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REMARKS ON METALLIC WARPED PRODUCT MANIFOLDS

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Abstract. We characterize the metallic structure on the product of two metallic manifolds in terms of metallic maps and provide a necessary and sufficient condition for the warped product of two locally metallic Riemannian manifolds to be locally metallic. We discuss a particular case of the product manifolds and we construct an example of the metallic warped product Riemannian manifold.
Keywords: Riemannian manifold, metallic warped product, projection mapping.

1. Introduction

Starting from a polynomial structure, which was generally defined by S. I. Goldberg, K. Yano and N. C. Petridis in ([8],[9]), we consider a polynomial structure on an m-dimensional Riemannian manifold (M, g), called by us a *metallic structure* ([6],[11],[7],[12]), determined by a $(1,1)$-tensor field J which satisfies the equation:

$$J^2 = pJ + qI, \tag{1.1}$$

where I is the identity operator on the Lie algebra of vector fields on M identified with the set of smooth sections $\Gamma(T(M))$ (and we will simply denote $X \in T(M)$) with p and q are non-zero natural numbers). From the definition, we easily get the recurrence relation:

$$J^{n+1} = g_{n+1} \cdot J + g_n \cdot I, \tag{1.2}$$

where $(\{g_n\}_{n\in\mathbb{N}^*})$ is the generalized secondary Fibonacci sequence defined by $g_{n+1} = pg_n + qg_{n-1}$, $n \geq 1$ with $g_0 = 0$, $g_1 = 1$ and p, $q \in \mathbb{N}^*$.

If (M, g) is a Riemannian manifold endowed with a metallic structure J such that the Riemannian metric g is J-compatible (i.e. $g(JX, Y) = g(X, JY)$, for any $X, Y \in T(M)$), then (M, g, J) is called a *metallic Riemannian manifold*. In this case:

$$g(JX, JY) = pg(X, JY) + qg(X, Y), \tag{1.3}$$

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for any $X, Y \in T(M)$.

It is known ([13]) that an almost product structure F on M induces two metallic structures:

$$J_{\pm} = \pm \frac{2\sigma_{p,q} - p}{2} F + \frac{p}{2} I \tag{1.4}$$

and, conversely, every metallic structure J on M induces two almost product structures:

$$F_{\pm} = \pm \frac{2}{2\sigma_{p,q} - p} J - \frac{p}{2\sigma_{p,q} - p} I, \tag{1.5}$$

where $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ is the metallic number, which is a positive solution of the equation $x^2 - px - q = 0$, for p and q non-zero natural numbers.

In particular, if the almost product structure F is compatible with the Riemannian metric, then J_+ and J_- are metallic Riemannian structures.

On a metallic manifold (M, J) there exist two complementary distributions $\mathcal{D}_l$ and $\mathcal{D}_m$ corresponding to the projection operators l and m ([13]) given by:

$$l = -\frac{1}{2\sigma_{p,q} - p} J + \frac{\sigma_{p,q}}{2\sigma_{p,q} - p} I, \quad m = \frac{1}{2\sigma_{p,q} - p} J + \frac{\sigma_{p,q} - p}{2\sigma_{p,q} - p} I. \tag{1.6}$$

The analogue concept of a locally product manifold is considered in the context of metallic geometry. Precisely, we say that the metallic Riemannian manifold (M, g, J) is *locally metallic* if J is parallel with respect to the Levi-Civita connection associated to g.

2. Metallic warped product Riemannian manifolds

2.1. Warped product manifolds

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds of dimensions n and m, respectively. Denote by p_1 and p_2 the projection maps from the product manifold $M_1 \times M_2$ onto M_1 and M_2 and by $\widetilde{\varphi} := \varphi \circ p_1$ the lift to $M_1 \times M_2$ of a smooth function φ on M_1. In this case, we call M_1 *the base* and M_2 *the fiber* of $M_1 \times M_2$. The unique element $\widetilde{X}$ of $T(M_1 \times M_2)$ that is p_1-related to $X \in T(M_1)$ and to the zero vector field on M_2 will be called the *horizontal lift of* X and the unique element $\widetilde{V}$ of $T(M_1 \times M_2)$ that is p_2-related to $V \in T(M_2)$ and to the zero vector field on M_1 will be called the *vertical lift of* V. Also denote by $\mathcal{L}(M_1)$ the set of all horizontal lifts of vector fields on M_1 and by $\mathcal{L}(M_2)$ the set of all vertical lifts of vector fields on M_2.

For $f > 0$ a smooth function on M_1, consider the Riemannian metric on $M_1 \times M_2$:

$$\widetilde{g} := p_1^* g_1 + (f \circ p_1)^2 p_2^* g_2. \tag{2.1}$$

Definition 2.1. ([4]) The product manifold of M_1 and M_2 together with the Riemannian metric $\widetilde{g}$ defined by (2.1) is called *the warped product* of M_1 and M_2 by the warping function f [and it is denoted by $(\widetilde{M} := M_1 \times_f M_2, \widetilde{g})$].

Note that if f is constant (equal to 1), the warped product becomes the usual product of the Riemannian manifolds.

For $(x,y) \in \widetilde{M}$, we shall identify $X \in T(M_1)$ with $(X_x, 0_y) \in T_{(x,y)}(\widetilde{M})$ and $Y \in T(M_2)$ with $(0_x, Y_y) \in T_{(x,y)}(\widetilde{M})$ ([3]).

The projection mappings of $T(M_1 \times M_2)$ onto $T(M_1)$ and $T(M_2)$, respectively, denoted by $\pi_1 =: Tp_1$ and $\pi_2 =: Tp_2$ verify:

$$\pi_1 + \pi_2 = I, \quad \pi_1^2 = \pi_1, \quad \pi_2^2 = \pi_2, \quad \pi_1 \circ \pi_2 = \pi_2 \circ \pi_1 = 0. \tag{2.2}$$

The Riemannian metric of the warped product manifold $\widetilde{M} = M_1 \times_f M_2$ equals to:

$$\widetilde{g}(\widetilde{X}, \widetilde{Y}) = g_1(X_1, Y_1) + (f \circ p_1)^2 g_2(X_2, Y_2), \tag{2.3}$$

for any $\widetilde{X} = (X_1, X_2), \widetilde{Y} = (Y_1, Y_2) \in T(\widetilde{M}) = T(M_1 \times_f M_2)$ and we notice that the leaves $M_1 \times \{y\}$, for $y \in M_2$, are totally geodesic submanifolds of $(\widetilde{M} = M_1 \times_f M_2, \widetilde{g})$.

If we denote by $\widetilde{\nabla}$, ${}^{M_1}\nabla$, ${}^{M_2}\nabla$ the Levi-Civita connections on $\widetilde{M}$, M_1 and M_2, we know that for any $X_1, Y_1 \in T(M_1)$ and $X_2, Y_2 \in T(M_2)$ ([14]):

$$\begin{aligned} \widetilde{\nabla}_{(X_1,X_2)}(Y_1, Y_2) = ({}^{M_1}\nabla_{X_1} Y_1 - \frac{1}{2} g_2(X_2, Y_2) \cdot grad(f^2), \\ {}^{M_2}\nabla_{X_2} Y_2 + \frac{1}{2f^2} X_1(f^2) Y_2 + \frac{1}{2f^2} Y_1(f^2) X_2). \end{aligned} \tag{2.4}$$

In particular:

$$\widetilde{\nabla}_{(X,0)}(0, Y) = \widetilde{\nabla}_{(0,Y)}(X, 0) = (0, X(\ln(f))Y).$$

Let R, R_{M_1}, R_{M_2} be the Riemannian curvature tensors on $\widetilde{M}$, M_1 and M_2 and $\widetilde{R_{M_1}}$, $\widetilde{R_{M_2}}$ the lift on $\widetilde{M}$ of R_{M_1} and R_{M_2}. Then:

Lemma 2.1. *([4]) If $(\widetilde{M} := M_1 \times_f M_2, \widetilde{g})$ is the warped product of M_1 and M_2 by the warping function f and $m > 1$, then for any X, Y, $Z \in \mathcal{L}(M_1)$ and any U, V, $W \in \mathcal{L}(M_2)$, we have:*

1. $R(X,Y)Z = \widetilde{R_{M_1}}(X,Y)Z$;
2. $R(U,X)Y = \frac{1}{f} H^f(X,Y)U$, *where H^f is the lift on $\widetilde{M}$ of $Hess(f)$;*
3. $R(X,Y)U = R(U,V)X = 0$;
4. $R(U,V)W = \widetilde{R_{M_2}}(U,V)W - \frac{|grad(f)|^2}{f^2}[g(U,W)V - g(V,W)U]$;
5. $R(X,U)V = \frac{1}{f} g(U,V) \widetilde{\nabla}_X grad(f)$.

Let S, S_{M_1}, S_{M_2} be the Ricci curvature tensors on $\widetilde{M}$, M_1 and M_2 and $\widetilde{S_{M_1}}$, $\widetilde{S_{M_2}}$ the lift on $\widetilde{M}$ of S_{M_1} and S_{M_2}. Then:

Lemma 2.2. *([4]) If $(\widetilde{M} := M_1 \times_f M_2, \widetilde{g})$ is the warped product of M_1 and M_2 by the warping function f and $m > 1$, then for any X, $Y \in \mathcal{L}(M_1)$ and any V, $W \in \mathcal{L}(M_2)$, we have:*

1. $S(X,Y) = \widetilde{S_{M_1}}(X,Y) - \frac{m}{f} H^f(X,Y)$, *where H^f is the lift on $\widetilde{M}$ of $Hess(f)$;*
2. $S(X,V) = 0$;
3. $S(V,W) = \widetilde{S_{M_2}}(V,W) - \left[\frac{\Delta(f)}{f} + (m-1)\frac{|grad(f)|^2}{f^2}\right] g(V,W)$.

Remark 2.1. For the case of product Riemannian manifolds:

i) the Riemannian curvature tensors verify ([2]):

$$R(\widetilde{X}, \widetilde{Y})\widetilde{Z} = (R_1(X_1, Y_1)Z_1, R_2(X_2, Y_2)Z_2), \tag{2.5}$$

for any $\widetilde{X} = (X_1, X_2), \widetilde{Y} = (Y_1, Y_2), \widetilde{Z} = (Z_1, Z_2) \in T(M_1 \times M_2)$, where R, R_1 and R_2 are respectively the Riemannian curvature tensors of the Riemannian manifolds $(M_1 \times M_2, \widetilde{g})$, (M_1, g_1) and (M_2, g_2);

ii) the Ricci curvature tensors verify ([2]):

$$S(\widetilde{X}, \widetilde{Y}) = S_1(X_1, Y_1) + S_2(X_2, Y_2), \tag{2.6}$$

for any $\widetilde{X} = (X_1, X_2), \widetilde{Y} = (Y_1, Y_2) \in T(M_1 \times M_2)$, where S, S_1 and S_2 are respectively the Ricci curvature tensors of the Riemannian manifolds $(M_1 \times M_2, \widetilde{g})$, (M_1, g_1) and (M_2, g_2).

Note that the Riemannian curvature tensor of a locally metallic Riemannian manifold has the following properties:

Proposition 2.1. *If (M, g, J) is a locally metallic Riemannian manifold, then for any $X, Y, Z \in T(M)$:*

$$R(X,Y)JZ = J(R(X,Y)Z), \tag{2.7}$$

$$R(JX,Y) = R(X,JY), \tag{2.8}$$

$$R(JX,JY) = qR(JX,Y) + pR(X,Y), \tag{2.9}$$

$$R(J^{n+1}X,Y) = g_{n+1} \cdot R(JX,Y) + g_n \cdot R(X,Y), \tag{2.10}$$

where $(\{g_n\}_{n\in\mathbb{N}^*})$ is the generalized secondary Fibonacci sequence defined by $g_{n+1} = pg_n + qg_{n-1}$, $n \geq 1$ with $g_0 = 0$, $g_1 = 1$ and p, $q \in \mathbb{N}^*$.

Proof. The locally metallic condition $\nabla J = 0$ is equivalent to $\nabla_X JY = J(\nabla_X Y)$, for any $X, Y \in T(M)$ and (2.7) follows from the definition of R. The relations (2.8), (2.9) and (2.10) follow from the symmetries of R and from the recurrence relation $J^{n+1} = g_{n+1} \cdot J + g_n \cdot I$. □

Theorem 2.1. *If* $(\widetilde{M} := M_1 \times_f M_2, \widetilde{g}, \widetilde{J})$ *is a locally metallic Riemannian warped product manifold, then* M_2 *is* $\widetilde{J}$*-invariant submanifold of* $\widetilde{M}$.

Proof. Applying (2.8) from Proposition 2.1 and Lemma 2.1, we obtain $H^f(X,Y)\widetilde{J}U = H^f(\widetilde{J}X,Y)U$, for any X, $Y \in \mathcal{L}(M_1)$ and any $U \in \mathcal{L}(M_2)$, where H^f is the lift on $\widetilde{M}$ of $Hess(f)$. □

2.2. Metallic warped product Riemannian manifolds

2.2.1. Metallic Riemannian structure on $(\widetilde{M}, \widetilde{g})$ induced by the projection operators

The endomorphism

$$F := \pi_1 - \pi_2 \tag{2.11}$$

verifies $F^2 = I$ and $\widetilde{g}(F\widetilde{X}, \widetilde{Y}) = \widetilde{g}(\widetilde{X}, F\widetilde{Y})$, thus F is an almost product structure on $M_1 \times M_2$.

By using relations (1.4) we can construct on $M_1 \times M_2$ two metallic structures, given by:

$$\widetilde{J}_{\pm} = \pm\frac{2\sigma_{p,q} - p}{2}F + \frac{p}{2}I. \tag{2.12}$$

Also from $\widetilde{g}(F\widetilde{X}, \widetilde{Y}) = \widetilde{g}(\widetilde{X}, F\widetilde{Y})$ follows $\widetilde{g}(\widetilde{J}_{\pm}\widetilde{X}, \widetilde{Y}) = \widetilde{g}(\widetilde{X}, \widetilde{J}_{\pm}\widetilde{Y})$. Therefore, we can state the following result:

Theorem 2.2. *There exist two metallic Riemannian structures* $\widetilde{J}_{\pm}$ *on* $(\widetilde{M}, \widetilde{g})$ *given by:*

$$\widetilde{J}_{\pm} = \pm\frac{2\sigma_{p,q} - p}{2}F + \frac{p}{2}I, \tag{2.13}$$

where $\widetilde{M} = M_1 \times_f M_2$ *and* $\widetilde{g}(\widetilde{X}, \widetilde{Y}) = g_1(X_1, Y_1) + (f \circ p_1)^2 g_2(X_2, Y_2)$, *for any* $\widetilde{X} = (X_1, X_2), \widetilde{Y} = (Y_1, Y_2) \in T(\widetilde{M}) = T(M_1 \times_f M_2)$.

Note that for $\widetilde{J}_+ = \frac{2\sigma_{p,q} - p}{2}F + \frac{p}{2}I$, the projection operators are $\pi_1 = m$, $\pi_2 = l$ and for $\widetilde{J}_- = -\frac{2\sigma_{p,q} - p}{2}F + \frac{p}{2}I$ we have $\pi_1 = l$, $\pi_2 = m$, where m and l are given by (1.6).

Remark 2.2. If we denote by $\widetilde{\nabla}$ the Levi-Civita connection on $\widetilde{M}$ with respect to $\widetilde{g}$, we obtain that $\widetilde{\nabla}F = 0$ [hence $\widetilde{\nabla}\widetilde{J}_{\pm} = 0$ and so $(\widetilde{M} = M_1 \times_f M_2, \widetilde{g}, \widetilde{J}_{\pm})$ is a locally metallic Riemannian manifold].

For the case of a product Riemannian manifold $(\widetilde{M} = M_1 \times M_2, \widetilde{g})$ with $\widetilde{g}$ given by (2.1) for $f = 1$ and $\widetilde{J}_{\pm}$ defined by (2.13), we deduce that the Riemann curvature of $\widetilde{\nabla}$ verifies (2.7), (2.8), (2.9), (2.10).

2.2.2. Metallic Riemannian structure on $(\widetilde{M}, \widetilde{g})$ induced by two metallic structures on M_1 and M_2

For any vector field $\widetilde{X} = (X, Y) \in T(M_1 \times M_2)$ we define a linear map $\widetilde{J}$ of tangent space $T(M_1 \times M_2)$ into itself by:

$$\widetilde{J}\widetilde{X} = (J_1 X, J_2 Y), \tag{2.14}$$

where J_1 and J_2 are two metallic structures defined on M_1 and M_2, respectively, with $J_i^2 = pJ_i + qI$, $i \in \{1, 2\}$ and p, q non zero natural numbers. It follows that:

$$\widetilde{J}^2\widetilde{X} = \widetilde{J}(J_1 X, J_2 Y) = (J_1^2 X, J_2^2 Y) = p(J_1 X, J_2 Y) + q(X, Y). \tag{2.15}$$

Also from $g_i(J_i X_i, Y_i) = g_i(X_i, J_i Y_i)$, $i \in \{1, 2\}$, we get $\widetilde{g}(\widetilde{J}\widetilde{X}, \widetilde{Y}) = \widetilde{g}(\widetilde{X}, \widetilde{J}\widetilde{Y})$. Therefore, we can state the following result:

Theorem 2.3. *If (M_1, g_1, J_1) and (M_2, g_2, J_2) are metallic Riemannian manifolds with $J_i^2 = pJ_i + qI$, $i \in \{1, 2\}$ and p, q non-zero natural numbers, then there exists a metallic Riemannian structure $\widetilde{J}$ on $(\widetilde{M}, \widetilde{g})$ given by:*

$$\widetilde{J}\widetilde{X} = (J_1 X, J_2 Y), \tag{2.16}$$

for any $\widetilde{X} = (X, Y) \in T(\widetilde{M})$, where $\widetilde{M} = M_1 \times_f M_2$ and $\widetilde{g}(\widetilde{X}, \widetilde{Y}) = g_1(X_1, Y_1) + (f \circ p_1)^2 g_2(X_2, Y_2)$, for any $\widetilde{X} = (X_1, X_2), \widetilde{Y} = (Y_1, Y_2) \in T(\widetilde{M}) = T(M_1 \times_f M_2)$.

For the case of a product Riemannian manifold $(\widetilde{M} = M_1 \times M_2, \widetilde{g})$ with $\widetilde{g}$ given by (2.1) for $f = 1$ and $\widetilde{J}_{\pm}$ defined by (2.13), we deduce that the Riemann curvature of $\widetilde{\nabla}$ verifies (2.7), (2.8), (2.9), (2.10).

Now we shall obtain a characterization of the metallic structure on the product of two metallic manifolds (M_1, J_1) and (M_2, J_2) in terms of *metallic maps*, that are smooth maps $\Phi : M_1 \to M_2$ satisfying:

$$T\Phi \circ J_1 = J_2 \circ T\Phi.$$

In a way similar to the case of Golden manifolds ([5]), we have:

Proposition 2.2. *The metallic structure $\widetilde{J} := (J_1, J_2)$ given by (2.16) is the only metallic structure on the product manifold $\widetilde{M} = M_1 \times M_2$ such that the projections p_1 and p_2 on the two factors M_1 and M_2 are metallic maps.*

A necessary and sufficient condition for the warped product of two locally metallic Riemannian manifolds to be locally metallic will be further provided:

Theorem 2.4. *Let* $(\widetilde{M} = M_1 \times_f M_2, \widetilde{g}, \widetilde{J})$ *(with* $\widetilde{g}$ *given by (2.1) and* $\widetilde{J}$ *given by (2.16)) be the warped product of the locally metallic Riemannian manifolds* (M_1, g_1, J_1) *and* (M_2, g_2, J_2)*. Then* $(\widetilde{M} = M_1 \times_f M_2, \widetilde{g}, \widetilde{J})$ *is locally metallic if and only if:*

$$\begin{cases} (df^2 \circ J_1) \otimes I = df^2 \otimes J_2 \\ g_2(J_1 \cdot, \cdot) \cdot grad(f^2) = g_2(\cdot, \cdot) \cdot J_1(grad(f^2)) \end{cases}.$$

Proof. Replacing the expression of $\widetilde{\nabla}$ from (2.4), under the assumptions ${}^{M_1}\nabla J_1 = 0$ and ${}^{M_2}\nabla J_2 = 0$ we obtain the conclusion. □

Theorem 2.5. *Let* $(\widetilde{M} = M_1 \times_f M_2, \widetilde{g}, \widetilde{J})$ *(with* $\widetilde{g}$ *given by (2.1) and* $\widetilde{J}$ *(2.16)) be the warped product of the metallic Riemannian manifolds* (M_1, g_1, J_1) *and* (M_2, g_2, J_2)*. If* M_1 *and* M_2 *have* J_1*- and* J_2*-invariant Ricci tensors, respectively (i.e.* $Q_{M_i} \circ J_i = J_i \circ Q_{M_i}$, $i \in \{1, 2\}$*), then* $\widetilde{M}$ *has* $\widetilde{J}$*-invariant Ricci tensor if and only if*

$$Hess(f)(J_1 \cdot, \cdot) - Hess(f)(\cdot, J_1 \cdot) \in \{0\} \times T(M_2).$$

Proof. If we denote by S, S_{M_1}, S_{M_2} the Ricci curvature tensors on $\widetilde{M}$, M_1 and M_2 and $\widetilde{S_{M_1}}$, $\widetilde{S_{M_2}}$ the lift on $\widetilde{M}$ of S_{M_1} and S_{M_2}, by using Lemma 2.2, for any X, $Y \in \mathcal{L}(M_1)$, we have:

$$S(\widetilde{J}X, Y) = \widetilde{S_{M_1}}(\widetilde{J}X, Y) - \frac{m}{f} H^f(\widetilde{J}X, Y) = \widetilde{S_{M_1}}(X, \widetilde{J}Y) - \frac{m}{f} H^f(\widetilde{J}X, Y) =$$

$$= S(X, \widetilde{J}Y) + \frac{m}{f} H^f(X, \widetilde{J}Y) - \frac{m}{f} H^f(\widetilde{J}X, Y),$$

where H^f is the lift on $\widetilde{M}$ of $Hess(f)$. Also, for any V, $W \in \mathcal{L}(M_2)$, we obtain:

$$S(\widetilde{J}V, W) = \widetilde{S_{M_2}}(\widetilde{J}V, W) - [f\Delta(f) + (m-1)|grad(f)|^2] g_2(J_2 V, W) =$$

$$= \widetilde{S_{M_2}}(V, \widetilde{J}W) - [f\Delta(f) + (m-1)|grad(f)|^2] g_2(V, J_2 W) = S(V, \widetilde{J}W).$$

□

Example 2.1. Consider $M := \{(u, \alpha_1, \alpha_2, ..., \alpha_n), u > 0, \alpha_i \in [0, \frac{\pi}{2}], i \in \{1, ..., n\}\}$ and let $f : M \to \mathbb{R}^{2n}$ be the immersion given by:

$$f(u, \alpha_1, ..., \alpha_n) := (u \cos \alpha_1, u \sin \alpha_1, ..., u \cos \alpha_n, u \sin \alpha_n). \tag{2.17}$$

We can find a local orthonormal frame of the submanifold M in $\mathbb{R}^{2n}$, spanned by the vectors:

$$Z_0 = \sum_{i=1}^{n} \left(\cos \alpha_i \frac{\partial}{\partial x_i} + \sin \alpha_i \frac{\partial}{\partial y_i} \right), \quad Z_i = -u \sin \alpha_i \frac{\partial}{\partial x_i} + u \cos \alpha_i \frac{\partial}{\partial y_i}, \tag{2.18}$$

for any $i \in \{1, ..., n\}$.

We remark that $\|Z_0\|^2 = n$, $\|Z_i\|^2 = u^2$, $Z_0 \perp Z_i$, for any $i \in \{1, ..., n\}$ and $Z_i \perp Z_j$, for $i, j \in \{1, ..., n\}$ with $i \neq j$.

In the next considerations, we shall denote by:

$$(X^1, Y^1, ..., X^k, Y^k, X^{k+1}, Y^{k+1}, ..., X^n, Y^n) =: (X^i, Y^i, X^j, Y^j),$$

for any $k \in \{2, ..., n-1\}$, $i \in \{1, ..., k\}$ and $j \in \{k+1, ..., n\}$.

Let $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the $(1,1)$-tensor field defined by:

$$J(X^i, Y^i, X^j, Y^j) := (\sigma X^i, \sigma Y^i, \overline{\sigma} X^j, \overline{\sigma} Y^j), \tag{2.19}$$

for any $k \in \{2, ..., n-1\}$, $i \in \{1, ..., k\}$ and $j \in \{k+1, ..., n\}$, where $\sigma := \sigma_{p,q}$ is the metallic number and $\overline{\sigma} = 1 - \sigma$. It is easy to verify that J is a metallic structure on $\mathbb{R}^{2n}$ (i.e. $J^2 = pJ + qI$).

Moreover, the metric $\overline{g}$, given by the scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{2n}$, is J-compatible and $(\mathbb{R}^{2n}, \overline{g}, J)$ is a metallic Riemannian manifold.

From (2.18) we get:

$$JZ_0 = \sigma \sum_{i=1}^{k} \left(\cos \alpha_i \frac{\partial}{\partial x_i} + \sin \alpha_i \frac{\partial}{\partial y_i} \right) + \overline{\sigma} \sum_{j=k+1}^{n} \left(\cos \alpha_j \frac{\partial}{\partial x_j} + \sin \alpha_j \frac{\partial}{\partial y_j} \right)$$

and, for any $k \in \{2, ..., n-1\}, i \in \{1, ..., k\}$ and $j \in \{k+1, ..., n\}$ we get:

$$JZ_i = \sigma Z_i, \quad JZ_j = \overline{\sigma} Z_j.$$

We can verify that JZ_0 is orthogonal to $span\{Z_1, ..., Z_n\}$ and

$$\cos(\widehat{JZ_0, Z_0}) = \frac{k\sigma + (n-k)\bar{\sigma}}{\sqrt{n(k\sigma^2 + (n-k)\bar{\sigma}^2)}}. \tag{2.20}$$

Consider the manifolds M_1 and M_2 with $TM_1 = span\{Z_0\}$ and $TM_2 = span\{Z_1, ..., Z_n\}$. Then $M := M_1 \times_u M_2$ with the Riemannian metric tensor $g = ndu^2 + u^2 \sum_{i=1}^{n} d\alpha_i^2$ is a warped product (semi-slant) submanifold of the metallic Riemannian manifold $(\mathbb{R}^{2n}, \langle \cdot, \cdot \rangle, J)$.

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