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PIECEWISE-LINEAR RADIAL COMPRESSION FUNCTION FOR PYRAMID TWO-DIMENSIONAL QUANTIZER

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Abstract. *In this paper a companding-type approach is presented to designing the pyramid two-dimensional quantizer whose cells are obtained by radial spreading of the cubic cells. For a memoryless Laplacian source, the optimal radial compression function and rate allocation between the radius and location quantizers are determined subject to the mean-squared error (MSE) criterion. The results also include formulation of a new method for linearization of compression function, based on a compression function derivative discretization. It is of special importance since the unclosed-form of optimal radial compression function causes certain difficulties in companding quantizer implementation.*

Key words: *two-dimensional quantization, companding quantization, compression function linearization, Laplacian source*

1. INTRODUCTION

The companding-type approach has been very successfully applied to scalar quantizer design by using Bennett's integral, which for the high resolution case specifies the average mean square error (the MSE distortion) as a function of the compression function [1], [2]. For dimension 2, the widely used companding model is a model of polar quantization where the radius of an input vector is quantized using a scalar compandor, while input vector phase is uniformly quantized. In polar quantization, the amplitude and phase can be quantized separately, which is characteristic of restricted polar quantization, or they can be quantized jointly, when the phase quantization is made dependent on the amplitude, specifying model of unrestricted polar quantization [1], [3]. These polar quantizers have been utilized to circularly-symmetric sources, such as a memoryless Gaussian source.

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The two-dimensional quantization of a memoryless Lapalcian source based on the companding principle, to the best of authors' knowledge, is not so widely applied. This motivates us to focus the analysis toward an N-point two-dimensional quantization of a memoryless Lapacian source with the aim to apply the companding-type approach in its designing. In order to do that, we assume that the space partition follows the source geometry in the sense that the cell size is inversely proportional to the source density. As a result we obtain that an amplitude output level consists of points with the same densities as well as that the cells are of the equal size on one amplitude level. In other words, we quantize an input vector in terms of its intensity, or radius and location vector, whereby the nonuniform spacing between amplitude levels is defined by means of a compression function. For the case when the number of cells per amplitude levels is optimized, the generalized optimal radial compression function was determined in [4], [5]. In [6] the two-dimensional quantization model was considered where the μ-low companding quantizer is used for radius quantization, while in [7], the generalized radial compression function was derived under the condition that the signal to quantization noise ratio (SQNR) is constant over a broad range of input signal variance. In [4], along with the unrestricted vector quantization, the pyramid vector quantization was considered based on the companding principle, i.e. the restricted vector quantization where vector intensity and location vector are quantized independently with the L-level scalar compandor and the M-level uniform scalar quantizer respectively, wherein holds $N = M \times L$. In this paper we perform an asymptotic analysis of pyramid two-dimensional quantization at which the quantization cells are different in respect to those from [4-7]. Particularly, in [4-7] the lattice cells are assumed, while our cells are obtained by performing the radial spreading of cubic cells that are uniformly distributed on the surface of the pyramid with unit radius. The proposed space partition eliminates the edge effect which appears due to overlapping of lattice cells and amplitude levels. It can also be noted that there exists certain analogy between the presented space partition and the space partition at the restricted polar quantization. The corresponding counterpart of the restricted polar quantizer is a pyramid two-dimensional quantizer presented here.

With respect to the MSE distortion, an optimization of pyramid two-dimensional quantization with cells that are obtained by radial spreading the cubic cells was somewhat researched in [8]. The first part of this paper, similarly to [8], is related to the determination of the asymptotically optimal number of levels and compression function for scalar compandor intended for quantization of input vector radius. In the second part of the paper we spread the analysis by researching the possibility for the simplification of the nonlinear quantization model defined in the first part of the paper. We develop a novel method for the compression function approximation with linear segments, based on a compression function derivative discretization. We opt for this kind of linearization in order to retain the quantizer structure in accordance with the geometric approach. Linearization of the optimal compression function for unrestricted two-dimensional quantization, based on segmentation of its first derivative was presented in [9]. Additionally, in this paper we assume that the size of segment steps changes in the manner to form geometric progression. The change of the step size of successive segments by a constant factor is the feature of a widely used standard G.711 [10], while as regards the Laplacian source, quantizers with geometric progression of step size were described in [11], [12].

2. ASYMPTOTICALLY OPTIMAL RATE ALLOCATION AND RADIAL COMPRESSION FUNCTION

Let us consider an N-point restricted two-dimensional quantization of a memoryless Laplacian source. Then the quantization input is a two-dimensional vector $\mathbf{x} = [x_1 \quad x_2]$ consisting of independent and identically distributed Laplacian variables x_i with zero mean and unit variance whose the joint probability density function (joint pdf) is:

$$f_{\mathbf{x}}(\mathrm{x}) = \prod_{k=1}^{2} f(x_k) = \frac{1}{2} \cdot \exp\left(-\sqrt{2}\sum_{k=1}^{2}|x_k|\right). \tag{1}$$

As discussed in Introduction, we consider quantization scheme in which an amplitude output level represents a contour of a constant pdf given by $\sum_{k=1}^{2}|x_k| = \hat{r},\ i = 1, 2, ..., L$, where L is a number of amplitude levels (see Fig. 1 [8]). Similarly, the amplitude decision level is $\sum_{k=1}^{2}|x_k| = r_i,\ i = 1, 2, ..., L$. The symmetry observed in the space partition imposes the input vector representation in terms of vector intensity, or radius r and the location vector $\mathbf{s}$ [8]:

$$\mathbf{x} = r\mathbf{s}, \tag{2}$$

where r is defined by:

$$r = \sum_{k=1}^{2}|x_k|, \tag{3}$$

while vector $\mathbf{s}$ is the normalized vector $\mathbf{x}$ according to the L^1 – norm $|\mathbf{x}|_1 = |x_1|+|x_2|$, so that it originates from (0,0) and ends on contour given by $\sum_{k=1}^{2}|x_k| = 1$. The radius r of input vector $\mathbf{x}$ is also a random variable having a pdf [13]:

$$f_r(r) = 2r\exp(-\sqrt{2}r),\ r \geq 0, \tag{4}$$

while the endpoint of location vector $\mathbf{s}$ is uniformly distributed on the unit pyramid [13].

The number of cells per amplitude levels is constant, here marked with M, so that it holds:

$$LM = N. \tag{5}$$

Hence, the output points are of the form:

$$\hat{\mathbf{x}}_{i,j} = \hat{r}_i\hat{\mathbf{s}}_j, \quad i = 1,\ldots,L, \quad j = 1,\ldots,M, \tag{6}$$

which enables the independent quantization of r and $\mathbf{s}$. We assume that the input vector radius is quantized by L-level scalar compandor whose compression function $h(r)$: $[0,+\infty) \rightarrow [0,1)$ maps nonuniform amplitude levels into uniform ones and enables the following representation for the step size of scalar compandor, Δ_i:

$$\Delta_i = 1/[Lh'(\hat{r}_i)], \quad i = 1, ..., L.. \tag{7}$$

The endpoint of location vector is quantized by M-level uniform scalar quantizer. Since the quantization rate is $R = 1/2 \times \log_2 N = 1/2 \times \log_2(L \times M)$ it holds:

$$R = (R_r + R_s)/2, \tag{8}$$

where $R_r = \log_2 L$ and $R_s = \log_2 M$ are rates of L-level radius and M-level location scalar quantizers, respectively.

Now it can be noted that for large L, the volume of the quantization cell $c_{i,j}$, $i=1,\ldots,L$, $j=1,\ldots,M$ can be correlated with the surface of cell S_j that contains the endpoint of $\mathbf{s}_j$:

$$\int_{c_{i,j}} d\mathrm{x} \approx \hat{r}_i \left(\int_{S_j} ds \right) \Delta_i = \hat{r}_i \frac{4\sqrt{2}}{M} \Delta_i, \quad i = 1,\ldots,L, \quad j = 1,\ldots,M \ , \tag{9}$$

where $\Delta_i = r_i - r_{i\text{-}1}$ (see Fig. 1). Also, Fig. 1 shows that the proposed space partition eliminates distortion degradation due to the edge effect unlike the two-dimensional quantization whose cells are rectangular and where Helmert transformation is not used. The figure points out that there is a certain analogy between the presented space partition and the space partition of the restricted polar quantization.

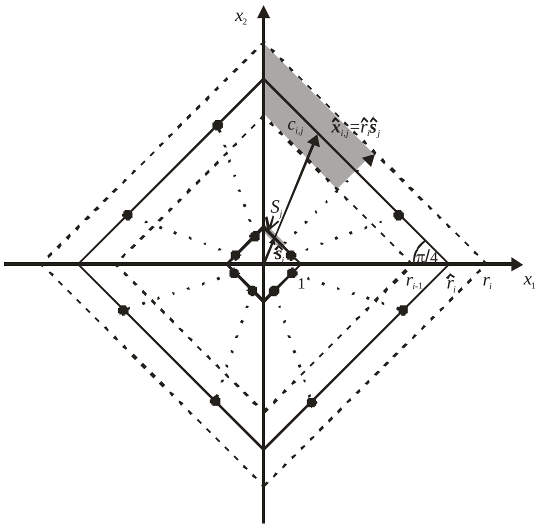

Fig. 1 An amplitude level in pyramid two-dimensional quantization with cells obtained by radial spreading the cubic cells

In general, the MSE distortion per dimension is [2]:

$$D = \frac{1}{2} \sum_{i=1}^{L} \sum_{j=1}^{M} \int_{c_{i,j}} (\mathbf{x} - \hat{\mathbf{x}}_{i,j})^2 f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x} \ , \tag{10}$$

where by combing the sine and cosine formulas the $(\mathbf{x} - \hat{\mathbf{x}}_{i,j})^2$ can be expressed through r and $\mathbf{s}$:

$$(\mathbf{x} - \hat{\mathbf{x}}_{i,j})^2 = \frac{(r - \hat{r}_i)^2}{2} + \hat{r}_i^2 (\mathbf{s} - \hat{\mathbf{s}}_j)^2 \ . \tag{11}$$

We conduct an asymptotic analysis, i.e. we assume the large L and that r and $f_{\mathbf{x}}(\mathbf{x})$ vary little inside a cell which allows their approximation with $\hat{r}_i$ and $f_{\mathbf{x}}(\hat{\mathbf{x}}_{i,j})$, respectively. By using these asymptotic approximations and by performing coordinate transformation ($d\mathbf{x}=\hat{r}_i\, dr d\mathbf{s}$) we can write that asymptotic distortion is:

$$D=\frac{1}{2}\sum_{i=1}^{L}\sum_{j=1}^{M} f_{\mathbf{x}}(\hat{\mathbf{x}}_{i,j})\int_{r_{i-1}}^{r_i}\int_{S_j}\left[\frac{(r-\hat{r}_i)^2}{2}+\hat{r}_i^2(\mathbf{s}-\hat{\mathbf{s}}_j)^2\right]\hat{r}_i dr d\mathrm{s}\,. \tag{12}$$

Integral solving in (12) gives:

$$D=\frac{1}{2}\sum_{i=1}^{L}\left(\frac{\Delta_i^2}{24}+\frac{8}{3M^2}\hat{r}_i^2\right)\sum_{j=1}^{M} f_{\mathbf{x}}(\hat{\mathbf{x}}_{i,j})\hat{r}_i\,\frac{4\sqrt{2}}{M}\Delta_i\,. \tag{13}$$

Utilizing (5), (7) and (9) yields:

$$D=\frac{1}{2}\sum_{i=1}^{L}\left(\frac{1}{24L^2}\frac{1}{[h'(\hat{r}_i)]^2}+\frac{8L^2}{3N^2}\hat{r}_i^2\right)P_i\,, \tag{14}$$

where P_i is the probability that vector $\mathbf{x}$ belongs to the ith amplitude level:

$$P_i=\sum_{j=1}^{M}\int_{c_{i,j}} f_{\mathbf{x}}(\mathbf{x})d\mathbf{x}\approx\sum_{j=1}^{M} f_{\mathbf{x}}(\hat{\mathbf{x}}_{i,j})\int_{c_{i,j}} d\mathbf{x}\,. \tag{15}$$

The P_i can also be expressed as probability that radius of input vector r lies between boundaries of the ith amplitude level:

$$P_i=\int_{r_{i-1}}^{r_i} f_r(r)dr\approx f_r(\hat{r}_i)\Delta_i\,. \tag{16}$$

By substituting (16) into (14), we get that the asymptotic MSE distortion per dimension can be written as follows:

$$D=\frac{1}{48L^2}\sum_{i=1}^{L}\frac{f_r(\hat{r}_i)}{[h'(\hat{r}_i)]^2}\Delta_i+\frac{4L^2}{3N^2}\sum_{i=1}^{L}\hat{r}_i^2 f_r(\hat{r}_i)\Delta_i\,. \tag{17}$$

By approximating the sums with integrals ($\hat{r}_i\to r,\Delta_i\to dr$), we obtain that the asymptotic distortion is:

$$D=\frac{1}{48L^2}\int_0^{\infty}\frac{f_r(r)}{[h'(r)]^2}dr+\frac{4L^2}{3N^2}\int_0^{\infty} r^2 f_r(r)dr\,. \tag{18}$$

Since $\int_0^{+\infty} r^2 f_r(r)dr=3$, (18) can be further transformed into:

$$D=l_0/L^2+4L^2/N^2\,, \tag{19}$$

where:

$$l_0 = \frac{1}{48}\int_0^{\infty} \frac{f_r(r)}{[h'(r)]^2}dr \ . \qquad (20)$$

By optimizing the distortion subject to L, i.e by differentiating D, given by (19), with respect to L and equalizing with zero, we find:

$$L = l_0^{1/4}(N/2)^{1/2} , \qquad (21)$$

$$D = 4l_0^{1/2}N^{-1} . \qquad (22)$$

Eq. (22) shows that for given N, D depends only on scalar variable l_0. If we further carefully observe the expression for l_0, we note that (20) corresponds to the Bennet's integral form for the case of scalar companding quantization [1], [2]. This enables us to apply the method from scalar quantization to get the optimal radial compression function of pyramid two-dimensional quantization:

$$h^*(r) = \frac{\int_0^r f_r^{1/3}(t)dt}{\int_0^{\infty} f_r^{1/3}(t)dt} = \frac{2^{2/3}}{3^{1/3}\Gamma(1/3)}\int_0^r t^{1/3}\exp(-\sqrt{2}t/3)dt , \qquad (23)$$

where $\Gamma(\cdot)$ is gamma function. Finally, by substituting (23) in (20), (21) and (22) we obtain that asymptotically optimal amplitude level number and distortion are:

$$L^* = 2^{-1}[\Gamma(1/3)/2]^{3/4}N^{1/2} , \qquad (24)$$

and

$$D^* = 2^{-1/2}[\Gamma(1/3)]^{3/2}N^{-1} , \qquad (25)$$

respectively. The expression for distortion coincides with the result from [4], while the numbers of amplitudes levels differ which results in small deviations of the corresponding values. The difference in number of amplitude levels originates because the cells of different shape are assumed in this paper and in [4]. We also compare the distortion, given by (25) with the distortion when the number of amplitude levels, as well as the number of cells per amplitude levels are optimized and when the cells are cubic (((n+2)/n)^(n+2)/6*N^(-2/n); where $n = 2$ [4], [5], [9]). Comparison shows that the performances are weaker in respect to the optimum for $10\log([\Gamma(1/3)/2]^{3/2}/12) = 0.6546$ dB.

With the L^* derivation we actually determine the asymptotically optimal relation between the rates of the employed scalar quantizers in the pyramid two-dimensional quantization:

$$R_s^* - R_r^* = \log_2[N/(L^*)^2] = 4[2/\Gamma(1/3)]^{3/2} \approx 2.58 . \qquad (26)$$

Equation (26) points out that the difference between rates of scalar quantizers is constant and does not depend on N. This dependence along with the equation (8) specifies the optimal allocation of rates between the radius and location scalar quantizers.

3. Compression Function Linearization Based on Compression Function Derivative Discretization

In order to simplify the companding model proposed in Section 2, we propose a novel method for the approximation of the nonlinear compression function (23) with linear segments. We assume L_0 segments inside which the constant slope of compression function is obtained by performing a discretization of compression function derivative. We also assume that slopes of linear segments form a geometric progression. As the first derivative of the optimal radial compression function:

$$\frac{dh(r)}{dr} = 2(\sqrt{2}/3)^{1/3}/\Gamma(1/3)r^{1/3}\exp(-\sqrt{2}r/3) \tag{27}$$

is not a monotonous function, we have to distinguish two ranges. The first one, for radii between $[0,1/\sqrt{2})$ and the second one, for radii between $[1/\sqrt{2},\infty)$, where $1/\sqrt{2}$ is the radius at which the compression function derivative has maximum $h'_{\max} = h'(1/\sqrt{2}) = 2/[3^{1/3}\Gamma(1/3)]\exp(-1/3)$. Since the step size should be inversely proportional to the first derivative of compression function, we assume the smallest step size $\Delta_{\min}$ for the segment that contains $1/\sqrt{2}$ at which the compression function derivative maximum is. We distribute the number of segments per previously defined ranges in the following manner: $L_0 = L_0'+1+L_0''$, where L_0' and L_0'' are numbers of segments in the first and the second region, respectively. We adopt that the number of segments in a range is proportional to the probability that the input vector belongs to that range. In that way we get that:

$$L_0' = L_0 \int_0^{1/\sqrt{2}} f_r(r)dr\,, \tag{28}$$

while $L_0'' = L_0 - L_0' - 1$. We assume that the step size is the smallest for segment $L_0'+1$ and that it increases by the scaling factor *s*, as the segment number increases or decreases in respect to $L_0'+1$. In other words, we define a model whose compression function slope per segments is:

$$h'_{\mathrm{PL}}(\hat{t}_i) = \begin{cases} h'_{\max}/s^{L_0'+1-i}, & i \le L_o'+1 \\ h'_{\max}/s^{i-L_0'-1}, & L_o'+1 \le i \le L_0 \end{cases}, \tag{29}$$

where *s* is the scaling factor that defines the geometric progression of the step size, while $\hat{t}_i$ denotes the input radius at which the piecewise-linear compression function, here denoted $h_{\mathrm{PL}}(r)$ and the nonlinear compression function, $h(r)$ are equal. Thus, for each segment except the first one, the condition $h_{\mathrm{PL}}(\hat{t}_i) = h(\hat{t}_i)$ provides us with specifying the piecewise-linear compression function as follows:

$$h^i_{\mathrm{PL}}(r) = h'_{\mathrm{PL}}(\hat{t}_i)(r-\hat{t}_i) + h(\hat{t}_i),\ 2 \le i \le L_0\,. \tag{30}$$

We obtain the compression function equation for the first segment by performing correction which enables that $h^1_{\mathrm{PL}}(0) = 0$:

$$h^1_{\mathrm{PL}}(r) = h'_{\mathrm{PL}}(\hat{t}_1)r\,. \tag{31}$$

The continuity of the piecewise-linear compression function implies that the segment boundaries, t_i are:

$$t_0 = 0,\ t_i = \frac{h(\hat{t}_{i-1}) - h(\hat{t}_i) + \hat{t}_i h'_{\mathrm{PL}}(\hat{t}_i) - \hat{t}_{i-1} h'_{\mathrm{PL}}(\hat{t}_{i-1})}{h'_{\mathrm{PL}}(\hat{t}_i) - h'_{\mathrm{PL}}(\hat{t}_{i-1})},\ 1 \le i \le L_0 - 1\,. \tag{32}$$

It should be noted that we linearize compression function that maps [0,+∞) into [0,1). This actually means that the piecewise-linear compression function must not exceed 1, which further implies that the segment boundary t_{L_0} cannot be infinity. Thus, we determine t_{L_0} from condition $h_{\mathrm{PL}}^{L_0}(t_{L_0}) = 1$, so that our piecewise-linear radial compression function maps [0, t_{L_0}) into [0, 1]. The input radii grater then t_{L_0} are quantized with the overload errors, i.e. t_{L_0} represent the boundary between granular and overload region, called the support limit of quantizer. Hence, the overall distortion of the linearized model consists of two components. By performing the similar analysis for distortion estimation as in Section 2, we obtain that the granular and overload distortions for the pyramid two-dimensional quantization defined with the proposed piecewise-linear radial compression function are:

$$D_{\mathrm{g}}^{\mathrm{PL}} = \frac{1}{48(L^* h'_{\max})^2}\left[s^{2(L_0'+1)} \sum_{i=1}^{L_0'} s^{-2i} \int_{t_{i-1}}^{t_i} f_r(r)dr + s^{-2(L_0'+1)} \sum_{i=L_0'+1}^{L_0} s^{2i} \int_{t_{i-1}}^{t_i} f_r(r)dr \right] + \frac{4(L^*)^2}{N^2}\,. \tag{33}$$

and

$$D_{\mathrm{o}}^{\mathrm{PL}} = \frac{1}{2}\int_{t_{L_0}}^{+\infty} \left[\frac{(r - t_{L_0})^2}{2} + \frac{8(L^*)^2 t_{L_0}^2}{3N^2} \right] f(r)dr\,, \tag{34}$$

respectively. Then the overall distortion is $D^{\mathrm{PL}} = D_g^{\mathrm{PL}} + D_o^{\mathrm{PL}}$.

4. Numerical Results

For given N and L_0, one can conclude from (29)-(34) that the distortion value only depends on scaling factor s, so that we provide the optimal value for the scaling factor, s^* by performing numerical minimization of D subject to s. For such determined s^* the piecewise-linear radial compression function is shown in Fig. 2. With the increase of L_0 the piecewise-linear compression function expectedly approaches the optimal nonlinear compression function. Fig. 2 also shows that support limit t_{L_0} increases with L_0 which along with the fact that the support limit increases with N, implies that when N is large and L_0 is small, the overload distortion can have a considerable contribution to the overall distortion. One of the reasons why it happens is that the geometric progression of the step size influences t_{L_0} and restricts its setting to an arbitrarily value. Thus, for a large number of amplitude levels and a small number of segments, one can expect not only granular distortion deterioration due to the poor approximation of the nonlinear compression function, but also the overload distortion deterioration due to the inappropriate value of the support limit.

In Table 1, for different pairs of (R = 1/2×log$_2N$, L_0) the optimal value for the scaling factor s^*, the support limit t_{L_0} and the resulting signal to quantization noise ratio calculated as $\mathrm{SQNR}^{\mathrm{PL}} = 10\times\log(1/D^{\mathrm{PL}})$ are tabulated. The last column in table refers to the SQNR of nonlinear pyramid two-dimensional quantization model, $\mathrm{SQNR}^* = 10\times\log(1/D^*)$, where D^* is given by (25). The results listed in Table 1 show that s^* decreases with the increase of L_0 and increases with R. One can also observe that for a given L_0 and a different R, the optimal scaling factors differ, which means that for each combination (R, L_0) the numerical optimization of SQNR in respect to s should be performed. One can see from Table 1 that for $L_0 = 3$ the difference between $\mathrm{SQNR}^{\mathrm{PL}}$ and SQNR^*, $\Delta\mathrm{SQNR} = \mathrm{SQNR}^* - \mathrm{SQNR}^{\mathrm{PL}}$ increases for about 0.5 dB when the rate increase is 0.5 bits/sample, which confirms observations derived from Fig. 2. Similarly, for any given L_0 the difference between $\mathrm{SQNR}^{\mathrm{PL}}$ and SQNR^* increases with R. That is why the adequate choice of segment number should be made in order to achieve small quality degradation due to compression function approximation with linear segments. On the other hand, the compression function linearization reduces the complexity of the quantizer. Actually, the optimal radial compression function is not of a closed-form nonlinear equation (see eq. (23)), which means that in order to define the corresponding expandor the transcendental equation must be solved. Unclosed-form characteristics of compressor and expandor imply certain difficulties in their implementation. Unlike this, the piecewise-linear compressor function and its expandor's counterpart, as linear blocks, can be easily implemented.

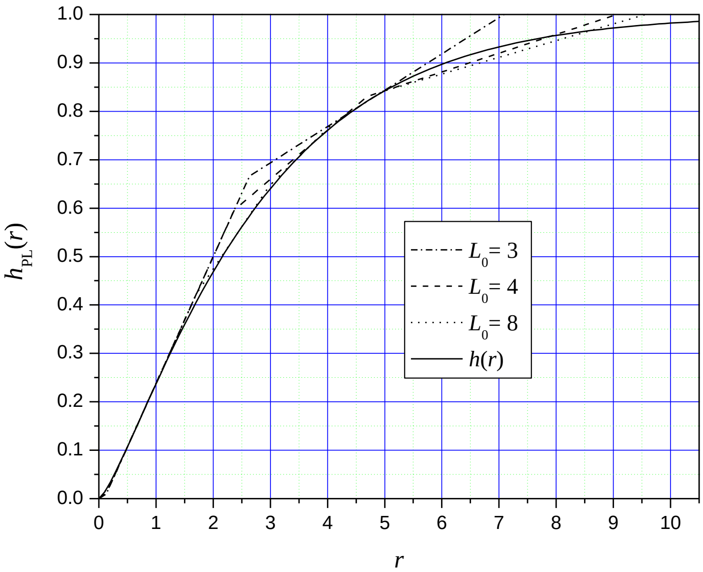

Fig. 2 The piecewise-linear compression function that approximates the optimal radial compression function

Table 1 Pyramid two-dimensional quantizer based on piecewise-linear radial compression function and its optimal nonlinear counterpart

R [bits/sam.]	L_0	s^*	t_{L_0}	SQNR^{PL} [dB]	SQNR* [dB]	ΔSQNR [dB]
5	3	2.74	6.50	24.664	25.189	0.525
	4	1.93	7.23	25.065		0.124
5.5	3	3.18	6.86	27.374	28.199	0.825
	4	2.11	7.67	27.953		0.246
	8	1.39	8.12	28.182		0.017
6	3	3.71	7.22	30.048	31.209	1.161
	4	2.29	8.15	30.836		0.373
	8	1.43	8.59	31.112		0.097
6.5	3	4.32	7.59	32.653	34.220	1.567
	4	2.46	8.63	33.710		0.510
	8	1.46	8.99	34.035		0.185
	16	1.19	9.07	34.079		0.141
7	3	5.03	8.02	35.258	37.230	1.972
	4	2.63	9.15	36.594		0.636
	8	1.49	9.410	36.953		0.277
	16	1.20	9.462	37.001		0.229
7.5	3	5.75	8.453	37.831	40.240	2.409
	4	2.78	9.64	39.470		0.770
	8	1.52	9.87	39.864		0.376
	16	1.21	9.89	39.915		0.325
8	3	6.48	8.89	40.385	43.251	2.866
	4	2.92	10.12	42.339		0.912
	8	1.55	10.37	42.766		0.485
	16	1.22	10.35	42.821		0.421

To evaluate the proposed model, in Fig. 3 we show SQNR as a function of R for our pyramid two-dimensional quantizer and other companding quantizers intended for Lapalcian source quantization. In Fig. 3 we show the performances of both pyramid two-dimensional quantizers, i.e. when the linearization is or is not performed. Although the number of segments is small ($L_0 = 8$), it is evident that for 5.5 bits/sample $\leq R \leq$ 10 bits/sample the SQNR deterioration does not exceed 1 dB, whereby the deterioration is smaller for lower R, which is explained above. In comparison with optimal unrestricted two-dimensional quantization, our pyramid quantization based on piecewise-linear compression function has SQNR up to 1.6 dB lower. On the other hand, the complexity of the proposed quantization scheme is reduced since the radius of input vector and the location vector are quantized independently by engaging simple two-component quantizer that consists of one scalar compandor and only one uniform scalar quantizer. Fig. 3 shows that the linearized pyramid two-dimensional quantizer outperforms logarithmic nonuniform two-dimensional quantizers proposed in [6] and [7]. Particularly, depending on R the SQNR of our pyramid quantizer is for 1 dB to 2 dB higher than that of unrestricted quantizer with quasilogarithmic compression characteristic ($\mu = 255$) [6]. In comparison with the SQNR of the restricted quantizer with semilogarithmic compression characteristic ($A = 87.6$) [7], the SQNR gain of our compandor amounts 2 to 2.5 dB for a given range of rates. One can notice that the proposed pyramid two-dimensional quantizer outperforms the optimal nonlinear scalar compandor in terms of SQNR, which is not the case with logarithmic two-dimensional quantizers from [6] and [7]. By comparing our two-dimensional quantizer having 8 segments with the piecewise-uniform scalar quantizer having 16 segments, whose geometric progression of step size is defined in [11], the SQNR gain of about 2.4 dB can be noted.

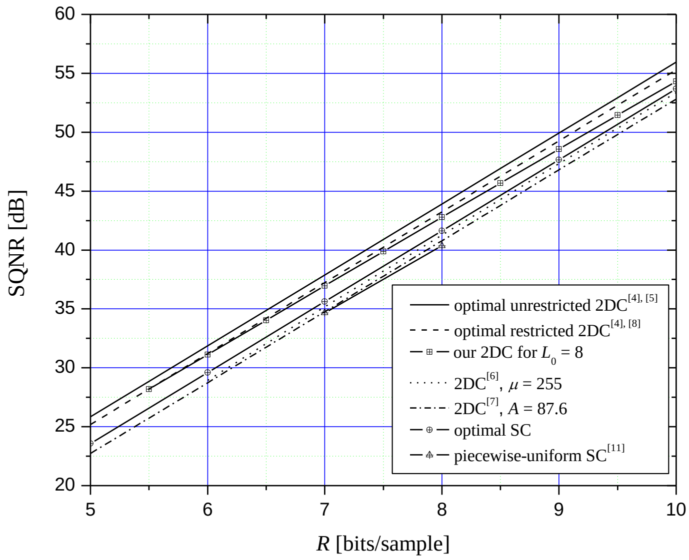

Fig. 3 SQNR versus bit rate for various models of companding quantization of Laplacian source (2DC-two-dimensional compandor, SC-scalar compandor)

5. Conclusion

In this paper, with respect to the MSE distortion we determine the asymptotically optimal rate allocation and radial compression function for the pyramid two-dimensional quantizer whose cells are obtained by radial spreading the cubic cells. The derived relation between the rates of radius and location quantizers $R_{\mathbf{s}}^{*} - R_r^{*} \approx 2.58$ along with relation $R_{\mathbf{s}}^{*} + R_r^{*} = R$ specifies the asymptotically optimal rate allocation. It is shown that the optimal radial compression function is an incomplete gamma function. This means that the asymptotically optimal pyramid two-dimensional quantizer has difficulties originating from the nonexistence of a closed-form expression for compression function. We eliminate these difficulties by performing linearization of the nonlinear compression function. We propose a new linearization method based on compression function derivative discretization, where the step size of successive segments changes by scaling factor s. We numerically optimize scaling factor s in respect to the MSE distortion. The proposed pyramid two-dimensional quantizer with the piecewise-linear radial compression function outperforms the optimal scalar compandor in terms of SQNR. Particularly, for $L_0 = 8$, the SQNR gain can attain 1.5 dB.

REFERENCES

[1] N. Jayant, P. Noll, *Digital Coding of Waveforms: Principles and Applications to Speech and Video*. Prentice Hall, Englewood Cliffs, New Jersey, 1984.

[2] Gersho, R. Gray, *Vector Quantization and Signal Compression*. Springer, 1991.

[3] P. Swaszek, T. Ku, "Asymptotic performances of unrestricted polar quantizer," IEEE Transaction on Information Theory, vol. 32, no. 2, pp. 330–333, 1986.

[4] P. Swaszek, "A vector quantizer for the Laplace source," IEEE Transactions on Information Theory, vol, 37, no. 5, pp. 1355–1364, 1991.

[5] Z. Perić, A. Jovanović, "Optimal piecewise linear radial compression function for Laplacian source," European Transactions on Telecommunications, vol. 19, no. 8, pp. 859–872, 2008. [Online]. Available: http://dx.doi.org/10.1002/ett.1314.

[6] Z. Perić, M. Petković, "Two-dimensional radial μ-law companding quantizer for Laplacian source," Transactions on Emerging Telecommunications Technologies, 2013. [Online]. Available: http://dx.doi.org/10.1002/ett.2669.

[7] Z. H. Perić, A. Ž. Jovanović, "Radial compression function for vector quantizer of Laplacian source with high dynamic variance range," European Transactions on Telecommunications, vol. 22, no. 3, pp. 113–124, 2011. [Online]. Available: http://dx.doi.org/10.1002/ett.1463.

[8] Z. Perić, A. Jovanović, O. Milanović, "Optimal product pyramid vector quantization of memoryless Laplacian source," in *Proceedings of the International Scientific Conference on Information, Communication and Energy Systems and Technologies, ICEST 2006*, Sofia, Bulgaria, pp. 200–203, 2006.

[9] Z. Perić, M. Novković, V. Despotović, "Linearization method for two-dimensional memoryless Laplass source," Electronics and Electrical Engineering, vol. 73, no. 1, pp. 41–44, 2007. [Online]. Available: http://www.ee.ktu.lt/journal/2007/1/09_T121_PERIC%20STR.pdf.

[10] ITU-T, Recommendation G.711, *Pulse Code Modulation (PCM) of Voice Frequencies*, International Telecommunication Union, 1972.

[11] Z. Perić, J. Nikolić, "Design of piecewise uniform scalar quantizer with geometric progression of segment width," Advances in Electrical and Computer Engineering, vol. 11, no. 4, pp. 37–40, 2011. [Online]. Available: http://dx.doi.org/10.4316/AECE.2011.04006.

[12] D. G. Jeong, J. D. Gibson, "Uniform and piecewise uniform lattice vector quantization for memoryless Gaussian and Laplacian sources," IEEE Transactions on Information Theory, vol. 39, no. 3, pp. 786–804. 1993.

[13] T. Fischer, "A pyramid vector quantizer," IEEE Transactions on Information Theory, vol. 32, no. 4, pp. 568–583, 1986.