le \varphi(M(x,y)) - \psi(N(x,y)), \tag{3.1}$$

where
$M(x,y) = \max\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \frac{d(x,Tx)[1+d(y,Ty\})]}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Ty\})]}{1+d(x,y)}, d(x,y)\}$
and
$N(x,y) = \max\{\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \frac{d(x,Tx)[1+d(y,Ty\})]}{1+d(x,y)}, d(x,y)\}$
for all $x, y \in X$ with $x \preceq y$.
Furthermore, assume that

(*i*) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(*ii*) if any non-decreasing sequence $\{x_n\}$ in X converges to x then $x_n \preceq x$ for all $n = 0,1,2,....$

Then T has a fixed point.

We now consider the following examples in support of our main results.

Example 3.1. Let $X = [0,3]$ with the usual metric. We define partial order $\preceq$ on X as follows:
$\preceq := \{(x,y) \in X\times X : x = y\} \cup \{(0,\frac{1}{2}),(0,\frac{3}{4}),(\frac{1}{2},\frac{3}{4})\}$, where $x \preceq y$ means $x \le y$ in the usual sense.
Then $(X,\preceq,d)$ is a partially ordered complete metric space. We define
$T : X \to X$ by $T(x) = \begin{cases} x+\frac{1}{2} & if\ x \in [0,1) - \{\frac{1}{2},\frac{3}{4}\} \\ \frac{3}{4} & if\ x = \frac{3}{4} \\ 2 & otherwise, \end{cases}$ and

$S : X \to X$ by $S(x) = \begin{cases} 2x & if\ x \in [0,1] - \{\frac{3}{4}, \frac{3}{8}\} \\ \frac{3}{2} & if\ x = \frac{3}{8} \\ \frac{3}{4} & if\ x = \frac{3}{4} \\ 2 & otherwise. \end{cases}$

Clearly $T(X) \subseteq S(X)$, SX is closed and T is S non-decreasing.
We choose $x_0 = 0 \in X$ then $Sx_0 \preceq Tx_0$. We define
$\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(t) = 2t^2, t \geq 0$, and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(t) = \frac{t}{4}$, $t \geq 0$.
We now verify the inequality (2.1).

Case (i) : Let $(x,\ y) = (0, \frac{1}{4})$ such that $S(0) \preceq S(\frac{1}{4})$.

In this case, $\varphi(d(T(0), T(\frac{1}{4}))) = \varphi(d(\frac{1}{2}, \frac{3}{4})) = \varphi(\frac{1}{4}) = \frac{1}{8}$,
$M(0, \frac{1}{4}) = \frac{1}{2}$ and $N(0, \frac{1}{4}) = \frac{1}{2}$;
Now $\varphi(M(0, \frac{1}{4})) = \varphi(\frac{1}{2}) = \frac{1}{2}, \psi(N(0, \frac{1}{4})) = \psi(\frac{1}{2}) = \frac{1}{8}$.
Therefore
$\varphi(d(T(0), T(\frac{1}{4}))) = \frac{1}{8} \leq \frac{1}{2} - \frac{1}{8} = \varphi(M(0, \frac{1}{4})) - \psi(N(0, \frac{1}{4}))$.

Case (ii) : Let $(x,\ y) = (0, \frac{3}{4})$ such that $S(0) \preceq S(\frac{3}{4})$.

In this case, $\varphi(d(T(0), T(\frac{3}{4}))) = \varphi(d(\frac{1}{2}, \frac{3}{4})) = \varphi(\frac{1}{4}) = \frac{1}{8}$,
$M(0, \frac{3}{4}) = \frac{3}{4}$ and $N(0, \frac{3}{4}) = \frac{3}{4}$;
Now $\varphi(M(0, \frac{3}{4})) = \varphi(\frac{3}{4}) = \frac{9}{8}, \psi(N(0, \frac{3}{4})) = \psi(\frac{3}{4}) = \frac{3}{16}$.
Therefore
$\varphi(d(T(0), T(\frac{3}{4}))) = \frac{1}{8} \leq \frac{9}{8} - \frac{3}{16} = \varphi(M(0, \frac{3}{4})) - \psi(N(0, \frac{3}{4}))$.

Case (iii) : Let $(x,\ y) = (\frac{1}{4}, \frac{3}{4})$ such that $S(\frac{1}{4}) \preceq S(\frac{3}{4})$.

In this case, $\varphi(d(T(\frac{1}{4}), T(\frac{3}{4}))) = \varphi(d(\frac{3}{4}, \frac{3}{4})) = \varphi(0) = 0$,
$M(\frac{1}{4}, \frac{3}{4}) = \frac{1}{4}$ and $N(\frac{1}{4}, \frac{3}{4}) = \frac{1}{4}$;
Now $\varphi(M(\frac{1}{4}, \frac{3}{4})) = \varphi(\frac{1}{4}) = \frac{1}{8}, \psi(N(\frac{1}{4}, \frac{3}{4})) = \psi(\frac{1}{4}) = \frac{1}{16}$.
Therefore
$\varphi(d(T(\frac{1}{4}), T(\frac{3}{4}))) = 0 \leq \frac{1}{8} - \frac{1}{16} = \varphi(M(\frac{1}{4}, \frac{3}{4})) - \psi(N(\frac{1}{4}, \frac{3}{4}))$.
In the remaining cases, the inequality (2.1) holds trivially.
Therefore S and T satisfy all the hypotheses of Theorem 2.1 and
S and T have infinitely many coincident points.
Furthermore, we note that clearly S and T are weakly compatible, and
$Sx = Tx \Rightarrow Sx \preceq SSx\ \forall x \in X$, so that $(i)$and (ii) of Theorem 2.2 hold and $\frac{3}{4}$ and 2 are common fixed points of S and T.
Further, we observe that S and T do not satisfy 'Condition H'.
For

Case (i) : If $u = 0$ then $Su = 0, Tu = \frac{1}{2}$, clearly $Su \preceq Tu$.

In this case, for any $x, y \in [0,3) - \{0, \frac{1}{4}, \frac{3}{4}\}$, neither Tx nor Ty is comparable to $\frac{1}{2} = Tu$.

Case (ii) : If $u = \frac{1}{4}$ then $Su = \frac{1}{2}, Tu = \frac{3}{4}$, clearly $Su \preceq Tu$.

In this case, for any $x, y \in [0,3) - \{0, \frac{1}{4}, \frac{3}{4}\}$, neither Tx nor Ty is comparable to $\frac{3}{4} = Tu$.

Case (iii) : If $u = \frac{3}{4}$ then $Su = \frac{3}{4}, Tu = \frac{3}{4}$, clearly $Su \preceq Tu$.

In this case, for any $x, y \in [0,3) - \{0, \frac{1}{4}, \frac{3}{4}\}$, neither Tx nor Ty is comparable to $\frac{3}{4} = Tu$.

Case (iv) : If $u = [1,3)$ then $Su = 2 = Tu$, clearly $Su \preceq Tu$.

In this case, for any $x, y \in [0,1) - \{\frac{1}{2}\}$, neither Tx nor Ty is comparable to $2 = Tu$.

Case (v) : If $u \in [0,3) - \{0, \frac{1}{4}, \frac{3}{4}\}$ then clearly $Su \npreceq Tu$.
Hence 'Condition(H)' fails to hold.

The following is an example in support of Theorem 2.2.

Example 3.2. Let $X = \{0, 1, 2, 5\}$ with the usual metric. We define partial order $\preceq$ on X as follows:
$\preceq := \{(0, 0), (1, 1), (2, 2), (5, 5), (0, 1), (0, 2), (0, 5), (1, 2), (1, 5), (2, 5)\}$, where $x \preceq y$ means $x \leq y$ in the usual sense.
Then $(X, \preceq, d)$ is a partially ordered metric space. We define $S, T : X \to X$ by $S0 = 0, S1 = 1, S2 = 5, S5 = 2$ and

$$T0 = T1 = T5 = 1, T2 = 2.$$

Clearly, $T(X) \subseteq S(X)$, and T is S non-decreasing.
We choose $x_0 = 0 \in X$. Then $Sx_0 \preceq Tx_0$. We define

$$\varphi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ by } \varphi(t) = t^3,\ t \geq 0, \quad \text{and}$$

$$\psi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ by } \psi(t) = \begin{cases} \frac{4}{5}t & if\ t \in \mathbb{Q}^+ \\ 1 & otherwise. \end{cases}$$

We now verify the inequality (2.1).
Case (i): Let $(x,\ y) = (1, 2)$ such that $S1 \preceq S2$.

In this case, $\varphi(d(T1, T2)) = \varphi(d(1, 2)) = \varphi(1) = 1$, $M(1, 2) = 4$ and $N(1, 2) = 4$. Now $\varphi(M(1, 2)) = \varphi(4) = 64$, $\psi(N(1, 2)) = \psi(4) = \frac{16}{5}$.
Therefore
$\varphi(d(T1, T2)) = 1 \leq 64 - \frac{16}{5} = \varphi(M(1, 2)) - \psi(N(1, 2))$.
Case (ii) : Let $(x,\ y) = (0, 2)$ such that $S0 \preceq S2$.

In this case, $\varphi(d(T0, T2)) = \varphi(d(1, 2)) = \varphi(1) = 1$, $M(0, 2) = 5$ and $N(0, 2) = 5$. Now $\varphi(M(0, 2)) = \varphi(5) = 125$, $\psi(N(0, 2)) = \psi(5) = 4$.
Therefore
$\varphi(d(T0, T2)) = 1 \leq 125 - 4 = \varphi(M(0, 2)) - \psi(N(0, 2))$.
Case (iii) : Let $(x,\ y) = (5, 2)$ such that $S5 \preceq S2$.

In this case, $\varphi(d(T5, T2)) = \varphi(d(1, 2)) = \varphi(1) = 1$, $M(5, 2) = 3$ and $N(5, 2) = 3$. Now $\varphi(M(5, 2)) = \varphi(3) = 27$, $\psi(N(5, 2)) = \psi(3) = \frac{12}{5}$.
Therefore
$\varphi(d(T5, T2)) = 1 \leq 27 - \frac{12}{5} = \varphi(M(5, 2)) - \psi(N(5, 2))$.
In the remaining cases the inequality (2.1) holds trivially.
Also, S and T are weakly compatible, and (ii) of Theorem 2.2 hold. Further, by choosing $u = 0$ with $S0 \preceq T0$ and $T0$ is comparable with Tx and Ty for all $x, y \in X$ so that 'Condition (H)' holds.
Therefore, S and T satisfy all the hypotheses of Theorem 2.2 and S and T have a unique common fixed point 1.

The following is an example in support of Theorem 2.3.

Example 3.3. Let $X = [0, 2]$ with the usual metric. We define partial order $\preceq$ on X as follows:
$\preceq := \{(x, y) \in X \times X : x = y\} \cup \{(\frac{1}{2^{2n}}, 0) : n \geq 1\}$, where $x \preceq y$ means $x \geq y$ in the usual sense.
Then $(X, \preceq, d)$ is a partially ordered complete metric space. We define

$$T : X \to X \text{ by } T(x) = \begin{cases} \frac{x^2}{4} & if\ x \in [0, 1) \\ 2 & if\ x \in [1, 2], \end{cases} \quad \text{and}$$

$$S : X \to X \text{ by } S(x) = \begin{cases} x^2 & if\ x \in [0, 1) \\ 2 & if\ x \in [1, 2]. \end{cases}$$

Clearly $T(X) \subseteq S(X)$, and T is S non-decreasing.
We choose $x_0 = 0 \in X$. Then $Sx_0 \preceq Tx_0$ and clearly S and T are reciprocally continuous

and the pair (S,T) is compatible.
We define $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ by $\varphi(t)=t^2,\ t\geq 0,$ and
$\psi:\mathbb{R}^+\to\mathbb{R}^+$ by $\psi(t)=\frac{3}{4}t^2\ if\ t\geq 0.$
We now verify the inequality (2.1).
Case (i) : Let $(x,\ y)=(\frac{1}{2^n},0)$ such that $S(\frac{1}{2^n})\preceq S(0)$, for n= 1,2,3, ...

In this case, $\varphi(d(T(\frac{1}{2^n})),T(0))=\varphi(d(\frac{1}{2^{2n+2}}),0)=\varphi(\frac{1}{2^{2n+2}})=(\frac{1}{2^{2n+2}})^2$,
$M(\frac{1}{2^n},0)=\frac{1}{2^{2n}}$ and $N(\frac{1}{2^n},0)=\frac{1}{2^{2n}}$.
Now $\varphi(M(\frac{1}{2^n},0))=\varphi(\frac{1}{2^{2n}})=(\frac{1}{2^{2n}})^2,\psi(N(\frac{1}{2^n},0))=\psi(\frac{1}{2^{2n}})=\frac{3}{4}\frac{1}{(2^{2n})^2}$.
Therefore
$\varphi(d(T(\frac{1}{2^n}),T(0)))=(\frac{1}{2^{2n+2}})^2\leq(\frac{1}{2^{2n}})^2-\frac{3}{4}\frac{1}{(2^{2n})^2}=\varphi(M(\frac{1}{2^n},0))-\psi(N(\frac{1}{2^n},0))$, for $n=1,2,3,\ldots$.
In the remaining cases, the inequality (2.1) holds trivially.
Therefore, S and T satisfy all the hypotheses of Theorem 2.3, and S and T have two common fixed points 0 and 2.
Further, we observe that S and T do not satisfy 'Condition H'.
For,
Case (i) : If $u=0$ then $Su=0=Tu$ so that $Su\preceq Tu$.

In this case for any $x,y\in(0,2]$, neither Tx nor Ty is comparable to $0=Tu$.
Case (ii) : If $u\in[1,2]$ then $Su=2=Tu$ so that $Su\preceq Tu$.

In this case for any $x,y\in[0,2)$, neither Tx nor Ty is comparable to $2=Tu$.
Case (iii) : If $u\in(0,1)$ then $Su\npreceq Tu$.
Hence 'Condition(H)' fails to hold.

Example 3.4. Let $X=\{1,2,4,5\}$ with the usual metric. We define partial order $\preceq$ on X as follows:
$\preceq:=\{(1,1),(2,2),(4,4)(5,5),(1,2),(1,4),(1,5),(2,4),(2,5)\}$, where
$x\preceq y$ means $x\leq y$ in the usual sense.
Then $(X,\preceq,d)$ is a partially ordered metric space. We define
$S,T:X\to X$ by $S1=1,S2=2,S4=5,S5=4$ and
$T1=T2=1,T4=T5=2.$
Clearly $T(X)\subseteq S(X)$, and T is S non-decreasing.
We choose $x_0=1\in X$. Then $Sx_0\preceq Tx_0$ and clearly S and T are compatible and reciprocally continuous.
We define $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ by $\varphi(t)=t^2,\ t\geq 0,$ and

$$\psi:\mathbb{R}^+\to\mathbb{R}^+ \text{ by } \psi(t)=\begin{cases} t & if\ t\in[0,1]\\ 2 & otherwise.\end{cases}$$

We now verify the inequality (2.1).
Case (i) :Let $(x,\ y)=(1,5)$ such that $S1\preceq S5$.

In this case, $\varphi(d(T1,T5))=\varphi(d(1,2))=\varphi(1)=1$, $M(1,5)=3$ and $N(1,5)=3$.
Now $\varphi(M(1,5))=\varphi(3)=9$, $\psi(N(1,5))=\psi(3)=2$.
Therefore
$\varphi(d(T1,T2))=1\leq 9-2=\varphi(M(1,5))-\psi(N(1,5))$.
Case (ii) : Let $(x,\ y)=(1,4)$ such that $S1\preceq S4$.

In this case, $\varphi(d(T1,T4))=\varphi(d(1,2))=\varphi(1)=1$, $M(1,4)=4$ and $N(1,4)=4$.
Now $\varphi(M(1,4))=\varphi(4)=16$, $\psi(N(1,4))=\psi(4)=2$.
Therefore
$\varphi(d(T1,T4))=1\leq 16-2=\varphi(M(1,4))-\psi(N(1,4))$.
Case (iii) : Let $(x,\ y)=(2,5)$ such that $S2\preceq S5$.

In this case, $\varphi(d(T2,T5)) = \varphi(d(1,2)) = \varphi(1) = 1$, $M(2,5) = 2$ and $N(2,5) = 2$. Now $\varphi(M(2,5)) = \varphi(2) = 4$, $\psi(N(2,5)) = \psi(2) = 2$.
Therefore
$\varphi(d(T2,T5)) = 1 \leq 4 - 2 = \varphi(M(2,5)) - \psi(N(2,5))$.
<u>*Case (iv)*</u> : Let $(x,\ y) = (2,4)$ such that $S2 \preceq S4$.

In this case, $\varphi(d(T2,T4)) = \varphi(d(1,2)) = \varphi(1) = 1$, $M(2,4) = 3$ and $N(2,4) = 3$. Now $\varphi(M(2,4)) = \varphi(3) = 9$, $\psi(N(2,4)) = \psi(3) = 2$.
Therefore
$\varphi(d(T2,T4)) = 1 \leq 9 - 2 = \varphi(M(2,4)) - \psi(N(2,4))$.
In the remaining cases the inequality (2.1) holds trivially.

Further, by choosing $u = 1$ with $S1 \preceq T1$ and $T1$ is comparable with Tx and Ty for all $x, y \in X$ so that 'Condition (H)' holds.
Therefore, S and T satisfy all the hypotheses of Theorem 2.3 and S and T have a unique common fixed point 1.

Example 3.5. Let $X = [0,1]$ with usual metric. We define partial order $\preceq$ on X as follows:
$\preceq := \{(\frac{1}{2^n}, \frac{1}{2^{n+k}})/n = 0,1,2,...,k = 1,2,3,....\} \cup \{(0,x)/x \in X\} \cup \Delta$, where $x \preceq y$ means $x \geq y$ in the usual sense.
Then $(X, \preceq, d)$ is a partially ordered complete metric space. We define
$S : X \to X$ by $Sx = \begin{cases} x & if\ x \in [0, \frac{1}{4}] \cup \{\frac{1}{2}, 1\} \\ 2x & if\ x \in (\frac{1}{4}, \frac{1}{2}) \\ \frac{x}{2} & if\ (\frac{1}{2}, 1) \text{ and} \end{cases}$
$T : X \to X$ by $Tx = \frac{x^2}{4}$ for all $x \in [0,1]$.
Clearly $T(X) \subseteq S(X)$, and T is S non-decreasing.
We choose $x_0 = \frac{1}{2} \in X$. Then $Sx_0 \preceq Tx_0$
We define $\varphi,\ \psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(t) = t,\ t \geq 0$, and $\psi(t) = \frac{t}{4},\ t \geq 0$.
We now verify the inequality (2.1).
<u>*Case (I)*</u> : Let $(x,y) = (\frac{1}{2^n}, \frac{1}{2^{n+k}})$ such that $Sx \preceq Sy$ for $n \geq 0$ and $k \geq 1$.
In this case, we have
$M(\frac{1}{2^n}, \frac{1}{2^{n+k}}) = \max\{a,b,c,d\}$ and $N(\frac{1}{2^n}, \frac{1}{2^{n+k}}) = \max\{a,b,d\}$,
where
$a = (\frac{d(S(\frac{1}{2^{n+k}}),T(\frac{1}{n+k}))[1+d(S(\frac{1}{2^n}),T(\frac{1}{2^n})]}{1+d(S(\frac{1}{2^n}),S(\frac{1}{2^{n+k}})}),\ b = (\frac{d(S(\frac{1}{2^n}),T(\frac{1}{2^n}))[1+d(S(\frac{1}{2^{n+k}}),T(\frac{1}{n+k}))]}{1+d(S(\frac{1}{2^n}),S(\frac{1}{2^{n+k}})})$,
$c = (\frac{d(S(\frac{1}{2^{n+k}}),T(\frac{1}{2^n})[1+d(S(\frac{1}{2^n}),T(\frac{1}{2^{n+k}})]}{1+d(S(\frac{1}{2^n}),S(\frac{1}{2^{n+k}})}),\ d = d(S(\frac{1}{2^n}), S(\frac{1}{2^{n+k}}))$.
We observe the following:

1. $a \leq b$ for all $k \geq 1$ and for all $n \geq 0$,
2. $c \leq b$ for all $k \leq n+2$,
3. $c \leq d$ for all $k \geq n+2$.

Hence $M(x,y) = N(x,y) = b$ or d.
<u>*Subcase (i)*</u> : $M(x,y) = N(x,y) = b$.

In this case, we have $(\frac{1}{2^{2n+2}} - \frac{1}{2^{2n+2k+2}}) \leq \frac{\frac{3}{4}(\frac{1}{2^n} - \frac{1}{2^{2n+2}})(1+\frac{1}{2^{n+k}} - \frac{1}{2^{2n+2k+2}})}{(1+\frac{1}{2^n} - \frac{1}{2^{n+k}})}$ for all $n \geq 0$ and $k \geq 1$, which implies that
$\varphi(d(Tx,Ty)) \leq b - \frac{b}{4} = \varphi(b) - \psi(b) = \varphi(M(x,y)) - \psi(N(x,y))$.
<u>*Subcase (ii)*</u> : $M(x,y) = N(x,y) = d$.

In this case, we have $(\frac{1}{2^{2n+2}} - \frac{1}{2^{2n+2k+2}}) \leq \frac{3}{4}(\frac{1}{2^n} - \frac{1}{2^{n+k}})$ which implies that
$\varphi(d(Tx,Ty)) \leq b - \frac{d}{4} = \varphi(d) - \psi(d) = \varphi(M(x,y)) - \psi(N(x,y))$.
In either case, the inequality (2.1) holds.
Case (II) : Let $(x,y) = (0,x)$ such that $S0 \preceq Sx$.

In this case, $M(0,x) = N(0,x) = \begin{cases} x & if\ x \in [0,\frac{1}{4}] \cup \{\frac{1}{2},1\} \\ 2x & if\ x \in (\frac{1}{4},\frac{1}{2}) \\ \frac{x}{2} & if\ x \in (\frac{1}{2},1). \end{cases}$

If $x \in [0,\frac{1}{4}] \cup \{\frac{1}{2},1\}$ then
$\varphi(d(T0,Tx)) = \frac{x^2}{4} \leq \frac{3x}{4} = x - \frac{x}{4} = \varphi(M(0,x)) - \psi(N(0,x))$.
Similarly, it is easy to see that the inequality (2.1) holds in all other cases.
Case (III) : Let $(x,y) \in \Delta$ such that $x = y$.
In this case, we note that
$M(x,x) = N(x,x) = d(Sx,Tx)(1 + d(Sx,Tx))$ for all $x \in X$.

$$\text{Now } \varphi(d(Tx,Tx)) = \varphi(0) \leq \frac{3}{4}M(x,x) = M(x,x) - \frac{N(x,x)}{4} = \varphi(M(x,y)) - \psi(N(x,x)) \text{ for all } x \in X.$$

Hence S and T satisfy the inequality (2.1).
Also, S,T are reciprocally continuous and compatible.
So let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$. Therefore, $x_n \to 0$ and $z = 0$. There exists $N \in \mathbb{Z}^+$ such that $n \geq N$ implies $x_n \leqslant \frac{1}{4}$. Therefore $Sx_n = x_n$ and $Tx_n = \frac{x_n^2}{4}$ for all $n \geq N$. Now $TSx_n = Tx_n = \frac{x_n^2}{4}$ and $STx_n = S(\frac{x_n^2}{4}) = \frac{x_n^2}{4}$ for all $n \geq N$.
Therefore $TSx_n = STx_n$ for all $n \geq N$. There is $d(TSx_n, STx_n) = 0$ for all $n \geq N$. Hence $\lim_{n\to\infty} d(TSx_n, STx_n) = 0$. Therefore, the pair (S,T) is compatible.
Also, $\lim_{n\to\infty} STx_n = \lim_{n\to\infty} \dfrac{x_n^2}{4} = 0 = S0$ and $\lim_{n\to\infty} TSx_n = \lim_{n\to\infty} \dfrac{x_n^2}{4} = 0 = T0$. Therefore, S,T are reciprocally continuous. We observe that 'condition (H)' holds, because by choosing $0 \in X$ we have $S0 \preceq T0$ and $T0 = 0$ is comparable with Tx and Ty for all $x,y \in X$. Hence all the hypotheses of Theorem 2.3 hold and S and T have a unique common fixed point 0.

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