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A NEW TYPE OF DISCRETE EULER-LAGRANGE EQUATION WITH APPLICATIONS IN OPTIMAL CONTROL *

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Abstract. *A new type of discrete Euler-Lagrange equation, suitable for generalization, is presented in this paper. Several forms of this equation can be found in references. They have different differential operators as well as combinations of them. The equation given in this paper uses only one differential operator providing easy generalizations of Euler-Lagrange equation. In the paper, generalizations for the case of more variables and more ordered differences in the functional which is optimized, are derived. The application in determining optimal control of a discrete system is also given.*

Key words: *Boundary conditions, Discrete Euler-Lagrange equation, Optimal control, Variational principle, Transversality conditions, Lagrange multipliers.*

1. INTRODUCTION

The first generalization of a variational principle for discrete systems appeared fifty years ago. Some optimization problems were solved and the first discrete analogues appeared. The discrete maximum principle was obtained [1] and discrete analogues of Euler-Lagrange equation were derived. In this way, variational principles were applied to discrete systems optimization. There are more forms of discrete Euler-Lagrange equation [1]. In [2], the discrete analogue of Euler-Lagrange equation was derived as well as a generalization of this equation:

$$F_y(n, r_n, r_{n-1}) + F_z(n+1, r_{n+1}, r_n) = 0\,. \tag{1}$$

For discrete functional $J(x) = \sum_{i=0}^{\infty} L(i, x_i, \Delta x_i)$, discrete equation has the following form [3]:

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$$L_x(i, x_i, \Delta x_i) - \Delta L_{\Delta x}(i-1, x_{i-1}, \Delta x_{i-1}) = 0 \,. \tag{2}$$

The discrete analogue of Euler-Lagrange equation is given in [4, 5]:

$$\frac{\partial L_D^{(k)}}{\partial q^{i(k)}} - \Delta_t \left(\frac{\partial L_D^{(k-1)}}{\partial(\Delta_i q^{i(k-1)})} \right) = 0 \,. \tag{3}$$

General approach to variational method application for discrete systems optimization is described in [6]. For the discrete functional:

$$J = \sum_{k=k_0}^{k_{f-1}} V(x(k), x(k+1), k) \,, \tag{4}$$

the equation is:

$$\frac{\partial V(x(k), x(k+1), k)}{\partial x(k)} + \frac{\partial V(x(k-1), x(k), k-1)}{\partial x(k)} = 0 \,. \tag{5}$$

Further development of variational principle for discrete systems led to some new forms of discrete Euler-Lagrange equation [7]:

$$\frac{\partial L_d}{\partial q_s} - D_{-h} \left(\frac{\partial L_d}{\partial q_{s,t}} \right) = 0 \,, \tag{6}$$

where D_{-h} is discrete operator.

All of these discrete analogues solve the problem of discrete functional extremization. One can notice that the difference between these equations is in using different difference operators. In this paper, the discrete analogue of Euler-Lagrange equation is derived using one difference operator. This form provides easy generalization of Euler-Lagrange equation and also its application for determining an optimal control of discrete systems where optimization methods developed for continuous systems can easily be used. One can easily notice transversality conditions which usually occur in practice. They can be related to either state coordinates or control parameters.

2. A Discrete Analogue of Euler-Lagrange Equation

Let discrete functional has the following form:

$$J = \sum_{i=0}^{N} f[x(i), \Delta x(i), i] \,, \tag{7}$$

where *f* - known continuous and differentiable function and $x(i)$ − unknown function with discrete argument *i*.

The task is to determine $x(i)$ so that *J* is minimal [8-10]. We define function $\delta(i)$ as:

$$\delta(0) = \delta(1) = \delta(N) = \delta(N+1) = 0 \,, \tag{8}$$

where $\delta(i)$ is an arbitrary function for interval $1 < i < N$.

In order to determine necessary conditions for minimum of the functional we will introduce variation $\varepsilon\delta(i)$ of the function $x(i)$:

$$x(i) = x^*(i) + \varepsilon\delta(i) . \tag{9}$$

After substitution (9) into (7) we have:

$$J = \sum_{i=0}^{N} f[x(i), \Delta x(i), i] = \sum_{i=0}^{N} f[x^*(i) + \varepsilon\delta(i), \Delta x^*(i) + \varepsilon\Delta\delta(i), i] . \tag{10}$$

Calculating the first derivative of J and equating it to zero:

$$\Delta\frac{\partial f}{\partial x(i)} + \frac{\partial f}{\partial x(i)} - \Delta\frac{\partial f}{\partial \Delta x(i)} = 0 . \tag{11}$$

Equation (11) represents discrete analogue of the well-known Euler-Lagrange equation. If the left and the right part of the function $x(i)$ is free, conditions for transversality are fulfilled:

$$\left.\frac{\partial f}{\partial(\Delta x)}\right|_{i=1} = 0, \ \left.\frac{\partial f}{\partial(\Delta x)}\right|_{i=N} = 0 . \tag{12}$$

When functional contains more order differences of state coordinates:

$$J = \sum_{i=0}^{N} f[x(i), \Delta x(i), \ldots, \Delta^k x(i), i] , \tag{13}$$

the discrete analogue of Euler-Lagrange equation can be obtained using the method described above:

$$(\Delta+1)^k \frac{\partial f}{\partial x(i)} + (\Delta+1)^{k-1}\Delta\frac{\partial f}{\partial \Delta x(i)} + \cdots + (-1)^k \Delta^k \frac{\partial f}{\partial \Delta^k x(i)} = 0 , \tag{14}$$

i.e.:

$$\begin{gathered}\sum_{l=0}^{k} (\Delta+1)^{k-l} \Delta^j (-1)^k \frac{\partial f}{\partial(\Delta^l x(i))} = 0, \\ \sum_{l=0}^{k}\sum_{p=0}^{k-j} \binom{k-l}{p} \Delta^{l+p} (-1)^l \frac{\partial f}{\partial(\Delta^l x(i))} = 0.\end{gathered} \tag{15}$$

When functional contains more unknown functions:

$$J = \sum_{i=0}^{N} f[x_1(i), x_2(i), \ldots, x_n(i), \Delta x_1(i), \Delta x_2(i), \ldots, \Delta x_m(i), i] , \tag{16}$$

in the same way we obtain n discrete analogues of Euler-Lagrange equation:

$$\begin{gathered}\Delta\frac{\partial f}{\partial x_1(i)} + \frac{\partial f}{\partial x_1(i)} - \Delta\frac{\partial f}{\partial \Delta x_1(i)} = 0, \\ \vdots \\ \Delta\frac{\partial f}{\partial x_n(i)} + \frac{\partial f}{\partial x_n(i)} - \Delta\frac{\partial f}{\partial \Delta x_n(i)} = 0.\end{gathered} \tag{17}$$

Finally, in the most general case, when there are more unknown functions in functional with more order differences:

$$J=\sum_{i=0}^{N} f[x_1(i),x_2(i),\ldots,x_n(i),\Delta x_1(i),\Delta x_2(i),\ldots,\Delta x_n(i),\Delta^k x_1(i),\ldots,\Delta^k x_n(i),i], \qquad (18)$$

we obtain the system of difference equations:

$$\sum_{l=0}^{k}\sum_{p=0}^{k-j}\binom{k-l}{p}\Delta^{l+p}(-1)^l \frac{\partial f}{\partial(\Delta^l x_r(i))}=0,\ r=1,2,\ldots,n\,. \qquad (19)$$

The system (19) represents a new discrete analogue of Euler-Lagrange equation. Its main quality is in the fact that only one type of difference operator Δ is used. This form is applicable in optimal control where traditional methods for continuous control can easily be used. Finally, this analogue of Euler-Lagrange equation is also applicable when transversality conditions are given for discrete systems.

3. Optimal Control of Discrete Systems

Lagrange multipliers are introduced to minimize J with differential restrictions, in the same way as continuous system control.

Let mathematical model of discrete system has the following form:

$$\Delta x_j = f_j(x_1,\ldots,x_n,u_1,\ldots,u_m,i),\ j=1,2,\ldots,n\,. \qquad (20)$$

We need to determine control $u(u_1,u_2,\ldots,u_m)$ that minimizes optimality criterion:

$$J=\sum_{i=1}^{N} f_0(x_1,\ldots,x_n,\Delta x_1,\cdot\cdot\cdot,\Delta x_n,u_1,\ldots,u_m)=\sum_{i=1}^{N} f_0(x(i),\Delta x(i),u(i))\,. \qquad (21)$$

Optimal control set is determined from condition for functional minimum:

$$J_1=\sum_{i=1}^{N} f_\lambda(x(i),\Delta x(i),u(i))\,, \qquad (22)$$

where:

$$f_\lambda = f_0+\sum_{j=1}^{N}\lambda_j(f_j-\Delta x_j)\,. \qquad (23)$$

Applying relations (17) we obtain:

$$\begin{aligned}
&\Delta\frac{\partial f_\lambda}{\partial x_j}+\frac{\partial f_\lambda}{\partial x_j}-\Delta\frac{\partial f_\lambda}{\partial \Delta x_j}=0,\ \ j=1,2,\ldots,n\\
&\Delta\frac{\partial f_\lambda}{\partial u_j}+\frac{\partial f_\lambda}{\partial u_j}=0,\ \ j=1,\ldots,m\\
&\Delta\frac{\partial f_\lambda}{\partial \lambda_j}+\frac{\partial f_\lambda}{\partial \lambda_j}=0,\ \ j=1,2,\ldots,n.
\end{aligned} \qquad (24)$$

The relations above are sufficient to determine optimal control $u_1, u_2, \ldots, u_m$, with the given boundary conditions:

$$x_j(1) = x_{j1},\ \ x_j(N) = x_{jN}\ . \tag{25}$$

4. Optimal Control of the Linear Discrete Systems

Determining optimal control of linear discrete systems is very important. State space model of the system and boundary conditions are:

$$\Delta X = AX + Bu,\ \ X(1) = X_1,\ \ X(N) = X_N\ . \tag{26}$$

We can apply (26) to determine optimal control of discrete systems. We use the most common criterion where we need to minimize the mean square error (related to state coordinates) and energy consumption during control:

$$J = \sum_{i=1}^{N} (X^T Q_1 X + \Delta x^T Q_2 \Delta x + u^T R u)\ , \tag{27}$$

where Q_1, Q_2, and R are diagonal matrix. Then we have:

$$f_\lambda = X^T Q_1 X + \Delta x^T Q_2 \Delta x + u^T R u + \lambda^T (AX + Bu - \Delta X). \tag{28}$$

The system of Euler-Lagrange equations is:

$$\begin{aligned}
&\Delta \frac{\partial f_\lambda}{\partial X} + \frac{\partial f_\lambda}{\partial X} - \Delta \frac{\partial f_\lambda}{\partial \Delta X} = 0,\\
&\Delta \frac{\partial f_\lambda}{\partial u} + \frac{\partial f_\lambda}{\partial u} = 0,\\
&\Delta \frac{\partial f_\lambda}{\partial \lambda} + \frac{\partial f_\lambda}{\partial \lambda} = 0,
\end{aligned} \tag{29}$$

where:

$$\begin{aligned}
&\frac{\partial (X^T Q_2 X)}{\partial X} = 2Q_1 X,\\
&\frac{\partial (\Delta X^T Q_2 \Delta X)}{\partial (\Delta X)} = 2Q_2 \Delta X,\\
&\frac{\partial (u^T R u)}{\partial u} = 2Ru,\\
&\frac{\partial f_\lambda}{\partial \lambda} = Ax + Bu - \Delta x.
\end{aligned} \tag{30}$$

After replacing (30) into (29) we obtain:

$$\begin{aligned}
&2Q_1 \Delta X + 2Q_1 X - 2Q_2 \Delta^2 X + \lambda^T A = 0\\
&2Ru + \lambda^T B = 0\\
&\Delta X - AX - Bu = 0.
\end{aligned} \tag{31}$$

From (31) we have:

$$\begin{aligned} \lambda^T &= -2RuB^{-1} \\ u &= B^{-1}(AX + \Delta X) \\ \lambda^T &= -2RB^{-1}(AX + \Delta X)B^{-1}. \end{aligned} \tag{32}$$

After substituting (32) into the first equation of the system (31) we obtain:

$$2Q_2\Delta^2 X - 2Q_1\Delta X + 2RB^{-1}\Delta XB^{-1}A - 2Q_1X + 2RB^{-1}AXB^{-1}A = 0\,. \tag{33}$$

Relation (33) is another form of Euler-Lagrange equation. This is $2n$-order equation, where n is the system order. Simultaneously solving this linear difference equations system and using boundary conditions, we can determine optimal control for the given optimality criterion. Optimal trajectory from the start to the destination point is also determined.

The same system of equations can also be used for optimization in the case of existing transversality conditions.

5. Numerical Study

This section demonstrates the effectiveness of the given form of Euler-Lagrange equation for determining optimal control of discrete system.

Let us consider the system described with a mathematical model:

$$\begin{aligned} \Delta x_1(i) &= 0.2x_2(i), \\ \Delta x_2(i) &= 2.1x_1(i) + u(i), \end{aligned} \tag{34}$$

with the given boundary conditions:

$$\begin{aligned} x_1(0) = 5,\ \ x_1(16) = 0, \\ x_2(0) = 0,\ \ x_2(16) = 0. \end{aligned} \tag{35}$$

We will determine optimal control $u(i)$ (for i =1,2,…,16)) which minimizes the optimality criterion:

$$J = \sum_{i=1}^{16}(u^2(i) + x_1^{\,2}(i) + x_2^{\,2}(i))\,. \tag{36}$$

Using (23) and (24) we obtain:

$$f_\lambda = u^2(i) + x_1^{\,2}(i) + x_2^{\,2}(i) + \lambda_1(0.2x_2(i) - \Delta x_1(i)) + \lambda_2(2.1x_1(i) - \Delta x_2(i) + u(i))\,, \tag{37}$$

$$\begin{aligned} 2\Delta x_1(i) + 2.1\lambda_2(i) + 2x_1(i) + 2.1\lambda_2 + \Delta\lambda_1 &= 0, \\ 2\Delta x_2(i) + 2.1\Delta\lambda_1(i) + 2x_2(i) + 0.2\lambda_1 + \Delta\lambda_2 &= 0, \end{aligned} \tag{38}$$

$$u(i) = -\frac{\lambda_2}{2}\,, \tag{39}$$

$$\begin{aligned}&\Delta x_1(i)-0.2x_2(i)=0,\\&\Delta x_2(i)-2.1x_1(i)-u(i)=0.\end{aligned} \tag{40}$$

After elimination $u(i),\lambda_1,\Delta\lambda_1,\lambda_2$ and $\Delta\lambda_2$, we finally obtain the difference equation:

$$\begin{aligned}&\Delta^4x_1(i)+3.2\Delta^3x_1(i)+2.62\Delta^3x_1(i)-0.5+\Delta x_1(i)+0.294x_1(i)=0,\\&x_2(i)=5\Delta x_1(i).\end{aligned} \tag{41}$$

Roots of characteristic equation (41) are: $s_1=-1.4,\ s_2=-0.6,\ s_3=-0.7,\ s_4=-0.5$, so general solution of (41) has the following form:

$$\begin{aligned}&x_1(i)=c_1(-0.4)^i+c_2(0.4)^i+c_3(0.3)^i+c_4(0.5)^i,\\&x_2(i)=-1.4c_1(-0.4)^i-0.6c_2(0.4)^i-0.7c_3(0.3)^i-0.5c_4(0.5)^i.\end{aligned} \tag{42}$$

Using boundary conditions (35), we obtain the following values for integration constants: c_1 = 3.7, c_2 = 3.1, c_3 = –1.4, c_4 = –0.4. Now, substituting these values into (42) we determine the optimal trajectory:

$$\begin{aligned}&x_1(i)=3.7(-0.4)^i+3.1(0.4)^i-1.4(0.3)^i-0.4(0.5)^i,\\&x_2(i)=-5.18(-0.4)^i+1.86(0.4)^i+0.98(0.3)^i+0.2(0.5)^i.\end{aligned} \tag{43}$$

Finally, the optimal control sequence is:

$$u^*(i)=15.02(-0.4)^i-7.63(0.4)^i-3.63(0.3)^i+0.74(0.5)^i,\ i=1,2,\ldots,16\,. \tag{44}$$

6. Conclusion

One form of discrete Euler-Lagrange equation using only one discrete operator is given in this paper. This form provides easy generalizations in the sense of large number of state coordinates and also in the sense of discrete operator order in the functional. Another advantage of the proposed form of Euler-Lagrange equation is its easy application for determining optimal control of discrete systems. Optimization methods developed for continuous systems can easily be used. This analogue is also applicable with the existing transversality conditions. Its effectiveness was verified for the certain discrete system.

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