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ON PARAMETRIZED HERMITE-HADAMARD TYPE INEQUALITIES

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Abstract. In recent years, many results have been devoted to the well-known Hermite-Hadamard inequality. This inequality has many applications in the area of pure and applied mathematics. In this paper, our main aim is to give a parametrized inequality of the Hermite-Hadamard type and its applications to f-divergence measures and means. First, we prove the identity associated with the right side of the Hermite-Hadamard inequality. By using this identity, the convexity of the function and some well-known inequalities, we obtain several results for the inequality. The inequalities derived here also point out some known results as their special cases.

Keywords. Hermite-Hadamard inequality; parametrized inequality; convex function.

1. Introduction

Almost no mathematician in applied mathematics, especially in nonlinear programming and optimization theory, can ignore the significant role of convex sets and convex functions. For the class of convex functions, many inequalities such as Jensen's, Hermite-Hadamard and Slater's inequalities have been introduced since this idea was introduced for the first time more than a century ago. Among the introduced inequalities, the most prominent is the so called Hermite-Hadamard's inequality. The statement of this inequality is (see [15]):

Let I be an interval in $\mathbb{R}$ and $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on I such that $a, b \in I$ with $a < b$. Then the inequalities

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

hold. If the function f is concave on I, then both the inequalities in (1.1) hold in the reverse direction. It gives an estimate from both sides of the mean value of a

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convex function and also ensure the integrability of the convex function. It is also a matter of great interest and one has to note that some of the classical inequalities for means can be obtained from Hadamard's inequality under the utility of peculiar convex functions f. These inequalities for convex functions play a crucial role in mathematical analysis and other areas of pure and applied mathematics.

For more recent results, generalizations, improvements and refinements related to Hermite-Hadamard inequality see [2, 3, 9, 10, 11, 12, 13, 14, 24, 30, 23, 22] and the references cited therein.
In 2010, Havva Kavurmaci et al. proved the following important lemma:

Lemma 1.1. [18] *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $f' \in L[a,b]$, then the following identity holds:*

$$\begin{aligned}
&\frac{(x-a)f(a)+(b-x)f(b)}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx \\
= \;& \frac{(x-a)^2}{b-a}\int_0^1 (t-1)f'(tx+(1-t)a)dt + \frac{(b-x)^2}{b-a}\int_0^1 (1-t)f'(tx+(1-t)b)dt.
\end{aligned} \tag{1.2}$$

Here I° denotes the interior of I.

The following results are the ultimate consequences of Lemma 1.1, which have been presented in [18] .

Theorem 1.1. *Under the assumptions of Lemma 1.1 and if $|f'|$ is convex on $[a,b]$, then we have the following inequality:*

$$\begin{aligned}
&\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{(x-a)^2}{b-a}\left[\frac{|f'(x)|+2|f'(a)|}{6}\right] + \frac{(b-x)^2}{b-a}\left[\frac{|f'(x)|+2|f'(b)|}{6}\right].
\end{aligned}$$

Theorem 1.2. *Suppose the conditions of Lemma 1.1 are satisfied and if the new mapping $|f'|^q$ $(q > 1)$ is convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq \;& \frac{1}{2}\left(\frac{1}{3}\right)^{\frac{1}{q}}\left[\frac{(x-a)^2\left[|f'(x)|^q+2|f'(a)|^q\right]^{\frac{1}{q}} + (b-x)^2\left[|f'(x)|^q+2|f'(b)|^q\right]^{\frac{1}{q}}}{b-a}\right].
\end{aligned}$$

Theorem 1.3. *Suppose the conditions of Lemma 1.1 hold and if the mapping* $|f'|^q$ $(q \geq 1)$ *is concave on* $[a, b]$, *then the following inequality is valid:*

$$\left| \frac{(x-a)f(a) + (b-x)f(b)}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{1}{2}\left[\frac{(x-a)^2 \left| f'\left(\frac{x+2a}{3}\right) \right| + (b-x)^2 \left| f'\left(\frac{x+2b}{3}\right) \right|}{b-a} \right].$$

The main purpose of this paper is to present a parametrized inequality of the Hermite-Hadamard type for functions whose first derivative absolute values are convex. We prove the identity for the right side of the inequality and discuss their particular case (Corollaries 2.2, 2.4, 2.6). By applying Jensen's inequality, power mean inequality and the convexity of functions in the identity, we obtain inequalities for the right side of the Hermite-Hadamard inequality. As applications, some new inequalities for f-divergence measures and means are established.

2. Main Results

In order to prove our main results, we need the following lemma.

Lemma 2.1. *Let* $\epsilon \in \mathbb{R}$ *and let* $f : I^\circ \to \mathbb{R}$ *be a differentiable function on* $I^\circ, a, b \in I^\circ$ *with* $a < b$. *If* $f' \in L[a, b]$, *then the following equality holds:*

$$\frac{(x-a)[(1-\epsilon)f(x) + \epsilon f(a)] + (b-x)[(1-\epsilon)f(x) + \epsilon f(b)]}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx$$

$$= \frac{(x-a)^2}{b-a}\int_0^1 (t-\epsilon)f'(tx + (1-t)a)dt + \frac{(b-x)^2}{b-a}\int_0^1 (\epsilon - t)f'(tx + (1-t)b)dt. \tag{2.1}$$

Proof. It suffices to note that

$$\begin{aligned} I_1 &= \frac{(x-a)^2}{b-a}\int_0^1 (t-\epsilon)f'(tx + (1-t)a)dt \\ &= \frac{(x-a)^2}{b-a}\left[\frac{(t-\epsilon)f(tx + (1-t)a)}{x-a}\Big|_0^1 - \int_0^1 \frac{f(tx + (1-t)a)dt}{x-a} \right] \\ &= \frac{(x-a)^2}{b-a}\left[\frac{(1-\epsilon)f(x) + \epsilon f(a)}{x-a} - \int_0^1 \frac{f(tx + (1-t)a)dt}{x-a} \right]. \end{aligned}$$

By substituting $u = tx + (1-t)a$ in (2.2) we have

$$I_1 = \frac{(x-a)^2}{b-a}\left[\frac{(1-\epsilon)f(x)+\epsilon f(a)}{x-a} - \int_a^x \frac{f(u)du}{(x-a)^2}\right]$$

$$= \frac{(x-a)[(1-\epsilon)f(x)+\epsilon f(a)]}{b-a} - \frac{1}{b-a}\int_a^x f(u)du, \tag{2.2}$$

similarly

$$I_2 = \frac{(b-x)^2}{b-a}\int_0^1 (\epsilon - t)f'(tx+(1-t)b)dt$$

$$= \frac{(b-x)[(1-\epsilon)f(x)+\epsilon f(b)]}{b-a} - \frac{1}{b-a}\int_x^b f(u)du, \tag{2.3}$$

now by adding (2.2) and (2.3) we get (2.1). □

Remark 2.1. If we choose $\epsilon = 1$, then from Lemma 2.1 we obtain Lemma 1.1.

Lemma 2.2. *Let ϵ be a real number. Then*

$$\int_0^1 |\epsilon - t|dt = \left\{\begin{array}{lr} \frac{2\epsilon-1}{2}, & \epsilon \geq 1 \\ \frac{2\epsilon^2-2\epsilon+1}{2}, & 0 < \epsilon < 1 \\ \frac{1-2\epsilon}{2}, & \epsilon \leq 0. \end{array}\right\}$$

Proof. **Case 1.** If $\epsilon \geq 1$, then $\int_0^1 |\epsilon - t|dt = \int_0^1 (\epsilon - t)dt = \frac{2\epsilon-1}{2}$.

Case 2. If $0 < \epsilon < 1$, then
$\int_0^1 |\epsilon - t|dt = \int_0^\epsilon (\epsilon - t)dt + \int_\epsilon^1 (t-\epsilon)dt = \frac{2\epsilon^2-2\epsilon+1}{2}$.

Case 3. If $\epsilon \leq 0$, then
$\int_0^1 |\epsilon - t|dt = \int_0^1 (t-\epsilon)dt = \frac{1-2\epsilon}{2}$.
□

Lemma 2.3. *Let ϵ be a real number. Then*

$$\int_0^1 |t-\epsilon|tdt = \left\{\begin{array}{lr} \frac{3\epsilon-2}{6}, & \epsilon \geq 1 \\ \frac{2\epsilon^3-3\epsilon+2}{6}, & 0 < \epsilon < 1 \\ \frac{2-3\epsilon}{6}, & \epsilon \leq 0. \end{array}\right\}$$

Lemma 2.4. *Let ϵ be a real number. Then*

$$\int_0^1 |t-\epsilon|(1-t)dt = \begin{cases} \frac{3\epsilon-1}{6}, & \epsilon \geq 1 \\ \frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}, & 0<\epsilon<1 \\ \frac{1-3\epsilon}{6}, & \epsilon \leq 0. \end{cases}$$

Theorem 2.1. *Let $\epsilon \in \mathbb{R}$ and let $f : I^\circ \to \mathbb{R}$ be a differentiable function on $I^\circ, a, b \in I^\circ$ with $a < b$ such that $f' \in L[a,b]$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds :*

$$\left|\frac{(x-a)[(1-\epsilon)f(x)+\epsilon f(a)]+(b-x)[(1-\epsilon)f(x)+\epsilon f(b)]}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{(x-a)^2}{b-a}\begin{cases} |f'(x)|\left(\frac{3\epsilon-2}{6}\right)+|f'(a)|\left(\frac{3\epsilon-1}{6}\right) & \text{if } \epsilon \geq 1 \\ |f'(x)|\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right)+|f'(a)|\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right) & \text{if } 0<\epsilon<1 \\ |f'(x)|\left(\frac{2-3\epsilon}{6}\right)+|f'(a)|\left(\frac{1-3\epsilon}{6}\right) & \text{if } \epsilon \leq 0 \end{cases}$$

$$+\frac{(b-x)^2}{b-a}\begin{cases} |f'(x)|\left(\frac{3\epsilon-2}{6}\right)+|f'(b)|\left(\frac{3\epsilon-1}{6}\right) & \text{if } \epsilon \geq 1 \\ |f'(x)|\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right)+|f'(b)|\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right) & \text{if } 0<\epsilon<1 \\ |f'(x)|\left(\frac{2-3\epsilon}{6}\right)+|f'(b)|\left(\frac{1-3\epsilon}{6}\right) & \text{if } \epsilon \leq 0. \end{cases}$$

Proof. It follows from the convexity of $|f'|$ and Lemma 2.1 that

$$\left|\frac{(x-a)[(1-\epsilon)f(x)+\epsilon f(a)]+(b-x)[(1-\epsilon)f(x)+\epsilon f(b)]}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{(x-a)^2}{b-a}\int_0^1 |t-\epsilon||f'(tx+(1-t)a)|dt + \frac{(b-x)^2}{b-a}\int_0^1 |\epsilon-t||f'(tx+(1-t)b)|dt$$

$$\leq \frac{(x-a)^2}{b-a}\int_0^1 |t-\epsilon|[t|f'(x)|+(1-t)|f'(a)|]dt$$

$$+\frac{(b-x)^2}{b-a}\int_0^1 |\epsilon-t|[t|f'(x)|+(1-t)|f'(b)|]dt$$

$$= \frac{(x-a)^2}{b-a}\begin{cases} |f'(x)|\left(\frac{3\epsilon-2}{6}\right)+|f'(a)|\left(\frac{3\epsilon-1}{6}\right) & \text{if } \epsilon \geq 1 \\ |f'(x)|\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right)+|f'(a)|\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right) & \text{if } 0<\epsilon<1 \\ |f'(x)|\left(\frac{2-3\epsilon}{6}\right)+|f'(a)|\left(\frac{1-3\epsilon}{6}\right) & \text{if } \epsilon \leq 0 \end{cases}$$

$$+\frac{(b-x)^2}{b-a}\begin{cases} |f'(x)|\left(\frac{3\epsilon-2}{6}\right)+|f'(b)|\left(\frac{3\epsilon-1}{6}\right) & \text{if } \epsilon \geq 1 \\ |f'(x)|\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right)+|f'(b)|\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right) & \text{if } 0<\epsilon<1 \\ |f'(x)|\left(\frac{2-3\epsilon}{6}\right)+|f'(b)|\left(\frac{1-3\epsilon}{6}\right) & \text{if } \epsilon \leq 0. \end{cases}$$

□

Corollary 2.1. *Under the assumption of Theorem 2.1 if we choose $x = \frac{a+b}{2}$, we have*

$$\left|\frac{\epsilon f(a) + \epsilon f(b) + 2(1-\epsilon) f\left(\frac{a+b}{2}\right)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{2}\left\{\begin{array}{l} |f'\left(\frac{a+b}{2}\right)|\left(\frac{3\epsilon-2}{6}\right) + (|f'(a)| + |f'(b)|)\left(\frac{3\epsilon-1}{6}\right) \ \text{if } \epsilon \geq 1, \\ |f'\left(\frac{a+b}{2}\right)|\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right) + (|f'(a)| + |f'(b)|)\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right) \\ \text{if } 0 < \epsilon < 1, \\ |f'\left(\frac{a+b}{2}\right)|\left(\frac{2-3\epsilon}{6}\right) + (|f'(a)| + |f'(b)|)\left(\frac{1-3\epsilon}{6}\right) \ \text{if } \epsilon \leq 0. \end{array}\right\}$$

Corollary 2.2. *Under the assumption of Theorem 2.1 if we choose $x = \frac{a+b}{2}$ and $\epsilon = 1$, we have*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{b-a}{12}\left(|f'(a)| + |f'\left(\frac{a+b}{2}\right)| + |f'(b)|\right) \\ &\leq \frac{b-a}{8}\left(|f'(a)| + |f'(b)|\right).\end{aligned}$$

Proof. The second inequality is obtained by using the convexity of $|f'|$. □

Theorem 2.2. *Let $\epsilon \in \mathbb{R}$ and let $f : I^\circ \to \mathbb{R}$ be a differentiable function on $I^\circ, a, b \in I^\circ$ with $a < b$ such that $f' \in L[a,b]$. If $|f'|^q, q \geq 1$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{(x-a)[(1-\epsilon)f(x) + \epsilon f(a)] + (b-x)[(1-\epsilon)f(x) + \epsilon f(b)]}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(x-a)^2}{b-a}\left\{\begin{array}{l} \left(\frac{2\epsilon-1}{2}\right)^{1-\frac{1}{q}}\left(|f'(x)|^q\left(\frac{3\epsilon-2}{6}\right) + |f'(a)|^q\left(\frac{3\epsilon-1}{6}\right)\right)^{\frac{1}{q}} \ \text{if } \epsilon \geq 1 \\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)^{1-\frac{1}{q}}\left(|f'(x)|^q\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right) + |f'(a)|^q\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right)\right)^{\frac{1}{q}} \\ \text{if } 0 < \epsilon < 1 \\ \left(\frac{1-2\epsilon}{2}\right)^{1-\frac{1}{q}}\left(|f'(x)|^q\left(\frac{2-3\epsilon}{6}\right) + |f'(a)|^q\left(\frac{1-3\epsilon}{6}\right)\right)^{\frac{1}{q}} \ \text{if } \epsilon \leq 0 \end{array}\right\}$$

$$+\frac{(b-x)^2}{b-a}\left\{\begin{array}{l}\left(\frac{2\epsilon-1}{2}\right)^{1-\frac{1}{q}}\left(|f'(x)|^q\left(\frac{3\epsilon-2}{6}\right)+|f'(b)|^q\left(\frac{3\epsilon-1}{6}\right)\right)^{\frac{1}{q}}\ \textit{if}\ \epsilon\geq 1\\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)^{1-\frac{1}{q}}\left(|f'(x)|^q\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right)+|f'(b)|^q\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right)\right)^{\frac{1}{q}}\\ \textit{if}\ 0<\epsilon<1\\ \left(\frac{1-2\epsilon}{2}\right)^{1-\frac{1}{q}}\left(|f'(x)|^q\left(\frac{2-3\epsilon}{6}\right)+|f'(b)|^q\left(\frac{1-3\epsilon}{6}\right)\right)^{\frac{1}{q}}\ \textit{if}\ \epsilon\leq 0.\end{array}\right\}$$

Proof. Using Lemma 2.1 and the Power mean inequality, we have

$$\begin{aligned}
&\left|\frac{(x-a)[(1-\epsilon)f(x)+\epsilon f(a)]+(b-x)[(1-\epsilon)f(x)+\epsilon f(b)]}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
\leq\ & \frac{(x-a)^2}{b-a}\int_0^1|t-\epsilon||f'(tx+(1-t)a)|dt+\frac{(b-x)^2}{b-a}\int_0^1|\epsilon-t||f'(tx+(1-t)b)|dt\\
\leq\ & \frac{(x-a)^2}{b-a}\left(\int_0^1|t-\epsilon|dt\right)^{1-\frac{1}{q}}\left(\int_0^1(|t-\epsilon|)|f'(tx+(1-t)a)|^q dt\right)^{\frac{1}{q}}\\
+\ & \frac{(b-x)^2}{b-a}\left(\int_0^1|\epsilon-t|dt\right)^{1-\frac{1}{q}}\left(\int_0^1|\epsilon-t||f'(tx+(1-t)b)|^q dt\right)^{\frac{1}{q}}\\
\leq\ & \frac{(x-a)^2}{b-a}\left(\int_0^1|t-\epsilon|dt\right)^{1-\frac{1}{q}}\left(\int_0^1|t-\epsilon|[t|f'(x)|^q+(1-t)|f'(a)|^q]dt\right)^{\frac{1}{q}}\\
+\ & \frac{(b-x)^2}{b-a}\left(\int_0^1|\epsilon-t|dt\right)^{1-\frac{1}{q}}\left(\int_0^1|\epsilon-t|[t|f'(x)|^q+(1-t)|f'(b)|^q]dt\right)^{\frac{1}{q}}\\
=\ & \frac{(x-a)^2}{b-a}\left\{\begin{array}{l}\left(\frac{2\epsilon-1}{2}\right)^{1-\frac{1}{q}}\left(|f'(x)|^q\left(\frac{3\epsilon-2}{6}\right)+|f'(a)|^q\left(\frac{3\epsilon-1}{6}\right)\right)^{\frac{1}{q}}\ \text{if}\ \epsilon\geq 1\\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)^{1-\frac{1}{q}}\left(|f'(x)|^q\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right)+|f'(a)|^q\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right)\right)^{\frac{1}{q}}\\ \text{if}\ 0<\epsilon<1\\ \left(\frac{1-2\epsilon}{2}\right)^{1-\frac{1}{q}}\left(|f'(x)|^q\left(\frac{2-3\epsilon}{6}\right)+|f'(a)|^q\left(\frac{1-3\epsilon}{6}\right)\right)^{\frac{1}{q}}\ \text{if}\ \epsilon\leq 0\end{array}\right\}
\end{aligned}$$

$$+\frac{(b-x)^2}{b-a}\left\{\begin{array}{l}\left(\frac{2\epsilon-1}{2}\right)^{1-\frac{1}{q}}\left(|f'(x)|^q\left(\frac{3\epsilon-2}{6}\right)+|f'(b)|^q\left(\frac{3\epsilon-1}{6}\right)\right)^{\frac{1}{q}} \text{ if } \epsilon\geq 1\\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)^{1-\frac{1}{q}}\left(|f'(x)|^q\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right)+|f'(b)|^q\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right)\right)^{\frac{1}{q}}\\ \text{if } 0<\epsilon<1\\ \left(\frac{1-2\epsilon}{2}\right)^{1-\frac{1}{q}}\left(|f'(x)|^q\left(\frac{2-3\epsilon}{6}\right)+|f'(b)|^q\left(\frac{1-3\epsilon}{6}\right)\right)^{\frac{1}{q}} \text{ if } \epsilon\leq 0.\end{array}\right\}$$

□

Corollary 2.3. *Under the assumption of Theorem 2.1 if we choose* $x=\frac{a+b}{2}$*, we have*

$$\left|\frac{\epsilon f(a)+\epsilon f(b)+2(1-\epsilon)f\left(\frac{a+b}{2}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq\frac{b-a}{2}\left\{\begin{array}{l}\left(\frac{2\epsilon-1}{2}\right)^{1-\frac{1}{q}}\left(|f'\left(\frac{a+b}{2}\right)|^q\left(\frac{3\epsilon-2}{6}\right)+|f'(a)|^q\left(\frac{3\epsilon-1}{6}\right)\right)^{\frac{1}{q}} \textit{ if } \epsilon\geq 1\\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)^{1-\frac{1}{q}}\left(|f'\left(\frac{a+b}{2}\right)|^q\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right)+|f'(a)|^q\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right)\right)^{\frac{1}{q}}\\ \textit{if } 0<\epsilon<1\\ \left(\frac{1-2\epsilon}{2}\right)^{1-\frac{1}{q}}\left(|f'\left(\frac{a+b}{2}\right)|^q\left(\frac{2-3\epsilon}{6}\right)+|f'(a)|^q\left(\frac{1-3\epsilon}{6}\right)\right)^{\frac{1}{q}} \textit{ if } \epsilon\leq 0\end{array}\right\}$$

$$+\frac{b-a}{2}\left\{\begin{array}{l}\left(\frac{2\epsilon-1}{2}\right)^{1-\frac{1}{q}}\left(|f'\left(\frac{a+b}{2}\right)|^q\left(\frac{3\epsilon-2}{6}\right)+|f'(b)|^q\left(\frac{3\epsilon-1}{6}\right)\right)^{\frac{1}{q}} \textit{ if } \epsilon\geq 1\\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)^{1-\frac{1}{q}}\left(|f'\left(\frac{a+b}{2}\right)|^q\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right)+|f'(b)|^q\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right)\right)^{\frac{1}{q}}\\ \textit{if } 0<\epsilon<1\\ \left(\frac{1-2\epsilon}{2}\right)^{1-\frac{1}{q}}\left(|f'\left(\frac{a+b}{2}\right)|^q\left(\frac{2-3\epsilon}{6}\right)+|f'(b)|^q\left(\frac{1-3\epsilon}{6}\right)\right)^{\frac{1}{q}} \textit{ if } \epsilon\leq 0.\end{array}\right\}$$

Corollary 2.4. *Under the assumption of Theorem 2.1 if we choose* $x=\frac{a+b}{2}$ *and* $\epsilon=1$*, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{8}\left(\frac{1}{3}\right)^{\frac{1}{q}}\left[\left(2|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(2|f'(b)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]$$
$$\leq \left(\frac{3^{1-\frac{1}{q}}}{8}\right)(b-a)(|f'(a)|+|f'(b)|).$$

Proof. The second inequality is obtained using the convexity of $|f'|^q$ and the fact that $\sum_{k=1}^{n}(a_k+b_k)^s \leq \sum_{k=1}^{n} a_k^s + \sum_{k=1}^{n} b_k^s$ for $0 \leq s < 1, a_1, a_2, ...a_n \geq 0,\ b_1, b_2, ...b_n \geq 0$. □

Theorem 2.3. *Let $\epsilon \in \mathbb{R}$ and let $f : I^\circ \to \mathbb{R}$ be a differentiable function on $I^\circ, a, b \in I^\circ$ with $a < b$ such that $f' \in L[a,b]$. If $|f'|^q, q \geq 1$ is concave on $[a,b]$, then the following inequality holds:*

$$\left|\frac{(x-a)[(1-\epsilon)f(x)+\epsilon f(a)]+(b-x)[(1-\epsilon)f(x)+\epsilon f(b)]}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{(x-a)^2}{b-a}\left\{\begin{array}{ll} \left(\frac{2\epsilon-1}{2}\right)\left|f'\left(\frac{(3\epsilon-2)x+(3\epsilon-1)a}{3(2\epsilon-1)}\right)\right| & \text{if } \epsilon \geq 1 \\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)\left|f'\left(\frac{(2\epsilon^3-3\epsilon+2)x+(-2\epsilon^3+6\epsilon^2-3\epsilon+1)a}{3(2\epsilon^2-2\epsilon+1)}\right)\right| & \text{if } 0 < \epsilon < 1 \\ \left(\frac{1-2\epsilon}{2}\right)\left|f'\left(\frac{(2-3\epsilon)x+(1-3\epsilon)a}{3(1-2\epsilon)}\right)\right| & \text{if } \epsilon \leq 0 \end{array}\right\}$$
$$+\frac{(b-x)^2}{b-a}\left\{\begin{array}{ll} \left(\frac{2\epsilon-1}{2}\right)\left|f'\left(\frac{(3\epsilon-2)x+(3\epsilon-1)b}{3(2\epsilon-1)}\right)\right| & \text{if } \epsilon \geq 1 \\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)\left|f'\left(\frac{(2\epsilon^3-3\epsilon+2)x+(-2\epsilon^3+6\epsilon^2-3\epsilon+1)b}{3(2\epsilon^2-2\epsilon+1)}\right)\right| & \text{if } 0 < \epsilon < 1 \\ \left(\frac{1-2\epsilon}{2}\right)\left|f'\left(\frac{(2-3\epsilon)x+(1-3\epsilon)b}{3(1-2\epsilon)}\right)\right| & \text{if } \epsilon \leq 0. \end{array}\right\}$$

Proof. By concavity of $|f'|^q$ and the power mean inequality we may write

$$|f'(\lambda x+(1-\lambda)y)|^q \geq \lambda|f'(x)|^q+(1-\lambda)|f'(y)|^q \geq (\lambda|f'(x)|+(1-\lambda)|f'(y)|)^q.$$

Hence

$$|f'(\lambda x+(1-\lambda)y)| \geq \lambda|f'(x)|+(1-\lambda)|f'(y)|,$$

so $|f'|$ is also concave. Now by applying triangular inequality and Lemma 2.1 we have:

$$\left|\frac{(x-a)[(1-\epsilon)f(x)+\epsilon f(a)]+(b-x)[(1-\epsilon)f(x)+\epsilon f(b)]}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{(x-a)^2}{b-a}\int_0^1 |t-\epsilon||f'(tx+(1-t)a)|dt + \frac{(b-x)^2}{b-a}\int_0^1 |\epsilon-t||f'(tx+(1-t)b)|dt$$

$$\leq \frac{(x-a)^2}{b-a}\left(\int_0^1 |t-\epsilon|dt\right)\left|f'\left(\frac{\int_0^1 |t-\epsilon|(tx+(1-t)a)dt}{\int_0^1 |t-\epsilon|dt}\right)\right|$$

$$+ \frac{(b-x)^2}{b-a}\left(\int_0^1 |\epsilon-t|dt\right)\left|f'\left(\frac{\int_0^1 |\epsilon-t|(tx+(1-t)b)dt}{\int_0^1 |\epsilon-t|dt}\right)\right|$$

$$= \frac{(x-a)^2}{b-a}\left\{\begin{array}{ll} \left(\frac{2\epsilon-1}{2}\right)\left|f'\left(\frac{(3\epsilon-2)x+(3\epsilon-1)a}{3(2\epsilon-1)}\right)\right| & \text{if } \epsilon \geq 1 \\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)\left|f'\left(\frac{(2\epsilon^3-3\epsilon+2)x+(-2\epsilon^3+6\epsilon^2-3\epsilon+1)a}{3(2\epsilon^2-2\epsilon+1)}\right)\right| & \text{if } 0<\epsilon<1 \\ \left(\frac{1-2\epsilon}{2}\right)\left|f'\left(\frac{(2-3\epsilon)x+(1-3\epsilon)a}{3(1-2\epsilon)}\right)\right| & \text{if } \epsilon \leq 0 \end{array}\right\}$$

$$+ \frac{(b-x)^2}{b-a}\left\{\begin{array}{ll} \left(\frac{2\epsilon-1}{2}\right)\left|f'\left(\frac{(3\epsilon-2)x+(3\epsilon-1)b}{3(2\epsilon-1)}\right)\right| & \text{if } \epsilon \geq 1 \\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)\left|f'\left(\frac{(2\epsilon^3-3\epsilon+2)x+(-2\epsilon^3+6\epsilon^2-3\epsilon+1)b}{3(2\epsilon^2-2\epsilon+1)}\right)\right| & \text{if } 0<\epsilon<1 \\ \left(\frac{1-2\epsilon}{2}\right)\left|f'\left(\frac{(2-3\epsilon)x+(1-3\epsilon)b}{3(1-2\epsilon)}\right)\right| & \text{if } \epsilon \leq 0. \end{array}\right\}$$

□

Corollary 2.5. *Under the assumption of Theorem 2.3 if we choose* $x = \frac{a+b}{2}$*, we have*

$$\left|\frac{\epsilon f(a)+\epsilon f(b)+2(1-\epsilon)f\left(\frac{a+b}{2}\right)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{2}\begin{cases} \left(\frac{2\epsilon-1}{2}\right)\left|f'\left(\frac{(9\epsilon-4)a+(3\epsilon-2)b}{6(2\epsilon-1)}\right)\right| & \text{if } \epsilon \geq 1 \\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)\left|f'\left(\frac{(-2\epsilon^3+12\epsilon^2-9\epsilon+4)a+(2\epsilon^3-3\epsilon+2)b}{6(2\epsilon^2-2\epsilon+1)}\right)\right| & \text{if } 0<\epsilon<1 \\ \left(\frac{1-2\epsilon}{2}\right)\left|f'\left(\frac{(4-9\epsilon)a+(2-3\epsilon)b}{6(1-2\epsilon)}\right)\right| & \text{if } \epsilon \leq 0 \end{cases}$$

$$+ \frac{b-a}{2}\begin{cases} \left(\frac{2\epsilon-1}{2}\right)\left|f'\left(\frac{(9\epsilon-4)b+(3\epsilon-2)a}{6(2\epsilon-1)}\right)\right| & \text{if } \epsilon \geq 1 \\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)\left|f'\left(\frac{(-2\epsilon^3+12\epsilon^2-9\epsilon+4)b+(2\epsilon^3-3\epsilon+2)a}{6(2\epsilon^2-2\epsilon+1)}\right)\right| & \text{if } 0<\epsilon<1 \\ \left(\frac{1-2\epsilon}{2}\right)\left|f'\left(\frac{(4-9\epsilon)b+(2-3\epsilon)a}{6(1-2\epsilon)}\right)\right| & \text{if } \epsilon \leq 0 \end{cases}$$

Corollary 2.6. *Under the assumption of Theorem 2.3 if we choose* $x = \frac{a+b}{2}$ *and* $\epsilon = 1$, *we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}\left[\left|f'\left(\frac{5a+b}{6}\right)\right| + \left|f'\left(\frac{a+5b}{6}\right)\right|\right]$$
$$\leq \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|.$$

Proof. The second inequality is obtained by using the concavity of $|f'|^q$. □

3. Applications to f-Divergence Measures

One of the basic problem in various applications of Probability Theory is finding an appropriate measure of distance between any two probability distributions. A lot of divergence measures for this purpose have been proposed and extensively studied by Kullback and Leibler [25], Renyi [29], Havrda and Charvat [16], Burbea and Rao [5], Lin [26], Csisźar[8], Ali and Silvey [1], Shioya and Da-te [31] and others (see for example [17] and the references therein). But here we will take only two of them and for this purpose define the following terms.

Let the set χ and the σ-finite measure μ be given and consider the set of all probability densities on μ to be defined on $\Omega := \{p| \, p : \chi \to \mathbb{R}, \, p(x) > 0, \, \int_\chi p(x)d\mu(x) = 1\}$.

Let $f:(0,\infty)\to\mathbb{R}$ be given function and consider $D_f(p,q)$ be defined by

$$D_f(p,q) := \int_\chi p(x) f\left[\frac{q(x)}{p(x)}\right] d\mu(x), \quad p,q\in\Omega. \tag{3.1}$$

If f is convex function, then (3.1) is known as the Csisźar f-divergence [8].

In [31], Shioya and Da-te introduced the Hermite-Hadamard (HH) divergence

$$D_{HH}^f(p,q) := \int_\chi p(x) \frac{\int_1^{\frac{q(x)}{p(x)}} f(t)dt}{\frac{q(x)}{p(x)}-1} d\mu(x), \quad p,\, q\in\Omega, \tag{3.2}$$

where f is convex function on $(0,\infty)$ with $f(1)=0$. In [31] the authors gave the property of HH divergence that $D_{HH}^f(p,q)\geq 0$ with the equality holds if and only if $p=q$.

Proposition 3.1. *Let all the assumptions of Theorem 2.1 hold with $I=(0,\infty)$ and $f(1)=0$. If $p,q\in\Omega$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{2}D_f(p,q) - D_{HH}^f(p,q)\right| \\ \le\ & \frac{1}{8}\left[|f'(1)|\int_\chi |q(x)-p(x)|d\mu(x) + \int_\chi |q(x)-p(x)|\left|f'\left(\frac{q(x)}{p(x)}\right)\right| d\mu(x)\right].\end{aligned} \tag{3.3}$$

Proof. Let $X_1=\{x\in\chi : q(x)>p(x)\}$, $X_2=\{x\in\chi : q(x)<p(x)\}$ and $X_3=\{x\in\chi : q(x)=p(x)\}$.

If $x\in X_3$, then obviously the equality holds in (3.3).

Now if $x\in X_1$, then by using Corollary 2.2 for $a=1$, $b=\frac{q(x)}{p(x)}$, multiplying both hand sides of the obtained results by $p(x)$ and then integrating over X_1, we get

$$\begin{aligned}&\left|\frac{1}{2}\int_{X_1} p(x) f\left[\frac{q(x)}{p(x)}\right] d\mu(x) - \int_{X_1} p(x)\frac{\int_1^{\frac{q(x)}{p(x)}} f(t)dt}{\frac{q(x)}{p(x)}-1} d\mu(x)\right| \\ \le\ & \frac{1}{8}\left[|f'(1)|\int_{X_1} |q(x)-p(x)|d\mu(x) + \int_{X_1} |q(x)-p(x)|\left|f'\left(\frac{q(x)}{p(x)}\right)\right| d\mu(x)\right].\end{aligned} \tag{3.4}$$

Similarly, if $x\in X_2$, then by using for $a=\frac{q(x)}{p(x)}$, $b=1$, multiplying both sides by $p(x)$ and then integrating over X_2, we get

$$\begin{aligned}&\left|\frac{1}{2}\int_{X_2} p(x) f\left[\frac{q(x)}{p(x)}\right] d\mu(x) - \int_{X_2} p(x)\frac{\int_1^{\frac{q(x)}{p(x)}} f(t)dt}{\frac{q(x)}{p(x)}-1} d\mu(x)\right| \\ \le\ & \frac{1}{8}\left[|f'(1)|\int_{X_2} |p(x)-q(x)|d\mu(x) + \int_{X_2} |p(x)-q(x)|\left|f'\left(\frac{q(x)}{p(x)}\right)\right| d\mu(x)\right].\end{aligned} \tag{3.5}$$

By adding the inequalities (3.4) and (3.5) and then using the triangular inequality we get (3.3). □

Proposition 3.2. *Let all the assumptions of Theorem 2.2 hold with* $I = (0, \infty)$ *and* $f(1) = 0$*. If* $p, q \in \Omega$*, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{2}D_f(p,q) - D^f_{HH}(p,q)\right| \\ &\le \left(\frac{3^{1-\frac{1}{q}}}{8}\right)\left[|f'(1)|\int_\chi |q(x)-p(x)|d\mu(x)\right. \\ &\left. + \int_\chi |q(x)-p(x)|\left|f'\left(\frac{q(x)}{p(x)}\right)\right| d\mu(x)\right]. \end{aligned} \tag{3.6}$$

Proof. The proof is similar to the proof of Proposition 3.1 but use Corollary 2.4 instead of Corollary 2.2. □

Proposition 3.3. *Let all the assumptions of Theorem 2.3 hold with* $I = (0, \infty)$ *and* $f(1) = 0$*. If* $p, q \in \Omega$*, then we have the inequality*

$$\begin{aligned} &\left|\frac{1}{2}D_f(p,q) - D^f_{HH}(p,q)\right| \\ &\le \frac{1}{4}\left[\int_\chi |q(x)-p(x)|\left|f'\left(\frac{p(x)+q(x)}{2p(x)}\right)\right| d\mu(x)\right]. \end{aligned} \tag{3.7}$$

Proof. The proof is similar to the proof of Proposition 3.1 but use Corollary 2.6 instead of Corollary 2.2. □

As in [10], we will consider the following particular means for any $a, b, c \in \mathbb{R}, a \neq b \neq c$ which are well known in the literature:

$$\begin{aligned} A(a,b,c;w_a,w_b,w_c) &= \frac{w_a a + w_b b + w_c c}{w_a + w_b + w_c} \qquad a,b,c>0, \\ \bar{L}(a,b) &= \frac{b-a}{\ln b - \ln a} \qquad a \neq b,\ b,b>0, \\ L_n(a,b) &= \left[\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right]^{\frac{1}{n}} \qquad a,b \in \mathbb{R}, a<b, n \neq -1,0, n \in \mathbb{R}. \end{aligned}$$

Proposition 3.4. *Let* $0 < a < b < c$, $n \in \mathbb{R}$,*and* $n > 2$*. Then the inequality*

$$\begin{aligned} &\left|A\left(a^n, b^n, \left(\frac{a+b}{2}\right)^n; \epsilon, \epsilon, 2(1-\epsilon)\right) - L_n(a,b)^n\right| \\ &\le \frac{n(b-a)}{2}\left\{\begin{array}{l} \left|\frac{a+b}{2}\right|^{n-1}\left(\frac{3\epsilon-2}{6}\right) + \left(|a|^{n-1}+|b|^{n-1}\right)\left(\frac{3\epsilon-1}{6}\right) \ \textit{if } \epsilon \ge 1, \\ \left|\frac{a+b}{2}\right|^{n-1}\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right) + \left(|a|^{n-1}+|b|^{n-1}\right)\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right) \\ \textit{if } 0<\epsilon<1, \\ \left|\frac{a+b}{2}\right|^{n-1}\left(\frac{2-3\epsilon}{6}\right) + \left(|a|^{n-1}+|b|^{n-1}\right)\left(\frac{1-3\epsilon}{6}\right) \ \textit{if } \epsilon \le 0, \end{array}\right\} \end{aligned}$$

holds.

Proof. By using the function $f(s) = s^n, s > 0, n > 2$, the proof can be obtained from Corollary 2.1. □

Proposition 3.5. *Let $0 < a < b < c$, $n \in \mathbb{R}$, and $n > 2$. Then the inequality*

$$|A(a^n, b^n, \left(\frac{a+b}{2}\right)^n; \epsilon, \epsilon, 2(1-\epsilon)) - L_n(a,b)^n|$$

$$\leq \frac{n(b-a)}{2}\left\{\begin{array}{l} \left(\frac{2\epsilon-1}{2}\right)^{1-\frac{1}{q}}\left(\left|\frac{a+b}{2}\right|^{(n-1)q}\left(\frac{3\epsilon-2}{6}\right)+|a|^{(n-1)q}\left(\frac{3\epsilon-1}{6}\right)\right)^{\frac{1}{q}} \quad \text{if } \epsilon \geq 1, \\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)^{1-\frac{1}{q}}\left(\left|\frac{a+b}{2}\right|^{(n-1)q}\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right)\right. \\ \left.+|a|^{(n-1)q}\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right)\right)^{\frac{1}{q}} \quad \text{if } 0 < \epsilon < 1, \\ \left(\frac{1-2\epsilon}{2}\right)^{1-\frac{1}{q}}\left(\left|\frac{a+b}{2}\right|^{(n-1)q}\left(\frac{2-3\epsilon}{6}\right)+|a|^{(n-1)q}\left(\frac{1-3\epsilon}{6}\right)\right)^{\frac{1}{q}} \quad \text{if } \epsilon \leq 0, \end{array}\right\}$$

$$+\frac{n(b-a)}{2}\left\{\begin{array}{l} \left(\frac{2\epsilon-1}{2}\right)^{1-\frac{1}{q}}\left(\left|\frac{a+b}{2}\right|^{(n-1)q}\left(\frac{3\epsilon-2}{6}\right)+|b|^{(n-1)q}\left(\frac{3\epsilon-1}{6}\right)\right)^{\frac{1}{q}} \quad \text{if } \epsilon \geq 1, \\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)^{1-\frac{1}{q}}\left(\left|\frac{a+b}{2}\right|^{(n-1)q}\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right)\right. \\ \left.+|b|^{(n-1)q}\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right)\right)^{\frac{1}{q}} \quad \text{if } 0 < \epsilon < 1, \\ \left(\frac{1-2\epsilon}{2}\right)^{1-\frac{1}{q}}\left(\left|\frac{a+b}{2}\right|^{(n-1)q}\left(\frac{2-3\epsilon}{6}\right)+|b|^{(n-1)q}\left(\frac{1-3\epsilon}{6}\right)\right)^{\frac{1}{q}} \quad \text{if } \epsilon \leq 0, \end{array}\right\}$$

holds.

Proof. By using the function $f(s) = s^n, s > 0, n > 2$, the proof can be obtained from Corollary 2.3. □

Proposition 3.6. *Let $0 < a < b < c$, $n \in \mathbb{R}$, and $1 < n < 2$. Then the inequality*

$$|A(a^n, b^n, \left(\frac{a+b}{2}\right)^n; \epsilon, \epsilon, 2(1-\epsilon)) - L_n(a,b)^n|$$

$$\leq \frac{|n|(b-a)}{2}\left\{\begin{array}{ll} \left(\frac{2\epsilon-1}{2}\right)\left|\frac{(9\epsilon-4)a+(3\epsilon-2)b}{6(2\epsilon-1)}\right|^{n-1} & \text{if } \epsilon \geq 1, \\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)\left|\frac{(-2\epsilon^3+12\epsilon^2-9\epsilon+4)a+(2\epsilon^3-3\epsilon+2)b}{6(2\epsilon^2-2\epsilon+1)}\right|^{n-1} & \text{if } 0 < \epsilon < 1, \\ \left(\frac{1-2\epsilon}{2}\right)\left|\frac{(4-9\epsilon)a+(2-3\epsilon)b}{6(1-2\epsilon)}\right|^{n-1} & \text{if } \epsilon \leq 0, \end{array}\right\}$$

$$+\quad \frac{|n|(b-a)}{2}\left\{\begin{array}{ll}\left(\frac{2\epsilon-1}{2}\right)\left|\frac{(9\epsilon-4)b+(3\epsilon-2)a}{6(2\epsilon-1)}\right|^{n-1} & \textit{if } \epsilon\geq 1,\\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)\left|\frac{(-2\epsilon^3+12\epsilon^2-9\epsilon+4)b+(2\epsilon^3-3\epsilon+2)a}{6(2\epsilon^2-2\epsilon+1)}\right|^{n-1} & \textit{if } 0<\epsilon<1,\\ \left(\frac{1-2\epsilon}{2}\right)\left|\frac{(4-9\epsilon)b+(2-3\epsilon)a}{6(1-2\epsilon)}\right|^{n-1} & \textit{if } \epsilon\leq 0,\end{array}\right\}$$

holds.

Proof. By using the function $f(s)=s^n, s>0, 1<n<2$, the proof can be obtained from Corollary 2.5. □

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