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SIMPSON'S TYPE INEQUALITY FOR F-CONVEX FUNCTION

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Abstract. In this paper, we obtain Simpson's type inequality for the function whose second derivatives absolute values are F-convex. Then, we give some special cases of the mappings F.

Keywords: Simpson's inequality, F-convex mapping

1. Introduction

The well-known [2] in the literatüre as Simpson's inequality is described by the following theorem:

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a, b) and $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$. Then, the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4.$$

For many years, many types of convexity have been defined, such as quasi-convex [1], pseudo-convex [5], strongly convex [6], ε-convex [4], s-convex [3], h-convex [9] and etc. Recently, a new convexity that depends on a certain function satisfying some axioms was defined by Samet in the paper [7] which generalizes different types of convexity, including ε-convex functions, α-convex functions, h-convex functions and many others.

Let us recall the family $\mathcal{F}$ of mappings $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R}\times [0, 1] \to \mathbb{R}$ satisfying the following axioms:

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(A1) If $u_i \in L^1(0,1)$, $i = 1,2,3$, then for every $\lambda \in [0,1]$, we have

$$\int_0^1 F(u_1(t), u_2(t), u_3(t), \lambda)dt = F\left(\int_0^1 u_1(t)dt, \int_0^1 u_2(t)dt, \int_0^1 u_3(t)dt, \lambda\right).$$

(A2) For every $u \in L^1(0,1)$, $w \in L^\infty(0,1)$ and $(z_1, z_2) \in \mathbb{R}^2$, we have

$$\int_0^1 F(w(t)u(t), w(t)z_1, w(t)z_2, t)dt = T_{F,w}\left(\int_0^1 w(t)u(t)dt, z_1, z_2\right),$$

where $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function that depends on (F, w), and it is nondecreasing with respect to the first variable.

(A3) For any $(w, u_1, u_2, u_3) \in \mathbb{R}^4$, $u_4 \in [0,1]$, we have

$$wF(u_1, u_2, u_3, u_4) = F(wu_1, wu_2, wu_3, u_4) + L_w$$

where $L_w \in \mathbb{R}$ is a constant that depends only on w.

Definition 1.1. Let $f : [a,b] \to \mathbb{R}$, $(a,b) \in \mathbb{R}^2$, $a < b$, be a given function. We say that f is a convex function with respect to some $F \in \mathcal{F}$ (or F-convex function) iff

$$F(f(tx + (1-t)y), f(x), f(y), t) \leq 0, \quad (x,y,t) \in [a,b] \times [a,b] \times [0,1].$$

One can obtain many types of convexity with the special cases of F. Some of them are listed below:

Remark 1.1. 1) If we choose the functions $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0,1] \to \mathbb{R}$ by

$$F(u_1, u_2, u_3, u_4) = u_1 - u_4u_2 - (1-u_4)u_3 - \varepsilon \tag{1.1}$$

and $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$T_{F,w}(u_1, u_2, u_3) = u_1 - \left(\int_0^1 tw(t)dt\right)u_2 - \left(\int_0^1 (1-t)w(t)dt\right)u_3 - \varepsilon, \tag{1.2}$$

then it is clear that $F \in \mathcal{F}$ for

$$L_w = (1-w)\varepsilon \tag{1.3}$$

and

$$F(f(tx + (1-t)y), f(x), f(y), t) = f(tx + (1-t)y) - tf(x) - (1-t)f(y) - \varepsilon \leq 0,$$

that is, f is an ε-convex function. Particularly, if we take $\varepsilon = 0$, then f is a convex function.

2) Let $h : J \to [0, \infty)$ be a given function which is not identical to 0, where J is an interval in $\mathbb{R}$ such that $(0,1) \subseteq J$. If we choose the functions $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0,1] \to \mathbb{R}$ by

$$F(u_1, u_2, u_3, u_4) = u_1 - h(u_4)u_2 - h(1-u_4)u_3 \tag{1.4}$$

and $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$T_{F,w}(u_1, u_2, u_3) = u_1 - \left(\int_0^1 h(t)w(t)dt\right) u_2 - \left(\int_0^1 h(1-t)w(t)dt\right) u_3, \tag{1.5}$$

then it is clear that $F \in \mathcal{F}$ for $L_w = 0$ and

$$F(f(tx + (1-t)y), f(x), f(y), t) = f(tx + (1-t)y) - h(t)f(x) - h(1-t)f(y) \leq 0,$$

that is, f is an h-convex function.

The following lemma obtained by Sarikaya et. al. in the paper [8] which motivates our main result.

Lemma 1.1. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If $f'' \in L^1[a,b]$, then the following equality holds:*

$$\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx = (b-a)^2 \int_0^1 k(t) f''(tb + (1-t)a)dt,$$

where

$$k(t) = \begin{cases} \frac{t}{2}\left(\frac{1}{3} - t\right), & t \in \left[0, \frac{1}{2}\right) \\ (1-t)\left(\frac{t}{2} - \frac{1}{3}\right), & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

2. A Simpson type inequality for F-convex function

In this part, we obtain Theorem related to Simpson's type inequality for functions whose second derivatives absolute values are F-convex. Then, we give special cases of this.

Theorem 2.1. *Let $I \subseteq \mathbb{R}$ be an interval, $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $(a,b) \in I^\circ \times I^\circ$, $a < b$. If $\left|f''\right|$ is F-convex on $[a,b]$, for some $F \in \mathcal{F}$ and the function $t \in [0,1] \to L_{w(t)}$ belongs to $L^1[0,1]$, then we have the following inequality*

$$T_{F,w}\left(\frac{1}{(b-a)^2}\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right|, \left|f''(b)\right|, \left|f''(a)\right|\right) + \int_0^1 L_{w(t)}dt \leq 0,$$

where $w(t) = |k(t)|$.

Proof. Since $|f''|$ is F-convex, we have

$$F\left(|f''(tb+(1-t)a)|, |f''(b)|, |f''(a)|, t\right) \leq 0, \ t \in [0,1].$$

Multiplying this inequality by $w(t) = |k(t)|$ and using axiom (A3), we get

$$F\left(w(t)|f''(tb+(1-t)a)|, w(t)|f''(b)|, w(t)|f''(a)|, t\right) + L_{w(t)} \leq 0,$$

for $t \in [0,1]$. Integrating over $[0,1]$ with respect to the variable t and using axiom (A2), we obtain

$$T_{F,w}\left(\int_0^1 w(t)|f''(tb+(1-t)a)|\,dt, |f''(b)|, |f''(a)|\right) + \int_0^1 L_{w(t)}dt \leq 0$$

for $t \in [0,1]$. On the other hand, using Lemma 1.1, we have

$$\begin{aligned} &\frac{1}{(b-a)^2}\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq & \int_0^1 |k(t)|\,|f''(tb+(1-t)a)|\,dt. \end{aligned}$$

Since $T_{F,w}$ is nondecreasing with respect to the first variable, we get

$$\begin{aligned} &T_{F,w}\left(\frac{1}{(b-a)^2}\left|\tfrac{1}{6}\left[f(a)+4f\left(\tfrac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|, |f''(b)|, |f''(a)|\right) \\ &+\int_0^1 L_{w(t)}dt \leq 0. \end{aligned}$$

This completes the proof. □

Corollary 2.1. *Under assumptions of Theorem 2.1, if we choose $F(u_1,u_2,u_3,u_4) = u_1 - u_4u_2 - (1-u_4)u_3 - \varepsilon$, then the function $|f''|$ is ε-convex on $[a,b]$, $\varepsilon \geq 0$ and we have the inequality*

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq & \frac{(b-a)^2}{162}\left[|f''(a)|+|f''(b)|\right]+\frac{1}{81}(b-a)^2\varepsilon. \end{aligned}$$

Proof. Using (1.3) with $w(t) = |k(t)|$, we obtain

$$\int_0^1 L_{w(t)}dt = \varepsilon\int_0^1(1-|k(t)|)dt = \varepsilon\left(\int_0^{\frac{1}{2}}(1-|k(t)|)dt + \int_{\frac{1}{2}}^1(1-|k(t)|)\,dt\right) = \frac{80}{81}\varepsilon.$$

From (1.2) with $w(t) = |k(t)|$, we get

$$\begin{aligned} T_{F,w}(u_1,u_2,u_3) &= u_1 - \left(\int_0^1 t\,|k(t)|\,dt\right) u_2 - \left(\int_0^1 (1-t)\,|k(t)|\,dt\right) u_3 - \varepsilon \\ &= u_1 - \frac{1}{162}\left[u_2 + u_3\right] - \varepsilon \end{aligned}$$

for $u_1, u_2, u_3 \in \mathbb{R}$. Hence,

$$\begin{aligned} & 0 \geq \\ & T_{F,w}\left(\frac{1}{(b-a)^2}\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|, \left|f''(b)\right|, \left|f''(a)\right|\right) \\ & + \int_0^1 L_{w(t)}dt \\ = & \frac{1}{(b-a)^2}\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ & -\frac{1}{162}\left[\left|f''(a)\right|+\left|f''(b)\right|\right]-\varepsilon+\frac{80}{81}\varepsilon \end{aligned}$$

that is

$$\begin{aligned} & \left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq & \frac{(b-a)^2}{162}\left[\left|f''(a)\right|+\left|f''(b)\right|\right]+\frac{1}{81}(b-a)^2\varepsilon. \end{aligned}$$

This completes the proof. □

Remark 2.1. Taking $\varepsilon = 0$ in Corollary 2.1, then the function $|f''|$ is convex and we have the inequality

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{162}\left[\left|f''(a)\right|+\left|f''(b)\right|\right]$$

which is given by Sarikaya et al. in [8].

Corollary 2.2. *Under the assumptions of Theorem 2.1, if we choose* $F(u_1,u_2,u_3,u_4) = u_1 - h(u_4)u_2 - h(1-u_4)u_3$, *then the function* $|f''|$ *is* h*-convex on* $[a,b]$ *and we have* *the inequality*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq (b-a)^2\left(\int_0^1 h(t)\left|k(t)\right|dt\right)\left[\left|f''(a)\right|+\left|f''(b)\right|\right]
\end{aligned}$$

Proof. From (1.5) with $w(t)=|k(t)|$, we have

$$\begin{aligned}
T_{F,w}(u_1,u_2,u_3) &= u_1-\left(\int_0^1 h(t)\left|k(t)\right|dt\right)u_2-\left(\int_0^1 h(1-t)\left|k(t)\right|dt\right)u_3 \\
&= u_1-\left(\int_0^1 h(t)\left|k(t)\right|dt\right)u_2-\left(\int_0^1 h(t)\left|k(1-t)\right|dt\right)u_3 \\
&= u_1-\left(\int_0^1 h(t)\left|k(t)\right|dt\right)(u_2+u_3)
\end{aligned}$$

for $u_1,u_2,u_3\in\mathbb{R}$. Then, by Theorem 2.1,

$$\begin{aligned}
&T_{F,w}\left(\frac{1}{(b-a)^2}\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|,\left|f''(b)\right|,\left|f''(a)\right|\right) \\
&= \frac{1}{(b-a)^2}\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\quad -\left(\int_0^1 h(t)\left|k(t)\right|dt\right)\left[\left|f''(a)\right|+\left|f''(b)\right|\right]\leq 0
\end{aligned}$$

that is,

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq (b-a)^2\left(\int_0^1 h(t)\left|k(t)\right|dt\right)\left[\left|f''(a)\right|+\left|f''(b)\right|\right]
\end{aligned}$$

which completes the proof. □

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