Title	クロピナ空間の超曲面の誘導的,内在的理論について
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Citation	北海道教育大学紀要. 第二部. A, 数学・物理学・化学・工学編, 34(1) : 1-11
Issue Date	1983-09
URL	http://s-ir.sap.hokkyodai.ac.jp/dspace/handle/123456789/6094
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Journal of Hokkaido University of Education (Section II A) Vol. 34, No. 1 September, 1983
北海道教育大学紀要（第２部Ａ）第34巻 第１号 昭和58年９月

On Induced and Intrinsic Theories of Hypersurfaces of Kropina Spaces.

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Abstract

Let $R^n = (M^n, \alpha)$ be an n-dimensional Riemannian space with a Riemannian metric $\alpha(x, dx) = (a_{ij}(x)dx^i dx^j)^{1/2}$ and let $F^n = (M^n, L)$ be a Kropina space with the fundamental function $L(x, y) = \alpha^2(x, y)/\beta(x, y)$, where $\beta(x, dx) = b_i(x)dx^i$.

The purpose of the present paper is to study the induced and intrinsic theories of hypersurface of a Kropina space.

§0. Introduction. The induced and intrinsic theories of the subspaces of a Finsler space have been studied by Davies ([3]) and Rund ([9]). The connection coefficients of a Kropina hypersurface can be written as the sum of Riemannian Christoffel symbols and other tensor. In this paper we compare the induced connection coefficients with intrinsic connection coefficients of a Kropina hypersurface and discuss whether they coincide or not. The notations and terminologies are refered to Matsumoto's monograph [7].

§1. Preliminaries. Let F^n be an n–dimensional Kropina space. Components g_{ij} of the fundamental tensor field are given by $g_{ij} = (\partial^2 L^2/\partial y^i \partial y^j)/2$, and the covariant components $y_i = g_{ij}y^j$ of the supporting element are given by $L\partial L/\partial y^i$. The angular metric tensor $h_{ij}(x,y)$ is defined as $h_{ij} = g_{ij} - l_i l_j$, $l_i = y_i/L$. The Riemannian space R^n with the metric $\alpha = (a_{ij}(x)y^i y^j)^{1/2}$ is called the *associated Riemannian space with F^n*. The Christoffel symbols of R^n are denoted by $\{{}^{\,i}_{jk}\}$ and this Riemannian connection is called the *associated one*. We denote by ∇_k the covariant differentiation with respect to x_k relative to the associated Riemannian connection. The fundamental tensor g_{ij} and the connection coefficients $F_j{}^i{}_k$ of the Cartan connection are given respectively by ([11]).

(1.1) $$g_{ij} = \tau(2a_{ij} - l_i b_j - l_j b_i) + l_i l_j, \qquad \tau = \alpha^2 \beta^{-2}$$

(1.2) $$F_j{}^i{}_k = \{{}_j{}^i{}_k\} + D_j{}^i{}_k.$$

The tensor D^i_{jk}, called the *difference tensor,* is given by ([11])

(1.3) $$D_j{}^i{}_k = -Q^{ir}(F_{rj} l_k + F_{rk} l_j) - E_{jk} Q^i - h^i{}_j \Phi_k$$
$$- h^i{}_k \Phi_j + h_{jk} \Phi^i + \lambda C_j{}^i{}_k,$$

where we put

(1.4)
$$\begin{aligned} &(1) \quad \nabla_k b_j = b_{jk}, \quad 2E_{jk} = b_{jk} + b_{kj}, \quad 2F_{jk} = b_{jk} - b_{kj}, \\ &(2) \quad b^i = a^{ij} b_j, \quad a^{ij} a_{jk} = \delta^i{}_k, \quad \rho = a_{ij} b^i b^j, \\ &\qquad Q^i = (2l^i - b^i)/\rho, \quad Q^{ir} = (a^{ir} + Q^i b^r)/2, \\ &(3) \quad \Phi_k = (\rho F_{0k}/\beta - \rho Q^r F_{rk} + 2b_{0k}/L + F_{r0} b^r l_k / L)/2, \\ &\qquad g^{ir} \Phi_r = \Phi^i, \quad \lambda = (E_{00}/L + F_{r0} b^r)/\rho, \quad \Phi_k y^k = \lambda. \end{aligned}$$

In (1.4) 3) and the remainder of the present paper the suffix " 0 " means the contraction by y^i. Contraction of (1.3) by y^k gives

(1.5) $$D_j{}^i{}_0 = -\{a^{ir}(LF_{rj} + F_{r0} l_j) + b^r(2l^i - b^i)(LF_{rj} +$$
$$+ F_{r0} l_j)/\rho\}/2 - E_{j0}(2l^i - b^i)/\rho - \lambda h^i{}_j,$$

where (1.4) was used.

Lemma 1([11]). *The difference tensor D^i_{jk} vanishes if and only if the covariant vector b_i is parallel with respect to the associated Riemannian connection, i.e.,* $\nabla_k b_i = 0$.

§ 2. Hypersurfaces of Kropina space and associated Riemannian space.

First, we are concerned with a hypersurface H^{n-1} of the underlying manifold M^n of a Kropina space $F^n = (M^n, L)$, which is represented parametrically by

(2.1) $$x^i = x^i(u^\sigma), \qquad \sigma = 1, 2, \cdots, n-1,$$

where u^σ are Gaussian coordinates on H^{n-1}. Introducing the notations

(2.2) $$B^i_\alpha = \partial x^i / \partial u^\alpha,$$

we shall assume that the matrix of these projection factors is of rank n−1. The following notations are also employed:

(2.3) $B^i_{\alpha\beta} = \partial^2 x^i/\partial u^\alpha \partial u^\beta, \quad B^{ij\cdots k}_{\alpha\beta\cdots\gamma} = B^i_\alpha B^j_\beta \cdots B^k_\gamma.$

The functions $B^i_\alpha(x)$ may be considered as components of $n-1$ linearly independent vectors tangent to H^{n-1}. Therefore any vector $\dot{x}^i$, tangent to H^{n-1}, may be written uniquely in the form

(2.4) $x^i = B^i_\alpha x^\alpha,$

where X^α are components of the vector relative to the u-coordinate system. In particular, we assume that the supporting element y^i is tangential to H^{n-1} so that

(2.5) $y^i(=\dot{x}^i) = B^i_\alpha \dot{u}^\alpha.$

The induced fundamental metric tensor $g_{\alpha\beta}(u, \dot{u})$ of the hypersurface H^{n-1} defined with respect to such a direction is given by

(2.6) $g_{\alpha\beta}(u, \dot{u}) = g_{ij}(x, y) B^{ij}_{\alpha\beta}.$

If L(x, y) represents the fundamental function of F^n for a direction y^i tangent to H^{n-1}, it follows from (2.5) that the corresponding fundamental function for H^{n-1} is given by $\overline{L}(u, \dot{u}) = L(x^i(u), B^i_\alpha \dot{u}^\alpha)$. For the Kropina space F^n, it follows from (2.1) and (2.5) that the fundamental function $\overline{L}$ is given by

(2.7) $\overline{L}(u, \dot{u}) = a_{\alpha\beta}(u)\dot{u}^\alpha \dot{u}^\beta / b_\beta \dot{u}^\beta, \quad a_{\alpha\beta} = a_{ij} B^{ij}_{\alpha\beta},$

in which $a_{\alpha\beta}(u)$ is the fundamental tensor of the Riemannian hypersurface R^{n-1} and $b_\alpha(u)$ is given by

(2.8) $b_\alpha = b_i B^i_\alpha.$

Thus, in virtue of (1.1), (2.7) and (2.8), the induced metric tensor $g_{\alpha\beta}$ in (2.6) is written by

(2.6′) $g_{\alpha\beta} = \overline{\tau}(2a_{\alpha\beta} - l_\alpha b_\beta - l_\beta b_\alpha) + l_\alpha l_\beta, \quad \overline{\tau} = \tau.$

Here we have

Proposition 1. *A hypersurface of a Kropina space is also a Kropina space.*

Remark. From the above proposition, the hypersurface of a Kropina space is called a *Kropina hypersurface.*

Further, we have

(2.9) $l^i = B^i_\alpha l^\alpha.$

As usual, $\det(g_{ij}) \neq 0$ is supposed. Thus according to our assumption the tensor $g_{\alpha\beta}(u, \dot{u})$ possesses the reciprocal tensor $g_{\alpha\beta}$ which is used to define a set of $n-1$ covariant vectors

(2.10) $B^\alpha_i(x, y) = g^{\alpha\beta}(u, \dot{u}) g_{ij}(x, y) B^j_\beta(x),$

which satisfy

(2.11) $B^i_\alpha B^\beta_i = \delta^\beta_\alpha$.

Another useful indentity ([3]) is

(2.12) $B^\alpha_i B^j_\alpha = \delta^j_i - N_i N^j$,

where the unit normal vector $N^i(x, y)$ is defined at each point of the Kropina hypersurface H^{n-1} with respect to the tangential supporting element y^i by a system of equations

(2.13) $N^i = g^{ij}(x, y) N_j, \quad g_{ij} N^i N^j = 1, \quad N_i B^i_\alpha = 0$,

which in turn imply

(2.14) $N^i B^\alpha_i = 0$.

Further we get

(2.15) $g_{ij} = g_{\alpha\beta} B^\alpha_i B^\beta_j + N_i N_j, \qquad g^{ij} = g^{\alpha\beta} B^i_\alpha B^j_\beta + N^j N^i$.

Next, we shall consider a hypersurface R^{n-1} of the associated Riemannian space with the metric $\alpha = (a_{ij}(x) y^i y^j)^{1/2}$ represented parametrically by the same equations as (2.1). Then u^α in (2.1) are Gaussian coordinates on R^{n-1}. And the function $B^i_\alpha(x)$ in (2.2) may be considered to be components of a set of $n-1$ linearly independent vectors tangent to R^{n-1}. The induced fundamental metric tensor of the Riemannian hypersurface R^{n-1} is given by $a_{\alpha\beta}$ in (2.7). The hypersurface of the associated Riemannian space R^n is called an *associated Riemannian hypersurface* $R^{n-1} = (M^{n-1}, \ \overline{\alpha} = (a_{\alpha\beta}(u) \dot{u}^\alpha \dot{u}^\beta)^{1/2})$.
The quantities $\overline{B}^\alpha_i(x)$ are uniquely defined along R^{n-1} by the equations

(2.16) $\overline{B}^\alpha_i(x) = a^{\alpha\beta}(u) a_{ij} B^j_\beta(x)$.

We denote the covariant components of a unit normal vector of R^{n-1} by $\overline{N}^i$. Then we have a field of linear frame $(B^i_1, \cdots, B^i_{n-1}; \ \overline{N}^i = a^{ij} \overline{N}_j)$ of R^n defined along R^{n-1} by

(2.17) $B^i_\alpha \overline{B}^\beta_i = \delta^\beta_\alpha, \qquad B^i_\alpha \overline{B}^\alpha_j = \delta^i_j - \overline{N}^i \overline{N}_j, \qquad \overline{N}^i \overline{B}^\alpha_i = 0$.

It follows from (2.17) that

(2.18) $a_{ij} = a_{\alpha\beta} \overline{B}^\alpha_i \overline{B}^\beta_j + \overline{N}_i \overline{N}_j$

Since $\overline{N}_i B^i_\alpha = 0$ and $B^i_\alpha \dot{u}^\alpha = y^i$, we see that the supporting element y^i is tangential to the associated Riemannian hypersurface R^{n-1}, that is $\overline{N}_i y^i = 0$, so that we have

(2.19) $\overline{N}^i Y_i = 0, \qquad Y_i = a_{ij} y^j$,

which will play an important role later on. The reciprocal tensor $g^{\alpha\beta}$ of $g_{\alpha\beta}$ is given by

(2.20) $g^{\alpha\beta} = [\bar{\rho} a^{\alpha\beta} + 2(1^\alpha b^\beta + 1^\beta b^\alpha) - b^\alpha b^\beta + 2(\bar{\rho}\bar{\tau} - 2) 1^\alpha 1^\beta] / 2\bar{\rho}\bar{\tau}$,

where we put

(2.21) (a) $\bar{\rho} = a_{\alpha\beta}b^\alpha b^\beta$, (b) $a^{\alpha\beta}b_\alpha = b^\beta$, (c) $l^\alpha = g^{\alpha\beta}l_\beta$.

With the help of relations (1.4)2), (2.7), (2.8) and (2.21)a), we can easily obtain

(2.22) $\bar{\rho} = \rho - (b_i\bar{N}^i)^2$.

It follows from (2.6′), (2.20), (2.21) and (2.22) that

(2.23)
(1) $Y^\alpha = (\dot{u}^\alpha)/\beta = \tau l^\alpha$, (2) $l_\alpha b^\alpha = 2 - \bar{\rho}\tau$,
(3) $a^{\alpha\varepsilon}l_\varepsilon = 2Y^\alpha - \tau b^\alpha$, (4) $b^j = b^\alpha B^j_\alpha + (b_i\bar{N}^i)\bar{N}^j$.

Further, in virtue of (2.10), (2.16) and (2.23) we have

(2.24) $B^\alpha_i = \bar{B}^\alpha_i + (b_m\bar{N}^m)(b^\alpha - 2l^\alpha)\bar{N}_i/\bar{\rho}$.

§ 3. Relation between induced and intrinsic connection parameters.

The Cartan connection coefficients of the Finsler space F^n are denoted by $F_j{}^i{}_k$. The induced connection parameters of hypersurface are defined by the relation ([8])

(3.1) $F_\gamma{}^\alpha{}_\beta = B^\alpha_i(B^i_{\beta\gamma} + F_j{}^i{}_k B^{jk}_{\beta\gamma})$.

And the intrinsic connection coefficients $\bar{F}_\beta{}^\alpha{}_\gamma$ are defined with respect to the induced metric (2.6) of hypersurface in a manner formally identical with the mode of definition of the coefficients $F_j{}^i{}_k$ in terms of the fundamental tensor g_{ij} of F^n.

On the other hand, for the h(hv)-torsion tensor C_{ijk} of a Finsler space we have ([2])

(3.2) $C_{ijk} = C_{\alpha\beta\gamma}B^\alpha_i B^\beta_j B^\gamma_k + M_{\alpha\beta}(B^\alpha_i B^\beta_j N_k + B^\alpha_j B^\beta_k N_i + B^\alpha_k B^\beta_i N_j)$
$+ M_\alpha(B^\alpha_i N_j N_k + B^\alpha_j N_k N_i + B^\alpha_k N_i N_j) + MN_iN_jN_k$,

where $C_{\alpha\beta\gamma}$ is the projection of C_{ijk} onto the hypersurface, M is the normal components of C_{ijk} and

(3.3) $M_{\alpha\beta} = C_{ijk}B^{ij}_{\alpha\beta}N^k$, $M_\alpha = C_{ijk}B^i_\alpha N^j N^k$.

The tensor $M_{\alpha\beta}$ in (4.2) will be called a *Brown tensor* over a hypersurface of a Finsler space. Let us denote the difference of induced and intrinsic connection coefficients of a hypersurface by $\Lambda_\beta{}^\alpha{}_\gamma$ ([9]). From (3.1), we have

(3.4) $\Lambda_\beta{}^\alpha{}_\gamma = \bar{F}_\beta{}^\alpha{}_\gamma - F_\beta{}^\alpha{}_\gamma$.

It is then shown [2] that

(3.5) $\Lambda_{\beta\alpha\gamma}\dot{u}^\beta = NM_{\alpha\gamma}$, $\Lambda_{\beta\alpha\gamma}\dot{u}^\beta\dot{u}^\gamma = M_{\alpha\gamma}\dot{u}^\gamma = 0$.

The following has been proved by Brown ((2]):

Lemma.2 *Assuming that $N \neq 0$, the induced and intrinsic connection coefficients coincide if and only if $M_{\alpha\beta} = 0$ over the Finsler hypersurface.*

§ 4. Normal unit vector of C-rebucible Finsler space.

In this section, we shall consider the normal unit vector of a C-reducible Finsler space which is defined by M. Matsumoto. [5].

Definition. A Finsler space $F^n (n \geqq 3)$ is called *C-reducible* if the h(hv)-torsion tensor C_{ijk} is written in the form

(4.1) $\quad C_{ijk} = (h_{ij}C_k + h_{jk}C_i + h_{ki}C_j)/(n+1).$

Remark. M. Matsumoto also indicated two certain metrics of a C-reducible Finsler space, namely Randers metric $(L = \alpha + \beta)$ and Kropina metric $(L = \alpha^2/\beta)$. Moreover, M. Matsumoto and S. Hōjō [6] have proved that the metric functions of C-reducible Finsler spaces are confined solely to the above metrics.

It is well-known ([10], [11]) that the h(hv)-tortion tensor C_{ijk} of a Kropina space and a Randers space is respectively given by

(4.2) $\quad \overset{k}{C}_{ijk} = (h_{ij}m_k + h_{jk}m_i + h_{ki}m_j)/2L, \qquad m_i = l_i - \tau b_i,$

(4.3) $\quad \overset{R}{C}_{ijk} = (h_{ij}L_k + h_{jk}L_i + h_{ki}L_j)/2L, \qquad L_i = (1+\mu)b_i - \mu l_i, \qquad \mu = \alpha^{-1}\beta.$

Since $N_k y^k = 0$ and $N_k B^k_\beta = 0$, from (3.3), (4.2) and (4.3), the Brown tensor $M_{\alpha\beta}$ of a C-reducible Finsler space F^n is given by

(4.4) $\quad M_{\alpha\beta} = N^k C_k h_{\alpha\beta}/(n+1).$

On the other hand, the torsion vector $\overset{k}{C}_k$ of a Kropina space (resp. $\overset{R}{C}_k$ of a Randers space) is given by

(4.5) $\quad \overset{k}{C}_k = (n+1)(l_k - \tau b_k)/2L, \qquad (\text{resp. } \overset{R}{C}_k = (n+1)(\mu l_k - \tau' b_k)/2L,$
$\qquad \tau' = 1+\mu), \ ([10], [11]).$

Therefore, $M_{\alpha\beta}$ of a C-reducible Finsler space F^n reduces to

(4.4′) $\quad M_{\alpha\beta} = \nu b_j N^j h_{\alpha\beta}/2L,$

where ν is some scalar. Consequently, we have

Theorem 1. *Let the covariant vector field b_i be tangential to the hypersurface of a C-reducible Finsler space. Then the induced and intrinsic connection coincide over the Finsler hypersurface.*

§ 5. Induced and intrinsic connection parameters of Kropina hypersurface.

In virtue of (1.2), the induced connection parameters $F_{\beta}{}^{\alpha}{}_{\gamma}$ of a Kropina hypersurface H^{n-1} is written in the form

$$F_{\beta}{}^{\alpha}{}_{\gamma} = B^{\alpha}_{i}(B^{i}_{\beta\gamma}+\{{}_{j}{}^{i}{}_{k}\}B^{jk}_{\beta\gamma})+B^{\alpha}_{i}D_{j}{}^{i}{}_{k}B^{jk}_{\beta\gamma}. \tag{5.1}$$

Since the induced and intrinsic Christoffel symbols of the associated Riemannian hypersurface R^{n-1} are equal, from (2.24) and (5.1) we have

$$F_{\beta}{}^{\alpha}{}_{\gamma} = \{{}_{\beta}{}^{\alpha}{}_{\gamma}\}+V_{\beta}{}^{\alpha}{}_{\gamma}+\Phi(b^{\alpha}-2l^{\alpha})\bar{\Omega}_{\beta\gamma}/\bar{\rho}, \tag{5.2}$$

where we put

$$\text{(5.3)}\qquad \text{(a)}\quad V_{\beta}{}^{\alpha}{}_{\gamma} = B^{\alpha}_{i}D_{j}{}^{i}{}_{k}B^{jk}_{\beta\gamma},$$
$$\text{(b)}\quad \{{}_{\beta}{}^{\alpha}{}_{\gamma}\} = \bar{B}^{\alpha}_{i}(B^{i}_{\beta\gamma}+\{{}_{j}{}^{i}{}_{k}\}B^{jk}_{\beta\gamma}),$$

and $\bar{\Omega}_{\beta\gamma}$ are components of the second fundamental tensor of the Riemannian hypersurface R^{n-1}. Contraction of (5.3)a) by $\dot{u}^{\gamma}$ yields

$$V_{\beta}{}^{\alpha}{}_{\gamma}\dot{u}^{\gamma} = B^{\alpha}_{i}D_{j}{}^{i}{}_{0}B^{j}_{\beta}. \tag{5.4}$$

The intrinsic connection parameters $\bar{F}_{\beta}{}^{\alpha}{}_{\gamma}$ of a Kropina hypersurface H^{n-1} are given by

$$\bar{F}_{\beta}{}^{\alpha}{}_{\gamma} = \{{}_{\beta}{}^{\alpha}{}_{\gamma}\}+D_{\beta}{}^{\alpha}{}_{\gamma}, \tag{5.5}$$

where we put

$$D_{\beta}{}^{\alpha}{}_{\gamma} = -Q^{\alpha\varepsilon}(F_{\varepsilon\beta}l_{\gamma}+F_{\varepsilon\gamma}l_{\beta})-E_{\beta\gamma}Q^{\alpha}-h^{\alpha}_{\gamma}\Phi_{\beta}-h^{\alpha}_{\beta}\Phi_{\gamma}+h_{\beta\gamma}\Phi^{\alpha}+\bar{\lambda}C_{\beta}{}^{\alpha}{}_{\gamma}, \tag{5.6}$$

and

$$2E_{\alpha\beta} = b_{\alpha\beta}+b_{\beta\alpha},\qquad 2F_{\alpha\beta} = b_{\alpha\beta}-b_{\beta\alpha},$$
$$b^{\alpha} = a^{\alpha\beta}b_{\beta},\qquad a^{\alpha\beta}a_{\beta\gamma} = \delta^{\alpha}_{\gamma},\qquad \bar{\rho} = a_{\alpha\beta}b^{\alpha}b^{\beta},$$
$$Q^{\alpha} = (2l^{\alpha}-b^{\alpha})/\bar{\rho},\qquad Q^{\alpha\varepsilon} = (a^{\alpha\varepsilon}+Q^{\alpha}b^{\varepsilon})/2,$$
$$\text{(5.7)}\quad \Phi_{\alpha} = (\bar{\rho}F_{0'\alpha}/\beta-\bar{\rho}Q^{\varepsilon}F_{\varepsilon\alpha}+2b_{0'\alpha}/\bar{L}+F_{\varepsilon 0'}b^{\varepsilon}l_{\alpha}/\bar{L})/2\bar{\rho},$$
$$g^{\alpha\varepsilon}\Phi_{\varepsilon} = \Phi^{\alpha},\quad \bar{\lambda} = (E_{0'0'}/\bar{L}+F_{\varepsilon 0'}b^{\varepsilon})/\bar{\rho},\quad \Phi_{\alpha}\dot{u}^{\alpha} = \bar{\lambda}.$$

The suffix "0′" means the contraction by $\dot{u}^{\alpha}$. Contracting (5.6) by $\dot{u}^{\gamma}$ and using the relations $C_{\beta}{}^{\alpha}{}_{\gamma}\dot{u}^{\gamma} = 0$, $h_{\gamma}{}^{\alpha}\dot{u}^{\gamma} = 0$ and (5.7) we obtain

$$\text{(5.8)}\qquad D_{\beta}{}^{\alpha}{}_{0'} = -\{a^{\alpha\varepsilon}(\bar{L}F_{\varepsilon\beta}+F_{\varepsilon 0'}l_{\beta})+b^{\varepsilon}(2l^{\alpha}-b^{\alpha})(\bar{L}F_{\varepsilon\beta}+F_{\varepsilon 0'}l_{\beta})/\bar{\rho}\}/2$$
$$-E_{0'\beta}(2l^{\alpha}-b^{\alpha})/\bar{\rho}-\bar{\lambda}h^{\alpha}_{\beta}.$$

Differentiating (2.8) covariantly with respect to u^{β} in the Riemannian hypersurface R^{n-1}, we get

$$\nabla_{\beta}b_{\alpha} = b_{\alpha\beta} = b_{ij}B^{ij}_{\alpha\beta}+b_{i}\bar{I}^{i}_{\alpha\beta}, \tag{5.9}$$

where $\bar{I}^{i}_{\alpha\beta}$ $(=\nabla_{\beta}B^{i}_{\alpha})$ is the normal curvature vector of R^{n-1}. Since the unit normal vector of $\overline{R}^{n-1}$ is $\overline{N}^i$, (5.9) may be written as

(5.10) $$b_{\alpha\beta} = b_{ij}B^{ij}_{\alpha\beta} + b_i\overline{N}^i\overline{\Omega}_{\alpha\beta}.$$

From (5.7) and (5.9), we have

(5.11) $$(1)\quad E_{\alpha\beta} = E_{ij}B^{ij}_{\alpha\beta} + b_i\overline{N}^i\overline{\Omega}_{\alpha\beta}, \qquad (2)\quad F_{\alpha\beta} = F_{ij}B^{ij}_{\alpha\beta},$$

where we have used the fact that $\overline{\Omega}_{\alpha\beta}$ is symmetric in α and β. Owing to (5.2) and (5.5) the difference $\Lambda_\beta{}^\alpha{}_\gamma$ of the induced and intrinsic connection coefficients of a Kropina hypersurface are given by

(5.12) $$\Lambda_\beta{}^\alpha{}_\gamma = \overline{F}_\beta{}^\alpha{}_\gamma - F_\beta{}^\alpha{}_\gamma = D_\beta{}^\alpha{}_\gamma - V_\beta{}^\alpha{}_\gamma - \phi(b^\alpha - 2l^\alpha)\overline{\Omega}_{\beta\gamma}/\bar{\rho}.$$

Multiplying (5.12) by $\dot{u}^\gamma$, using (3.5), (5.4) and (5.8) we obtain $NM_{\alpha\beta}$ of the Kropina hypersurface H^{n-1}:

(5.13) $$\begin{aligned} NM_{\alpha\beta} &= (\lambda-\lambda')h_{\alpha\beta} - (2l_\alpha - g_{\alpha\gamma}b^\gamma)\{b^\varepsilon(\overline{L}F_{\varepsilon\beta} - F_{0'\varepsilon}l_\beta)/2 + E_{0'\beta}\}/\bar{\rho} \\ &- g_{\alpha\gamma}a^{\gamma\varepsilon}(\overline{L}F_{\varepsilon\beta} + F_{\varepsilon 0'}l_\beta)/2 + (2l_\alpha - g_{\alpha\gamma}B^\gamma_i b^i)\{b^r(LF_{rj}B^j_\beta + F_{r0}l_\beta)/2 + E_{j0}B^j_\beta\}/\rho \\ &+ g_{\alpha\gamma}B^\gamma_i a^{ir}(LF_{rj}B^j_\beta + F_{r0}l_\beta)/2 + \phi(2l_\alpha - g_{\alpha\gamma}b^\gamma)\overline{\Omega}_{\beta 0'}/\bar{\rho}. \end{aligned}$$

On direct calculation with the help of relations (1.1), (2.5), (2.6′), (2.8), (2.9), (2.19), (5.7) and (5.11), we get

(5.14) $$\begin{aligned} &2l_\alpha - g_{\alpha\beta}b^\beta = \bar{\rho}\bar{\tau}(2l_\alpha - \bar{\tau}b_\alpha), \quad 2l_\alpha - g_{\alpha\beta}B^\beta_j b^j = \tau\rho(2l_\alpha - \tau b_\alpha), \\ &b^\varepsilon F_{\varepsilon 0'} = b^j F_{j0} - \phi F_{j0}\overline{N}^j, \quad \phi = b_j\overline{N}^j, \quad b_{0'0'} = E_{00} + \Phi\overline{\Omega}_{0'0'}. \end{aligned}$$

Consequently, in virtue of (2.23), (2.24) and (5.14), $NM_{\alpha\beta}$ is written in the form

(5.13′) $$NM_{\alpha\beta} = (b_i\overline{N}^i)\{b_r\overline{N}^r(E_{00}/L + F_{r0}b^r)/\rho + \overline{\Omega}_{0'0'}/L - F_{r0}\overline{N}^r\}h_{\alpha\beta}.$$

From (5.13′) we have to discuss the two cases given by

(5.15) $$(A)\quad b_i\overline{N}^i = 0, \qquad (B)\quad b_i\overline{N}^i \neq 0.$$

First, we consider the case (A). In this case R^{n-1} is called a *tangential associated hypersurface,* because the covariant vector field b_i is tangential to the associated Riemannian hypersurface R^{n-1}. From (5.13′) and lemma 2 we can state

Theorem 2. *On a tangential associated Riemannian hypersurface, the induced and intrinsic connections coincide with each other.*

For a tangential associated hypersurface R^{n-1}, from (5.10) we obtain $b_{\alpha\beta} = b_{ij}B^{ij}_{\alpha\beta}$, so that it follows that

(5.16) $$b_{\alpha\beta}B^\alpha_j B^\beta_k = b_{hi}H^h_j H^i_k,$$

where we put $H_{hj} = a_{hj} - \overline{N}_h\overline{N}_j$ and $H^h_j = a^{hm}H_{mj}$. Since $\phi = b_i\overline{N}^i = 0$, if $\nabla_j\overline{N}^i = 0$, (5.16) yields $b_{\alpha\beta}B^{\alpha\beta}_{jk} = b_{jk}$. Thus we have

Theorem 3. *Assume that an associated Riemannian hypersurface R^{n-1} be tangential and the unit normal vector fiejd $\bar{N}^i$ of R^{n-1} is parallel with respect to the associated Riemannian connection. Then* $\nabla_j b_i = 0$ *if and only if* $\nabla_\alpha b_\beta = 0$.

In (5.10), if the vector field b_i is parallel with respect to the associated Riemannian connection, that is $b_{ij} = 0$, then we get

(5.17) $$b_{\alpha\beta} = b_i \bar{N}^i \Omega_{\alpha\beta}.$$

Here we can state

Theorem 4. *Assume that the covariant vector* b_i *be parallel with respect to the associated Riemannian connection and the associated Riemannian hypersurface R^{n-1} be not totally geodesic. Then an associated Riemannian hypersurface R^{n-1} is tangential if and only if $\nabla_\alpha b_\beta = 0$.*

Definition. A Finsler space is called an *affinely connected* space if the Berwald connection coefficients are functions of position only, such a space will be called a *Berwald sqace.*

Lemma 3 ([11]). *If the covariant vector field b_i is parallel with respect to the associated Riemannian connection, then the Kropina space is the Berwald one.*

From (5.17) and the above lemma, we have

Theorem 5. *If the vector field b_i is tangential to the Riemannian hypersurface R^{n-1}, then the Kropina hypersurface H^{n-1} is a Berwald space, provided that $b_{ij} = 0$.*

Next we consider the case $b_i \bar{N}^i \neq 0$. In virtue of (5.17), we have the following

Theorem 6. *Assume that the vector field b_i be parallel with respect to the associated Riemannian connection and $b_i \bar{N}^i \neq 0$. Then the associated hypersurface R^{n-1} is totally geodesic if and only if* $\nabla_\alpha b_\beta = 0$.

From the above theorem and the lemma 3, we obtain

Corollary. *Assume that the vector field b_i be parallel with respect to the associated Riemannian connection and* $b_i \bar{N}^i \neq 0$. *If the associated hypersurface R^{n-1} is totally geodesic, the Kropina hypersurface H^{n-1} is a Berwald space.*

Further from (5.13′) we get

Teeorem 7. *Assume that the vector field b_i be parallel with respect to the associated Riemannian connection. If the associated hypersurface R^{n-1} is totally geodesic, then the induced and intrinsic* *connections of a Kropina hypersurface coincide with each other, provided that* $N \neq 0$.

Next, we assume that the vector field b_i is gradient, that is $2F_{ij} = b_{ij} - b_{ji} = 0$. Then (5.13′) yields

(5.18) $$NM_{\alpha\beta} = (b_i\bar{N}^i)\{(b_r\bar{N}^r)b_{00}/\rho + \bar{\Omega}_{0'0'}\}/L.$$

Here we get

Theorem 8. *Assume that the vector field* b_i *be gradient and* $N \neq 0$, $b_i\bar{N}^i \neq 0$. *Then the induced and intrinsic connections of a Kropina hypersurface coincide with each other if and only if the relation*

(5.19) $$(b_i\bar{N}^i)b_{00}/\rho + \bar{\Omega}_{0'0'} = 0$$

holds.

Also, the following lemma has been proved by Brown [2] :

Lemma 4. *A geodesic of a Finsler hypersurface is a geodesic of a Finsler space if and only if* $N = \Omega_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta = 0$ *along the curve, where* $\Omega_{\alpha\beta}$ *are to be considered as the components of the second fundamental tensor of the Finsler hypersurface.*

Using the above lemma and (5.18), we get

Theorem 9. *Assume that the vector field* b_i *be gradient and* $M_{\alpha\beta} \neq 0$, $b_i\bar{N}^i \neq 0$. *Then a geodesic of a Kropina hypersurface* H^{n-1} *is a geodesic of a Kropina space* F^n *if and only if the* relation (5.19) holds.

From the above and (5.17) we can state

Theorem 10. *Assume that the vector field* b_i *be parallel with respect to the associated Riemannian connection and* $M_{\alpha\beta} \neq 0$, $b_i\bar{N}^i \neq 0$. *If* $\nabla_\alpha b_\beta = 0$, *then a geodesic of the Kropina hypersurface* H^{n-1} *is a geodesic of a Kropina space* F^n.

Acknowledgement. We wish to express our gratitude to Prof. Dr. M. Matsumoto for his valuable advice and criticism. The third author is extremely grateful to Dr. B.N. Prasad for his encouragement during the preparation of this paper.

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