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EXISTENCE AND STABILITY RESULTS FOR IMPULSIVE INTEGRO-DIFFERENTIAL EQUATIONS*

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Abstract. In this paper, we study a new class of impulsive integro-differential equations for which the impulses are not instantaneous. By using fixed point approach and techniques of analysis, we present the existence and uniqueness theorem and derive an interesting stability result in the sense of generalized Ulam-Hyers-Rassias.

1. Introduction and Preliminaries

Differential equations with instantaneous impulses have been treated in several works, see for instance the monographs [1, 2, 3]. It seems that the classical differential equations models with instantaneous impulses could not characterize the dynamics of evolution processes completely in pharmacotherapy. For example, consider the hemodynamic equilibrium of a person, the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous process. In fact, this situation should be characterized by a new case of impulsive action, which starts at an arbitrary fixed point and stays active on a finite time interval. Recently, the authors [4, 5] introduced a new class of abstract semilinear impulsive differential equations with not instantaneous impulses and obtained some interesting existence and uniqueness theorems.

The famous stability of functional equations was raised by Ulam [6] in 1940 at University of Wisconsin. In the past seventy years, Ulam's type stability problems have been taken up by a large number of mathematicians and the study of this area has grown to be one of the most important subjects in the mathematical analysis area. For a quite long time, Ulam stability problem has been attracted by many famous researchers since it is quite useful in many applications such as numerical

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analysis, optimization, biology and economics, where finding the exact solution is quite difficult. For more details, the readers can refer to the monographs of [7, 8, 9, 10, 11, 12, 13] and other recent contribution [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31] by means of fixed point approach and classical analysis methods.

The main objective of this paper is to apply a fixed point theorem of the alternative, for contractions on a generalized complete metric space to derive existence and a generalized Ulam-Hyers-Rassias stability results for a new class of impulsive integro-differential equations of the form:

$$\begin{cases} x'(t) = f\left(t, x(t), \int_0^t k(s, x(s))ds\right), \ t \in (s_i, t_{i+1}], \ i = 0, 1, 2, \ldots, m, \\ x(t) = g_i\left(t, x(t), \int_0^t l(s, x(s))ds\right), \ t \in (t_i, s_i], \ i = 1, 2, \ldots, m, \end{cases} \tag{1.1}$$

where $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \cdots < t_m \leq s_m < t_{m+1} = T$ are pre-fixed numbers, $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $k, l : [0, T] \times \mathbb{R} \to \mathbb{R}$ are continuous, and $g_i : [t_i, s_i] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous for all $i = 1, 2, \ldots, m$, which is called not instantaneous impulses.

Let $J = [0, T]$ and $C(J, \mathbb{R})$ be the space of all continuous functions from J into $\mathbb{R}$. We also recall the piecewise continuous functions space

$$PC(J, \mathbb{R}) := \{x : J \to \mathbb{R} : x \in C((t_k, t_{k+1}], \mathbb{R}), \ k = 0, 1, \ldots, m$$

and there exist $x(t_k^-)$ and $x(t_k^+), k = 1, \ldots, m,$ with $x(t_k^-) = x(t_k)\}$. Meanwhile, we set

$$PC^1(J, \mathbb{R}) := \{x \in PC(J, \mathbb{R}) : x' \in PC(J, \mathbb{R})\}.$$

A function $x \in PC^1(J, \mathbb{R})$ is called a classical solution of the impulsive Cauchy problem

$$\begin{cases} x'(t) = f\left(t, x(t), \int_0^t k(s, x(s))ds\right), \ t \in (s_i, t_{i+1}], \ i = 0, 1, 2, \ldots, m, \\ x(t) = g_i\left(t, x(t), \int_0^t l(s, x(s))ds\right), \ t \in (t_i, s_i], \ i = 1, 2, \ldots, m, \\ x(0) = x_0 \in \mathbb{R}, \end{cases} \tag{1.2}$$

if x satisfies

$$x(0) = x_0, \ x(t) = g_i\left(t, x(t), \int_0^t l(s, x(s))ds\right), \ t \in (t_i, s_i], \ i = 1, 2, \ldots, m,$$

and

$$\begin{aligned} x(t) &= x_0 + \int_0^t f\left(s, x(s), \int_0^s k(\tau, x(\tau))d\tau\right)ds, \ t \in [0, t_1], \\ x(t) &= g_i\left(s_i, x(s_i), \int_0^{s_i} l(s, x(s))ds\right) + \int_{s_i}^t f\left(s, x(s), \int_0^s k(\tau, x(\tau))d\tau\right)ds, \\ &\quad t \in (s_i, t_{i+1}], \ i = 1, 2, \ldots, m. \end{aligned}$$

Motivated by the concepts of stability in [26, 29], we can introduce a generalized Ulam-Hyers-Rassias stability concept for the equation (1.1).

Let $\psi \geqslant 0$ and $\varphi \in PC(J, \mathbb{R}_+)$ is nondecreasing. Consider

$$\begin{cases} \left|y'(t) - f\left(t, y(t), \int_0^t k(s, y(s))ds\right)\right| \leq \varphi(t), \\ t \in (s_i, t_{i+1}],\ i = 0, 1, 2, \ldots, m, \\ \left|y(t) - g_i\left(t, y(t), \int_0^t l(s, y(s))\right)\right| \leq \psi, \\ t \in (t_i, s_i],\ i = 1, 2, \ldots, m. \end{cases} \tag{1.3}$$

Definition 1.1. The equation (1.1) is generalized Ulam-Hyers-Rassias stable with respect to (φ, ψ) if there exists $c_{f,g_i,\varphi} > 0$ such that for each solution $y \in PC^1(J, \mathbb{R})$ of the inequality (1.3) there exists a solution $x \in PC^1(J, \mathbb{R})$ of the equation (1.1) with

$$|y(t) - x(t)| \leq c_{f,g_i,\varphi}(\varphi(t) + \psi),\ t \in J.$$

Obviously, if $y \in PC^1(J, \mathbb{R})$ is a solution of the inequality (1.3) then y is a solution of the following integral inequality

$$\begin{cases} \left|y(t) - g_i\left(t, y(t), \int_0^t l(s, y(s))\right)\right| \leq \psi, \\ t \in (t_i, s_i],\ i = 1, 2, \ldots, m; \\ \left|y(t) - y(0) - \int_0^t f\left(s, y(s), \int_0^s k(\tau, y(\tau))d\tau\right)ds\right| \leq \int_0^t \varphi(s)ds, \\ t \in [0, t_1]; \\ \left|y(t) - g_i\left(s_i, y(s_i), \int_0^{s_i} l(s, y(s))ds\right)\right. \\ \left. - \int_{s_i}^t f\left(s, y(s), \int_0^s k(\tau, y(\tau))d\tau\right)ds\right| \leq \psi + \int_{s_i}^t \varphi(s)ds, \\ t \in (s_i, t_{i+1}],\ i = 1, 2, \ldots, m. \end{cases} \tag{1.4}$$

In order to apply a fixed point theorem of the alternative, for contractions on a generalized complete metric space to derive our main result, we need to collect the following fact.

For a nonempty set X, a function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if and only if d satisfies
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.1. *(see [32]) Let (X, d) be a generalized complete metric space. Assume that $T : X \to X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(T^{k+1}x, T^k x) < +\infty$ for some $x \in X$, then the*

followings are true:
(i) The sequence $\{T^n x\}$ *converges to a fixed point* x^* *of* T;
(ii) x^* *is the unique fixed point of* T *in*

$$X^* = \left\{ y \in X \mid d(T^k x, y) < \infty \right\};$$

(iii) If $y \in X^*$, *then*

$$d(y, x^*) \leq \frac{1}{1-L} d(Ty, y).$$

2. Existence and Stability

We introduce the following assumptions:

(H_1) $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

(H_2) There exists a positive constant L_f such that

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq L_f(|u_1 - u_2| + |v_1 - v_2|)$$

for each $t \in J$, and all $u_1, u_2, v_1, v_2 \in \mathbb{R}$.

(H_3) $g_i \in C([t_i, s_i] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there are positive constants L_{g_i}, $i = 1, 2, \ldots, m$ such that

$$|g_i(t, u_1, v_1) - g_i(t, u_2, v_2)| \leq L_{g_i}(|u_1 - u_2| + |v_1 - v_2|),$$

for each $t \in (t_i, s_i]$, and all $u_1, u_2, v_1, v_2 \in \mathbb{R}$.

(H_4) $k \in C(J \times \mathbb{R}, \mathbb{R})$ and there is a positive constant K such that

$$|k(t, u_1) - k(t, u_2)| \leq K|u_1 - u_2|, \text{ for each } t \in J, \text{ and all } u_1, u_2 \in \mathbb{R}.$$

(H_5) $l \in C(J \times \mathbb{R}, \mathbb{R})$ and there is a positive constant L, such that

$$|l(t, u_1) - l(t, u_2)| \leq L|u_1 - u_2|, \text{ for each } t \in J, \text{ and all } u_1, u_2 \in \mathbb{R}.$$

(H_6) Let $\varphi \in C(J, \mathbb{R}_+)$ be a nondecreasing function. There exists $c_\varphi > 0$ such that

$$\int_0^t \varphi(s) ds \leq c_\varphi \varphi(t), \text{ for each } t \in J.$$

Now, we discuss existence and stability of the equation (1.1) by using the concept of generalized Ulam-Hyers-Rassias in the above section.

Theorem 2.1. *Assume that* $(H_1), (H_2), (H_3), (H_4), (H_5), (H_6)$ *are satisfied and a function* $y \in PC^1(J, \mathbb{R})$ *satisfies (1.3). Then there exists a unique solution* $y_0 : J \to \mathbb{R}$ *such that*

$$y_0(t) = \begin{cases} x(0) + \int_0^t f\left(s, y_0(s), \int_0^s k(\tau, y_0(\tau))d\tau\right)ds, \\ \quad t \in [0, t_1], \\ g_i\left(t, y_0(t), \int_0^t l(s, y_0(s))ds\right), \\ \quad t \in (t_i, s_i],\ i = 1, 2, \dots, m, \\ g_i\left(s_i, y_0(s_i), \int_0^{s_i} l(s, y_0(s))ds\right) \\ + \int_{s_i}^t f\left(s, y_0(s), \int_0^s k(\tau, y_0(\tau))d\tau\right)ds, \\ \quad t \in (s_i, t_{i+1}],\ i = 1, 2, \dots, m, \end{cases} \tag{2.1}$$

and

$$|y(t) - y_0(t)| \le \frac{(1 + c_\varphi)(\varphi(t) + \psi)}{1 - \rho},\ t \in J, \tag{2.2}$$

provided that

$$\begin{aligned} \rho : \ &= \ \max\Big\{(Lc_\varphi + LT + 1)L_{g_i} \\ &\quad +(KT^2 + Kc_\varphi^2 + c_\varphi)L_f \mid\ i = 1, 2, \dots, m\Big\} < 1. \end{aligned} \tag{2.3}$$

Proof. Consider the space of piecewise continuous functions

$$X = \{g : J \to \mathbb{R} \mid g \in PC(J, \mathbb{R})\}, \tag{2.4}$$

endowed with the generalized metric on X defined by

$$\begin{aligned} d(g, h) \ &= \ \inf\Big\{C_1 + C_2 \in [0, +\infty] \mid |g(t) - h(t)| \\ &\quad \le (C_1 + C_2)(\varphi(t) + \psi) \text{ for all } t \in J\Big\}, \end{aligned} \tag{2.5}$$

where

$$C_1 \in \{C \in [0, +\infty] \mid |g(t) - h(t)| \le C\varphi(t) \text{ for all } t \in (s_i, t_{i+1}],\ i = 0, 1, 2, \dots, m\},$$

and

$$C_2 \in \{C \in [0, +\infty] \mid |g(t) - h(t)| \le C\psi \text{ for all } t \in (t_i, s_i],\ i = 1, 2, \dots, m\}.$$

It is easy to verify that (X, d) is a complete generalized metric space.

Define an operator $\Lambda : X \to X$ by

$$(\Lambda x)(t) = \begin{cases} x_0 + \int_0^t f\left(s, x(s), \int_0^s k(\tau, x(\tau))d\tau\right)ds, \\ \quad t \in [0, t_1]; \\ g_i\left(t, x(t), \int_0^t l(s, x(s))ds\right), \\ \quad t \in (t_i, s_i],\ i = 1, 2, \ldots, m; \\ g_i\left(s_i, x(s_i), \int_0^{s_i} l(s, x(s))ds\right) \\ + \int_{s_i}^t f\left(s, x(s), \int_0^s k(\tau, x(\tau))d\tau\right)ds, \\ \quad t \in (s_i, t_{i+1}],\ i = 1, 2, \ldots, m. \end{cases} \tag{2.6}$$

for all $x \in X$ and $t \in [0, T]$. Clearly, Λ is a well defined operator according to (H_1), (H_3), (H_4), (H_5).

Next, we shall verify that Λ is strictly contractive on X. Note that the definition of (X, d), for any $g, h \in X$, it is possible to find a $C_1, C_2 \in [0, \infty]$ such that

$$|g(t) - h(t)| \quad \leq \quad \begin{cases} C_1\varphi(t),\ t \in (s_i, t_{i+1}],\ i = 0, 1, 2, \ldots, m, \\ C_2\psi, \quad t \in (t_i, s_i],\ i = 1, 2, \ldots, m. \end{cases} \tag{2.7}$$

From the definition of Λ in (2.6), (H_2), (H_3), (H_4), (H_5) and (2.7), we obtain that
Case 1: $t \in [0, t_1]$,

$$\begin{aligned} & |(\Lambda g)(t) - (\Lambda h)(t)| \\ \leq\ & \int_0^t \left| f\left(s, g(s), \int_0^s k(\tau, g(\tau))d\tau\right) - f\left(s, h(s), \int_0^s k(\tau, h(\tau))d\tau\right)\right| ds \\ \leq\ & L_f \int_0^t \left(|g(s) - h(s)| + \left|\int_0^s k(\tau, g(\tau))d\tau - \int_0^s k(\tau, h(\tau))d\tau\right|\right)ds \\ \leq\ & L_f \int_0^t \left(C_1\varphi(s) + \int_0^s |k(\tau, g(\tau)) - k(\tau, h(\tau))|d\tau\right)ds \\ \leq\ & L_f \int_0^t \left(C_1\varphi(s) + K\int_0^s |g(\tau) - h(\tau)|d\tau\right)ds \\ \leq\ & L_f \int_0^t \left(C_1\varphi(s) + KC_1\int_0^s \varphi(\tau)d\tau\right)ds \\ \leq\ & L_f C_1 \int_0^t \left(\varphi(s) + Kc_\varphi\varphi(s)\right)ds \\ =\ & L_f C_1(1 + Kc_\varphi)\int_0^t \varphi(s)ds \\ \leq\ & L_f C_1(1 + Kc_\varphi)c_\varphi\varphi(t). \end{aligned}$$

Case 2: for $t \in (t_i, s_i]$,

$$\begin{aligned}
& |(\Lambda g)(t) - (\Lambda h)(t)| \\
\leq\ & L_{g_i}\left(|g(t) - h(t)| + \left|\int_0^t l(s, g(s))ds - \int_0^t l(s, h(s))ds\right|\right) \\
\leq\ & L_{g_i}\left(C_2\psi + \int_0^t |l(s, g(s)) - l(s, h(s))|ds\right) \\
\leq\ & L_{g_i}\left(C_2\psi + L\int_0^t |g(s) - h(s)|ds\right) \\
\leq\ & L_{g_i}\left(C_2\psi + L(C_1 + C_2)\int_0^t (\varphi(s) + \psi)ds\right) \\
\leq\ & L_{g_i}\left(C_2\psi + L(C_1 + C_2)(c_\varphi\varphi(t) + \psi T)\right).
\end{aligned}$$

Case 3: for $t \in (s_i, t_{i+1}]$,

$$\begin{aligned}
& |(\Lambda g)(t) - (\Lambda h)(t)| \\
\leq\ & \left|g_i\left(s_i, g(s_i), \int_0^{s_i} l(s, g(s))ds\right) - g_i\left(s_i, h(s_i), \int_0^{s_i} l(s, h(s))ds\right)\right| \\
& + \left|\int_{s_i}^t f\left(s, g(s), \int_0^s k(\tau, g(\tau))d\tau\right)ds - \int_{s_i}^t f\left(s, h(s), \int_0^s k(\tau, h(\tau))d\tau\right)ds\right| \\
\leq\ & L_{g_i}\left(C_2\psi + L\int_0^{s_i} |g(s) - h(s)|ds\right) \\
& + L_f\int_{s_i}^t \left(C_1\varphi(s) + K\int_0^s |g(\tau) - h(\tau)|d\tau\right)ds \\
\leq\ & L_{g_i}\left(C_2\psi + L(C_1 + C_2)\int_0^{s_i} (\varphi(s) + \psi)ds\right) \\
& + L_f\int_{s_i}^t \left(C_1\varphi(s) + K(C_1 + C_2)\int_0^s (\varphi(\tau) + \psi)d\tau\right)ds \\
\leq\ & L_{g_i}\left(C_2\psi + L(C_1 + C_2)(c_\varphi\varphi(s_i) + \psi s_i)\right) \\
& + L_f\int_{s_i}^t \left(C_1\varphi(s) + K(C_1 + C_2)(c_\varphi\varphi(s) + \psi s)\right)ds \\
\leq\ & L_{g_i}\left(C_2\psi + L(C_1 + C_2)(c_\varphi\varphi(t) + \psi T)\right) \\
& + L_f\left(C_1c_\varphi\varphi(t) + K(C_1 + C_2)(c_\varphi^2\varphi(t) + \psi T^2)\right) \\
=\ & \left(L_{g_i}L(C_1 + C_2)c_\varphi + L_fC_1c_\varphi + L_fK(C_1 + C_2)c_\varphi^2\right)\varphi(t)
\end{aligned}$$

$$
\begin{aligned}
& +\left(L_{g_i}C_2 + L_{g_i}L(C_1+C_2)T + L_fK(C_1+C_2)T^2\right)\psi \\
\leq \quad & \left((Lc_\varphi + LT + 1)L_{g_i} + (KT^2 + Kc_\varphi^2 + c_\varphi)L_f\right)(C_1+C_2)(\varphi(t)+\psi).
\end{aligned}
$$

From above, we have

$$
|(\Lambda g)(t) - (\Lambda h)(t)| \quad \leq \quad \rho(C_1+C_2)(\varphi(t)+\psi),\ t \in J,
$$

that is,

$$
d(\Lambda g, \Lambda h) \leq \rho(C_1+C_2)(\varphi(t)+\psi).
$$

Hence, we can conclude that

$$
d(\Lambda g, \Lambda h) \leq \rho d(g,h)
$$

for any $g, h \in X$, and since the condition (2.3), the strictly continuous property is shown.

Let us take $g_0 \in X$. From the piecewise continuous property of g_0 and Λg_0, it follows that there exists a constant $0 < G_1 < \infty$ such that

$$
\begin{aligned}
& |(\Lambda g_0)(t) - g_0(t)| \\
= \quad & \left|x(0) + \int_0^t f\left(s, g_0(s), \int_0^s k(\tau, g_0(\tau))d\tau\right)ds - g_0(t)\right| \\
\leq \quad & G_1\varphi(t) \\
\leq \quad & G_1(\varphi(t)+\psi),\ t \in [0, t_1].
\end{aligned}
$$

There exists a constant $0 < G_2 < \infty$ such that

$$
\begin{aligned}
& |(\Lambda g_0)(t) - g_0(t)| \\
= \quad & \left|g_i\left(t, g_0(t), \int_0^t l(s, g_0(s))ds\right) - g_0(t)\right| \\
\leq \quad & G_2\psi \\
\leq \quad & G_2(\varphi(t)+\psi),\ t \in (t_i, s_i],\ i = 1, 2, \ldots, m.
\end{aligned}
$$

There exists a constant $0 < G_3 < \infty$ such that

$$
\begin{aligned}
& |(\Lambda g_0)(t) - g_0(t)| \\
= \quad & \left|g_i\left(s_i, g_0(s_i), \int_0^{s_i} l(s, g_0(s))ds\right) + \int_{s_i}^t f\left(s, g_0(s), \int_0^s k(\tau, g_0(\tau))d\tau\right)ds - g_0(t)\right| \\
\leq \quad & G_3(\varphi(t)+\psi),\ t \in (s_i, t_{i+1}],\ i = 1, 2, \ldots, m,
\end{aligned}
$$

since $f, g_i, g_0, \int_0^{s_i} l(s, g_0(s))ds$ and $\int_0^s k(\tau, g_0(\tau))d\tau$ are bounded on J and $\varphi(\cdot) + \psi > 0$. Thus, (2.5) implies that

$$
d(\Lambda g_0, g_0) < \infty.
$$

By using Banach fixed point theorem, there exists a continuous function $y_0 : J \to \mathbb{R}$ such that $\Lambda^n g_0 \to y_0$ in (X, d) as $n \to \infty$ and $\Lambda y_0 = y_0$, that is, y_0 satisfies equation (2.1) for every $t \in J$.

Next, we check that

$$\{g \in X \mid d(g_0, g) < \infty\} = X.$$

For any $g \in X$, since g and g_0 are bounded on J and $\min_{t\in J}(\varphi(t) + \psi) > 0$, there exists a constant $0 < C_g < \infty$ such that

$$|g_0(t) - g(t)| \leq C_g(\varphi(t) + \psi),$$

for any $t \in J$. Hence, we have $d(g_0, g) < \infty$ for all $g \in X$, that is,

$$\{g \in X \mid d(g_0, g) < \infty\} = X.$$

Hence, we conclude that y_0 is the unique continuous function with the property (2.1). On the other hand, from (1.4) and (H_6) it follows that

$$d(y, \Lambda y) \leq 1 + c_\varphi. \tag{2.8}$$

Summarizing, we have

$$d(y, y_0) \quad \leq \quad \frac{d(\Lambda y, y)}{1-\rho} \quad \leq \quad \frac{1 + c_\varphi}{1-\rho},$$

which means that (2.2) is true for $t \in J$. The proof is done. $\square$

3. An example

In this section, we give an example to illustrate the above results.

Consider

$$\begin{cases} x'(t) = \frac{1}{5+t^2}\left(|x(t)| + \int_0^t \frac{|x(s)|}{10+s^2}ds\right), \ t \in (0,1], \\ x(t) = \frac{1}{(5+(t-1)^2)(1+|x(t)|)}\left(|x(t)| + \int_0^t \frac{|x(s)|}{15+s^2}ds\right), \ t \in (1,2], \end{cases}$$

and

$$\begin{cases} \left|y'(t) - \frac{1}{5+t^2}\left(|y(t)| + \int_0^t \frac{|y(s)|}{10+s^2}ds\right)\right| \leq e^t, \ t \in [0,1], \\ \left|y(t) - \frac{1}{(5+(t-1)^2)(1+|y(t)|)}\left(|y(t)| + \int_0^t \frac{|y(s)|}{15+s^2}ds\right)\right| \leq 1, \ t \in (1,2]. \end{cases}$$

Let $J = [0,2]$ and $0 = t_0 = s_0 < t_1 = 1 < s_1 = 2$. Denote $k(t, x(t)) = \frac{|x(t)|}{10+t^2}$ by $K = \frac{1}{10}$, and

$$f(t, x(t), \int_0^t k(s, x(s))ds) = \frac{1}{5+t^2}\left(|x(t)| + \int_0^t \frac{|x(s)|}{10+s^2}ds\right)$$

by $L_f = \frac{1}{5}$ for $t \in [0,1]$, and $l(t,x(t)) = \frac{|x(t)|}{15+t^2}$ with $L = \frac{1}{15}$, and

$$g_1(t,x(t),\int_0^t l(s,x(s))ds) = \frac{1}{(5+(t-1)^2)(1+|x(t)|)}\left(|x(t)| + \int_0^t \frac{|x(s)|}{15+s^2}ds\right)$$

with $L_{g_1} = \frac{1}{5}$ for $t \in (1,2]$. We put $\varphi(t) = e^t$ and $\psi = 1$. Set $c_\varphi = 1$, we have $\int_0^t e^s ds \le e^t$. Obviously,

$$(Lc_\varphi + LT + 1)L_{g_1} + (KT^2 + Kc_\varphi^2 + c_\varphi)L_f = \frac{27}{50} < 1.$$

By Theorem 2.1, there exists a unique solution $y_0 : [0,2] \to \mathbb{R}$ such that

$$y_0(t) = \begin{cases} x(0) + \int_0^t \frac{1}{5+s^2}\left(|y_0(s)| + \int_0^s \frac{|y_0(\tau)|}{10+\tau^2}d\tau\right)ds, & t \in [0,1], \\ \frac{1}{(5+(t-1)^2)(1+|y_0(t)|)}\left(|y_0(t)| + \int_0^t \frac{|y_0(s)|}{15+s^2}ds\right), & t \in (1,2], \end{cases}$$

and

$$|y(t) - y_0(t)| \le \frac{100}{23}(e^t + 1),\ t \in [0,2].$$

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